On the Holroyd-Talbot Conjecture for Sparse Graphs

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Abstract

Given a graph G, let $\mu(G)$ denote the size of the smallest maximal independent set of G. A family of sets is called a *star* if some element is in every set of the family. A *split* vertex has degree at least 3. Holroyd and Talbot conjectured the following Erdős-Ko-Rado-type statement about intersecting families of independent sets of graphs: if $1 \leq r \leq \mu(G)/2$ then there is an intersecting family of independent r-sets of maximum size that is a star. In this paper we prove similar statements for sparse graphs on n vertices: roughly, for graphs of bounded average degree with $r \leq O(n^{1/3})$, for graphs of bounded degree with $r \leq O(n^{1/2})$, and for trees having a bounded number of split vertices with $r \leq O(n^{1/2})$.

1 Introduction

For $0 \leq r \leq n$, let $\binom{[n]}{r}$ denote the family of r-element subsets (r-*sets*) of $[n] = \{1, 2, \ldots, n\}$. For any family $\mathcal{F}$ of sets, define the shorthand $\cap\mathcal{F} = \cap_{S \in \mathcal{F}} S$.

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If $\cap\mathcal{F} \neq \emptyset$, we say that $\mathcal{F}$ is a *star*; in this case, any $x \in \cap\mathcal{F}$ is called a *center*. The family $\mathcal{F}_x = \{S \in \mathcal{F} \mid x \in S\}$ is called the *full star of $\mathcal{F}$ at x*. Furthermore, we define the notation $\mathcal{F}^r = \{S \in \mathcal{F} \mid |S| = r\}$. The family $\mathcal{F}$ is *intersecting* if every pair of its members intersects.

Erdős, Ko, and Rado [11] proved the following classical theorem of central importance in extremal set theory.

Theorem 1. (Erdős-Ko-Rado, 1961) *If $\mathcal{F} \subseteq \binom{[n]}{r}$ is intersecting for $r \leq n/2$, then $|\mathcal{F}| \leq \binom{n-1}{r-1}$. Moreover, if $r < n/2$, equality holds if and only if $\mathcal{F} = \binom{[n]}{r}_x$ for some $x \in [n]$.*

Hilton and Milner [16] proved the following stronger stability result.

Theorem 2. (Hilton-Milner, 1967) *If $\mathcal{F} \subseteq \binom{[n]}{r}$ is intersecting for $r \leq n/2$, and $\mathcal{F}$ is not a star, then $|\mathcal{F}| \leq \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$.*

For a graph G, let $\mathcal{I}(G)$ denote the family of independent sets of G. We write $s_r(v) = |\mathcal{I}_v^r(G)|$ when G is understood. Let $\mathcal{F} \subseteq \mathcal{I}^r(G)$ be an intersecting subfamily of maximum size. We say that G is r-EKR if some v satisfies $s_r(v) = |\mathcal{F}|$, and *strictly* r-EKR if every such $\mathcal{F}$ equals $\mathcal{I}_v^r(G)$ for some v.

Write $\alpha(G)$ for the independence number of G. Let $\mu(G)$ denote the size of a smallest maximal independent set of G. Equivalently, $\mu(G)$ is the size of the smallest independent dominating set of G. Holroyd and Talbot [18] made the following conjecture.

Conjecture 3. (Holroyd-Talbot, 2005) *For any graph G, if $1 \leq r \leq \mu(G)/2$ then G is r-EKR.*

Of course, this conjecture is true for the empty graph by Theorem 1. While not explicitly stated in graph-theoretic terms, earlier results by Berge [2], Deza and Frankl [10], and Bollobás and Leader [4] support the conjecture. For example, the case of G equal to a disjoint union of k complete graphs of sizes

$n_1 \leq \cdots \leq n_r$ was verified (in fact for all $r \leq \alpha(G)$) in [4, 10] for the uniform case $2 \leq n_1 = \cdots = n_k$, in [17] for the non-uniform case $2 \leq n_1 \leq \cdots \leq n_k$, and in [3] for the general case. The cases of G being a power of either a path [17] or a cycle [22], or a *special chain* (essentially, a path of complete graphs of increasing size) or the disjoint union of two special chains [19], were both verified for all $r \leq \alpha(G)$ as well. The conjecture has been proven for $\mu(G)$ sufficiently large in terms of r [5], and also for various graph classes, for example, disjoint unions of complete graphs, paths, and cycles containing at least one isolated vertex [7, 17], disjoint unions of complete multipartite graphs containing at least one isolated vertex [8], disjoint unions of length-2 paths [14], chordal graphs containing an isolated vertex [19], and others. In fact, for the cases of complete graphs and cycles just mentioned, [7] extends the range of r beyond $\mu(G)/2$ to $\alpha(G)/2$. One can observe, for example, that the complete k-partite graph $G = K_{n_1,\ldots,n_k}$ is r-EKR for all $r \leq \alpha(G)/2$, because every independent set is contained in some part. However, G is not r-EKR for $\alpha(G)/2 < r \leq \alpha(G)$.

For vertices u and v in a graph G, we use the notations $\deg_G(u)$ and $\mathsf{dist}_G(u, v)$ for the degree of vertex u and the distance between u and v in G, respectively; we may omit the subscript if the context is clear.

2 Results

Here we prove the following theorem.

Theorem 4. *Let r and d be positive integers. Suppose that G is a graph on $n > \frac{27}{8}dr^2$ vertices, having maximum degree less than d. Then G is r-EKR.*

We can expand the class of graphs beyond bounded degree to bounded average degree at the cost of reducing the range of r from $O(n^{1/2})$ to $O(n^{1/3})$, as follows.

Theorem 5. *Given a positive integer r, let $c \geq e/36$ be a constant. Suppose that G is a graph on $n > 18cr^3$ vertices, having at most cn edges. Then G is r-*EKR*.*

It is likely that a quadratic bound on n is possible for Theorem 5 as well. Note that the case $c = 1$ in Theorem 5 is especially relevant for trees. In this case, we can retrieve a quadratic lower bound for n for one special class of trees.

A *split vertex* in a graph is a vertex of degree at least three. A *spider* is a tree with exactly one split vertex. For a spider S with split vertex w and leaves $v_1, \ldots, v_k$, we write $S = S(\ell_1, \ldots, \ell_k)$, where $\ell_i = \mathsf{dist}(w, v_i)$. The notation is written in *spider order* when the following conditions hold:

- if ℓ_i and ℓ_j are both odd and $\ell_i < \ell_j$ then $i < j$;
- if ℓ_i and ℓ_j are both even and $\ell_i < \ell_j$ then $i > j$; and
- if ℓ_i is odd and ℓ_j is even then $i < j$.

Notice that, since every independent set of $S(1, 1, \ldots, 1)$ is a subset of its leaves, Conjecture 3 is true for $S(1, 1, \ldots, 1)$. In an attempt to prove the Holroyd-Talbot conjecture for spiders by induction, the authors of [20] proved the following result.

Theorem 6. (Hurlbert-Kamat, 2022) *Suppose that $S = S(\ell_1, \ldots, \ell_k)$ is a spider written in spider order. Let w be the split vertex of S, for each i let u_i be any vertex on the wv_i-path, and suppose that $r \leq \alpha(S)$. Then*

1. $s_r(w) \leq s_r(v_i)$ *for all* i,
2. $s_r(u_i) \leq s_r(v_i)$ *for all* i, *and*
3. $s_r(v_j) \leq s_r(v_i)$ *for all* $i < j$.

Estrugo and Pastine [12] call a tree T r-HK if $s_r(v)$ is maximized at a leaf of T (and HK if r-HK for all $r \leq \alpha(T)$). It is proved in [19] that every tree is r-HK for $r \leq 4$, but Baber [1], Borg [6], and Feghali, Johnson, and Thomas [13] each found counterexamples when $r \geq 5$. However, parts 1 and 2 of Theorem 6 together imply that every spider S is HK. Theorem 5 shows that spiders are r-EKR for $r < (n/18)^{1/3}$. Unfortunately, $\mu/2$ for spiders is roughly $n/6$, so there remains a big gap. Our next theorem shrinks that gap somewhat.

Theorem 7. *Let $S = S(\ell_1, \ldots, \ell_k)$ be a spider on n vertices, with split vertex w and leaves $v_1, \ldots, v_k$. Suppose that $r \leq \sqrt{n \ln 2} - (\ln 2)/2$. Then S is r-EKR.*

We note that every spider S has $\alpha(S) = 1 > \sqrt{n \ln 2} - (\ln 2)/2$ for $n \leq 2$, $\alpha(S) = 2 > \sqrt{n \ln 2} - (\ln 2)/2$ for $n = 3$, and $\alpha(S) \geq (n-1)/2 > \sqrt{n \ln 2} - (\ln 2)/2$ for $n \geq 4$. In other words, the hypothesis of Theorem 7 implies that $r \leq \alpha(S)$ for all n.

Finally, we prove the following similar result for more general trees.

Theorem 8. *Let T be a tree on n vertices, with exactly $s > 1$ split vertices. Suppose that $1 < s < r/2$ and $r \leq \sqrt{n \ln c} - (\ln c)/2$, where $c = 2 - 2s/r$. Then T is r-EKR.*

3 Technical Lemmas

Proposition 9. *If $0 \leq x \leq 2k/(k+1)^2$ for some $k \geq 1$, then $e^{-x} < 1 - \left(\frac{k}{k+1}\right) x$.*

Proof. Let $0 \leq x \leq 2k/(k+1)^2$ for some $k \geq 1$. Then $|x| < 1$, and so $e^{-x} = \sum_{i \geq 0} (-x)^i / i! < 1 - x + x^2/2$. Also, $(k+1)x < 2$, which implies that $x^2/2 < x/(k+1) = [1 - k/(k+1)]x$. Thus $e^{-x} < 1 - x + x^2/2 < 1 - \left(\frac{k}{k+1}\right) x$. □

Corollary 10. *If $0 \leq y \leq 2k^2/(k+1)^3$ for some $k \geq 1$, then $1 - y > e^{-\left(\frac{k+1}{k}\right) y}$.*

Proof. Set $x = \left(\frac{k+1}{k}\right) y$ and apply Proposition 9. □

Lemma 11. *If* $r \geq 2$, $d \geq 2$, *and* $n \geq \frac{27}{8}dr^2$, *then* $\prod_{i=1}^{r-1}\left(1 - \frac{r+id}{n}\right) > \frac{r}{n}$.

Proof. We begin with

$$\prod_{i=1}^{r-1}\left(1 - \frac{r+id}{n}\right) \geq 1 - \sum_{i=1}^{r-1}\frac{r+id}{n} = 1 - \frac{r(r-1) + d\binom{r}{2}}{n} = 1 - \frac{(d+2)\binom{r}{2}}{n}.$$

Since $d \geq 2$, and by using Corollary 10 with $y = dr^2/n$ and $k = 2$, we have

$$1 - \frac{(d+2)\binom{r}{2}}{n} > 1 - \frac{dr^2}{n} > e^{-3dr^2/2n} > e^{-4/9} > .64\ .$$

In addition, we calculate

$$\frac{r}{n} \leq \frac{8}{27dr} \leq \frac{2}{27} < .08\ ,$$

which completes the proof. □

Claim 12. *Let* G *be a graph with* n *vertices and maximum degree less than* d. *Then every vertex* v *satisfies*

$$s_r(v) \geq \frac{1}{(r-1)!}(n-d)(n-2d)\cdots(n-(r-1)d).$$

Proof. Let W_0 be the set of vertices of G, and set $w_0 = v$. For each $0 < i < r$, choose $w_i \in W_i$, where $W_{i+1} = W_i - N[w_i]$. Then by induction we have $|W_i| \geq n - id$ for each such i. The resulting set $\{w_0, \ldots, w_{r-1}\}$ is independent in G and there are at least $\prod_{0<i<r}(n - id)$ ways to choose such sets, ignoring replication. Accounting for replication, we obtain the result. □

Lemma 13. *Let* H *be a graph with at least* $m = n(1 - 1/3r)$ *vertices and maximum degree less than* d. *Suppose that* $1/3r + rd/n \leq 2k^2/(k+1)^3$ *for some*

$k \geq 1$. *Then every vertex v satisfies*

$$s_r(v) \geq \frac{n^{r-1}}{(r-1)!} e^{-(r-1)2k/(k+1)^2}.$$

Proof. We use Claim 12 and Corollary 10 with $y = 1/3r + rd/n$ to obtain

$$\begin{aligned} s_r(v) &\geq \frac{1}{(r-1)!} \prod_{0<i<r} (m - id) \geq \frac{n^{r-1}}{(r-1)!} \prod_{0<i<r} \left(1 - \frac{1}{3r} - \frac{id}{n}\right) \\ &\geq \frac{n^{r-1}}{(r-1)!} \prod_{0<i<r} \left[1 - \left(\frac{1}{3r} + \frac{rd}{n}\right)\right] \geq \frac{n^{r-1}}{(r-1)!} \prod_{0<i<r} e^{-\left(\frac{k+1}{k}\right)\left(\frac{1}{3r} + \frac{rd}{n}\right)} \\ &\geq \frac{n^{r-1}}{(r-1)!} e^{-(r-1)\left(\frac{k+1}{k}\right)\left(\frac{1}{3r} + \frac{rd}{n}\right)} \geq \frac{n^{r-1}}{(r-1)!} e^{-(r-1)2k/(k+1)^2}. \end{aligned}$$

□

4 Proof of Theorem 4

We use the following result of Frankl [15]. For $\mathcal{F} \subseteq \binom{[n]}{r}$, define $\overline{\mathcal{F}_x} = \mathcal{F} - \mathcal{F}_x$.

Theorem 14. (Frankl, 2020) *If $\mathcal{F} \subseteq \binom{[n]}{r}$ is intersecting and $r < n/72$, then there is some x such that $|\overline{\mathcal{F}_x}| \leq \binom{n-3}{r-2}$.*

Proof of Theorem 4. The result is trivial for $r = 1$ or $d = 1$, so we assume $r \geq 2$ and $d \geq 2$. Let x be as in Theorem 14, and select $E \in \overline{\mathcal{F}_x}$, which we may assume to be nonempty. Via the same counting method as in Claim 12, we have at least

$$\frac{1}{(r-1)!}(n-r-d)(n-r-2d)\cdots(n-r-(r-1)d) \tag{1}$$

r-sets $F \in \mathcal{I}_r(x)$ with $F \cap E = \emptyset$. Since $\mathcal{F}$ is intersecting, these sets are not in

$\mathcal{F}_x$. Therefore, using Theorem 14 and the bound in (1), we have

$$\begin{aligned} |\mathcal{F}| &= |\mathcal{F}_x| + |\overline{\mathcal{F}_x}| \\ &\leq |\mathcal{I}_r(x)| - \frac{(n-r-d)\cdots(n-r-(r-1)d)}{(r-1)!} + \binom{n-3}{r-2}. \end{aligned}$$

This upper bound is at most $|\mathcal{I}_r(x)|$ precisely when

$$\binom{n-3}{r-2} \leq \frac{1}{(r-1)!}\prod_{i=1}^{r-1}(n-r-id),$$

which we rewrite as

$$\prod_{i=1}^{r-1}(n-r-id) \geq (r-1)!\binom{n-3}{r-2} = (r-1)\prod_{i=1}^{r-2}(n-2-i).$$

This inequality will follow from showing that

$$\prod_{i=1}^{r-1}(n-r-id) \geq rn^{r-2},$$

which holds by Lemma 11, and which completes the proof.

5 Proof of Theorem 5

The result is trivial for $r=1$, so we may assume that $r \geq 2$. Let V_0 be the set of vertices of G. For each $i \geq 0$, choose $v_i \in V(G_i)$ such that $\deg_{G_i}(v_i) \geq 3cr$, where $G_{i+1} = G_i - v_i$. Let t be minimum such that $\Delta(G_t) < 3cr$. The number of edges removed in this process is at least $3tcr$, which must be at most the number of edges of G; thus $t \leq n/3r$. Hence $V(G_t) = n - t \geq n(1-1/3r)$.

Now we set $d = 3cr$, $k = 4r-7 \geq 1$, and calculate that

$$(k+3) + \left(\frac{3k+1}{k^2}\right) \leq k+7 = 4r,$$

so that $(k+1)^3 \leq 4k^2 r$, which implies that

$$\frac{1}{3r} + \frac{rd}{n} < \frac{1}{3r} + \frac{3cr^2}{18cr^3} = \frac{1}{2r} \leq \frac{2k^2}{(k+1)^3}.$$

This allows the use of Lemma 13 with $H = G_t$, $m = n(1 - 1/3r)$, and $d = 3cr$. We obtain that each vertex v of G_t has $s_r(v)$ at least

$$\frac{n^{r-1}}{(r-1)!} e^{-(r-1)2k/(k+1)^2}. \tag{2}$$

Now we use Theorem 2 to show that any intersecting family $\mathcal{F}$ of independent r-sets that is not a star has size less than (2). First, we note the combinatorial identity $\binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1 = 1 + \binom{n-2}{r-2} + \binom{n-3}{r-2} + \cdots + \binom{n-r-1}{r-2}$. Second, we observe the inequality $r^2/n < e^{-(r-1)2k/(k+1)^2}$. Indeed,

$$\frac{r^2}{n} < \frac{1}{18cr} \leq e^{-1} \leq e^{-(r-1)(8r-14)/(4r-6)^2} = e^{-(r-1)2k/(k+1)^2},$$

because $c \geq e/36$ and $(4r-6)^2 > (r-1)(8r-14)$ (since $r \geq 2$).

Finally, if $\mathcal{F}$ is as above, then we have

$$|\mathcal{F}| < r\binom{n-2}{r-2} = \frac{r(r-1)}{n-1}\binom{n-1}{r-1} < \frac{r^2}{n} \cdot \frac{n^{r-1}}{(r-1)!} < \frac{n^{r-1}}{(r-1)!} e^{-(r-1)2k/(k+1)^2}.$$

This finishes the proof.

6 Proof of Theorem 7

Lemma 15. *Let $S = S(\ell_1, \ldots, \ell_k)$ be a spider on n vertices and let v be a leaf of S. Suppose that $r \leq \alpha(S)$. Then*

$$s_r(v) \geq \binom{n-r-1}{r-1} + \binom{n-k-r-2}{r-2}.$$

Proof. Let $S = S(\ell_1, \ldots, \ell_k)$, in spider order. We may assume that $v = v_k$ and then use Theorem 6 for the other leaves. For $S(1, 1, \ldots, 1)$ we have $s_r(v) = \binom{n-2}{r-1}$ and $k = n - 1$, so that $\binom{n-k-r-2}{r-2} = 0$ and $\binom{n-2}{r-1} \geq \binom{n-r-1}{r-1}$. Thus we may assume that $\ell_k \geq 2$, implying that v and w are not adjacent.

We first count the number of independent r-sets containing v that do not contain the split vertex w. The number of such sets is

$$|\mathcal{I}_v^r(S - w)| = |\mathcal{I}_v^r(\cup_{i=1}^k P_{\ell_i})|,$$

where P_{ℓ_i} denotes the path on ℓ_i vertices.

Next we add edges to the disjoint union of paths, joining the many paths together to form one long path, thus reducing the number of independent r-sets that contain v but not w. For each $1 \leq i \leq k$, let u_i be the neighbor of w on the wv_i-path in S; that is, the endpoint of the i^{th} path of $S - w$ that is different from v_i. Now, for each $1 \leq i < k$, add the edge $v_i u_{i+1}$. Finally, remove v and its unique neighbor, resulting in the graph P_m, for $m = n - 3$. This results in the inequality

$$|\mathcal{I}_v^r(\cup_{i=1}^k P_{\ell_i})| \geq |\mathcal{I}^{r-1}(P_m)|.$$

We relabel the vertices of P_m as $x_1, \ldots, x_m$, in order. Observe that $\{x_{a_1}, x_{a_1+a_2}, \ldots, x_{a_1+\cdots+a_{r-1}}\}$ is independent in P_m if and only if

$$\sum_{i=1}^{r} a_i = m, \; a_1 \geq 1, \; a_i \geq 2 \text{ for } 1 < i < r, \text{ and } a_r = m - a_{r-1} \geq 0. \tag{3}$$

Set $b_1 = a_1 - 1$, $b_i = a_i - 2$ for $1 < i < r$, and $b_r = a_r$. Then system (3) can be rewritten as

$$\sum_{i=1}^{r} b_i = m - 2r + 3 = n - 2r, \text{ with } b_i \geq 0, \text{ for all } 1 \leq i \leq r. \tag{4}$$

It is well known that the number of integer solutions to system (4) equals

$$\binom{n-2r+r-1}{r-1} = \binom{n-r-1}{r-1}.$$

Second, we count the number of independent r-sets containing v that also contain the split vertex w. The number of such sets equals

$$|\mathcal{I}_v^{r-1}(S - N[w])| = |\mathcal{I}_v^{r-1}(\cup_{i=1}^k P_{\ell_i - 1})|.$$

As above, we add edges to the disjoint union of paths, to reduce the number of independent r-sets that contain v and w. For each $1 \leq i \leq k$, let u_i' be the neighbor of u_i other than w on the wv_i-path in S. Now, for each $1 \leq i < k$, add the edge $v_i u_{i+1}'$. Finally, remove v and its unique neighbor, resulting in the graph $P_{m'}$, for $m' = n - 3 - k$. This results in the inequality

$$|\mathcal{I}_v^{r-1}(\cup_{i=1}^k P_{\ell_i - 1})| \geq |\mathcal{I}^{r-2}(P_{m'})|.$$

Counting via the same method as above, we obtain

$$|\mathcal{I}^{r-2}(P_{m'})| = \binom{n-k-r-2}{r-2}$$

such sets, which completes the proof. □

Proof of Theorem 7. It is easy to check that $r \leq \sqrt{n \ln 2} - (\ln 2)/2$ implies that $r^2 \leq (n-r)\ln 2$. We use this in the calculations below.

Using Lemma 15 with Theorem 2, as in the proof of Theorem 5, the result will follow from proving the inequality

$$\binom{n-1}{r-1} < 2\binom{n-r-1}{r-1}. \tag{5}$$

To accomplish this, we denote $m^{\underline{t}} = m!/(m-t)!$ and calculate the ratio

$$\begin{aligned}\binom{n-1}{r-1}\Big/\binom{n-r-1}{r-1} &= \frac{(n-1)^{\underline{r-1}}}{(n-r-1)^{\underline{r-1}}} \leq \frac{(n-r+1)^{r-1}}{(n-2r+1)^{r-1}} \\ &= \left(\frac{n-2r+1}{n-r+1}\right)^{-(r-1)} = \left(1-\frac{r}{n-r+1}\right)^{-(r-1)} \\ &\leq e^{r(r-1)/(n-r+1)} < e^{r^2/(n-r)} \qquad (6) \\ &\leq e^{\ln 2} = 2,\end{aligned}$$

which finishes the proof.

7 Proof of Theorem 8

Lemma 16. *Let T be a tree on n vertices with exactly $s > 1$ split vertices, and let v be a leaf of T. Suppose that $r \leq \alpha(T)$. Then*

$$s_r(v) \geq \binom{n-r-s}{r-1} + 1.$$

Proof. Let W denote the set of split vertices of T. We need only count the number of independent r-sets containing v that do not contain any split vertex. The number of such sets equals

$$\mathcal{I}_v^r(T-W)| > |\mathcal{I}_v^r(P_{n-s})| = |\mathcal{I}^{r-1}(P_{n-s-2})| = \binom{n-r-s}{r-1},$$

as in the proof of Lemma 15.

The strict inequality comes from the existence of at least one independent r-set of $T-W$ that is not independent in P_{n-s} because of the joining of the many paths that create P_{n-s}. For example, let P' and P'' be two paths in $T-W$ that are consecutive in P_{n-s}, with endpoints $u' \in P'$ and $u'' \in P''$ such that u' is adjacent to u'' in P_{n-s}. Let $A \in \mathcal{I}_v^r(P_{n-s})$, define a' to be the vertex in A that is closest to u', a'' to be the vertex in $A - \{a'\}$ that is closest to u'', and $A' = (A - \{a', a''\}) \cup \{u', u''\}$. Then $A' \in \mathcal{I}_v^r(T - W) - \mathcal{I}_v^r(P_{n-s})$. □

Proof of Theorem 8. As in the proof of Theorem 7, we use Lemma 16 and Theorem 2, which reduces the proof to certifying the inequality

$$\binom{n-1}{r-1} \leq \binom{n-r-1}{r-1} + \binom{n-r-s}{r-1}. \tag{7}$$

Suppose that $1 < s < r/2$ and $r \leq \sqrt{n \ln c} - (\ln c)/2$, where $c = 2 - 2s/r$. Let $a = \frac{r^2}{\ln c} + r$ and $b = \frac{(r+2)^3}{2(r+1)} + r + s - 1$. By rearranging the given condition on r, we obtain $n \geq a + \frac{\ln c}{4} > a$. Now let $d = 2(r+1)\ln c$ so that we have

$$\begin{aligned}
d(a-b) &= 2(r+1)r^2 - (\ln c)\left[(r+2)^3 + 2(r+1)(s-1)\right] \\
&> 2(r+1)r^2 - (\ln 2)\left[(r+2)^3 + (r+1)(2s-2)\right] \\
&> 2(r+1)r^2 - 0.7\left[(r+2)^3 + (r+1)(r-2)\right] \\
&= 2r^3 + 2r^2 - 0.7\left(r^3 + 7r^2 + 11r + 6\right) \\
&= 1.3r^3 - 2.9r^2 - 7.7r - 4.2 \\
&> 0
\end{aligned}$$

since $r \geq 5$. Because $a - b > 0$ and $n > a$, we have $n > b$, which is equivalent to

$$\frac{r+1}{n-r-s+1} < \frac{2(r+1)^2}{(r+2)^3}. \tag{8}$$

Next, we derive the following estimates, using Inequality 8 to access Corollary

10 with $y = (r+1)/(n-r-s+1)$ and $k = r+1$.

$$\begin{aligned}
\frac{\binom{n-r-1}{r-1}+\binom{n-r-s}{r-1}}{\binom{n-r-1}{r-1}} &= 1+\frac{(n-2r)^{\underline{s-1}}}{(n-r-1)^{\underline{s-1}}} \geq 1+\left(\frac{n-2r-s+2}{n-r-1-s+2}\right)^{s-1} \\
&> 1+\left(\frac{n-2r-s}{n-r-s+1}\right)^{s} = 1+\left(1-\frac{r+1}{n-r-s+1}\right)^{s} \\
&> 1+e^{-\left(\frac{r+2}{r+1}\right)\left(\frac{r+1}{n-r-s+1}\right)s} > 1+e^{-\left(\frac{r+2}{r+1}\right)\left(\frac{2(r+1)^2}{(r+2)^3}\right)s} \\
&= 1+e^{-\left(\frac{2(r+1)}{(r+2)^2}\right)s} > 1+e^{-2s/r} > 2-2s/r.
\end{aligned}$$

The assumption that $s < r/2$ makes the final result greater than 1. Finally, we follow Inequality (6), since $r \leq \sqrt{n \ln c} - (\ln c)/2$ implies that $r \leq \sqrt{n \ln 2} - (\ln 2)/2$, and calculate the ratio

$$\binom{n-1}{r-1} \Big/ \binom{n-r-1}{r-1} < e^{r^2/(n-r)} \leq e^{\ln(2-2s/r)} = 2-2s/r,$$

which finishes the proof.

8 Questions and Remarks

It is clear that improving the orders of magnitude in the upper bound on r in our results will require techniques other than comparison to the Hilton-Milner bounds. To that end, the specificity of spider structure and the knowledge of the location of their biggest stars begs for a proof that they are r-EKR for $r \leq \mu/2$ (or possibly $r \leq \alpha$).

Along these lines, consider the family $\mathcal{T}$ of all trees having no vertex of degree 2. The authors of [20] conjecture that every tree in $\mathcal{T}$ is HK. Naturally, we believe that such trees are r-EKR for all $r \leq \mu(T)$ as well. As a first step in this direction, for $i \in \{1,2,3\}$, let $T_i(h)$ be a complete binary tree of depth h (i.e. having $2^{h+1}-1$ vertices), with root vertex v_i. Note that v_i is the unique degree-

2 vertex in $T_i(h)$. Now define the tree $T(h)$ by $V(T(h)) = \{w\} \cup_{i=1}^{3} V(T_i(h))$, with w adjacent to each v_i. Then $T(h) \in \mathcal{T}$.

Problem 17. *Show that $T(h)$ is r-EKR for all $r \le \mu(T(h))/2$.*

Finally, we say that a family $\mathcal{F}$ of sets is EKR if it has the property that if $\mathcal{H}$ is an intersecting subfamily of $\mathcal{F}$ then there is some element x such that $|\mathcal{H}| \le |\mathcal{F}_x|$, and that a graph G is EKR if $\mathcal{I}(G)$ is EKR. We observe that the non-uniform case — considering $\mathcal{I}(G)$ instead of $\mathcal{I}^r(G)$ — has yet to be studied specifically for graphs. Of course, this is a special case of Chvátal's conjecture (see [9]) that every subset-closed family $\mathcal{F}$ of sets is EKR. For example, by a result of [21], every graph with an isolated vertex is EKR. Also, powers of paths or cycles (resp. special chains) are EKR by the results of [17, 22] (resp. [19]) for fixed r because we can use the same star center for each r. The same holds for disjoint unions of complete graphs because the star center is either an isolated vertex, if it exists, or a vertex in a smallest component. Any vertex-transitive graph G that is r-EKR for all $r \le \alpha(G)$ would also be EKR. It is conjectured in [14] that if G is a disjoint union of length-2 paths then it is r-EKR for all $\mu(G)/2 < r \le \alpha(G)/2$. It may also be true for all $r \le \alpha(G)$, which would imply that G is EKR because the largest star is always centered on a leaf, and all leaves look alike. Additionally, if one could prove that every spider S is r-EKR for all $r \le \alpha(S)$ then it would follow from Theorem 6 that spiders are EKR. Of course, while complete k-partite graphs G are not r-EKR for $\alpha(G)/2 < r \le \alpha(G)$, that does not mean that they are not EKR.

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