Zero Measure, Infinite Count?

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Abstract

Although infinity cannot be shown physically, we can visualize it the same way as we can imagine the existence of a formula, an irrational number like π, or a perfect cube. According to Hans Hahn, "There are no perfect cubes in the physical world, only in the world of ideas." It is here then, in the world of ideas that we will tap the peculiarities of infinity to show that a set can have a measure of zero while being uncountably infinite.

Procedure

In this paper, arguments will be constructed using basic concepts of set theory. Most readers will already be familiar with these concepts, but since they are necessary to the final proof, we prefer their inclusion so the reader may follow each step.

After an initial introduction to sets and operations, it will be shown that two sets can have equal cardinality. From this we will derive a rule that determines equipotence (equal cardinality), apply it to infinite sets, and show $|\mathbf{Q}| = |\mathbf{N}|$. Note that throughout the paper we will use the symbol **Q** and $\mathbb{Q}$, **N** and $\mathbb{N}$, **R** and $\mathbb{R}$, and **C** and $\mathbb{C}$ interchangeably. Furthermore, by investigating the cardinality of set $\mathbb{R}$, the set of real numbers via two methods, we will discover the existence of other types of infinity and how as this set approaches zero, it can still be uncountably infinite.

Introduction

When we refer to "sets" we simply mean a collection of objects. A set could be a collection of coins, a set of molecules in a cup of coffee, or a set of dishes. These sets are much different than describing a bag of coins, a cup of coffee, or china, because they are considered as a whole rather than a collection of items.

Georg Cantor, the mathematician who founded the set theory branch of mathematics, defines a set as, "…a collection into a whole of definite distinct objects of our intuition or thought. The objects are called elements (members) of the set." Although we do not always know what grants membership into a set, we do know that the individual elements of a set are related by definition, by function, or by property.

I. Set Operations

In this paper we will be utilizing two operators - "proper subset" represented by the symbol "$\subset$" and the "union operation" represented by the symbol "$\cup$". A proper subset, **Y,** of set **X**, exists when all elements in **Y**, are also in **X,** and $\mathbf{Y} \neq \mathbf{X}$. It is represented symbolically as $\mathbf{Y} \subset \mathbf{X}$. New sets can be created from the union of two sets, in which the union function combines the elements of two sets without repeats. An example of the union of two finite sets is shown below:

Let **A** = {a , b , c , …, z}; **B** = {1, 2, 3, …, 26} then **A** union **B**

$= \{a, b, c, \ldots, z\} \cup \{1, 2, 3, \ldots, 26\}$

$= \{a, 1, b, 2, c, 3, \ldots, z, 26\}$

Note when these two operations are used together within a statement, they illustrate the following relationships where each part is less than the whole:

$\mathbf{A} \subset (\mathbf{A} \cup \mathbf{B})$, if $\mathbf{B} \neq \emptyset$ $\qquad$ $\mathbf{B} \subset (\mathbf{A} \cup \mathbf{B})$, if $\mathbf{B} \neq \emptyset$

If we use the union and subset notations on the most widely used mathematical sets we observe the following relationships:

$$\emptyset \subset \mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C};$$

$\mathbb{C} = \mathbb{C} \cup \{a + bi \mid a,b \in \mathbb{R}\}$

(where $\mathbb{C}$ is the set of complex numbers)

$\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} - \mathbb{Q})$

(where $(\mathbb{R}-\mathbb{Q})$ is the set of irrational numbers)

$\mathbb{Q} = \mathbb{Z}^{-} \cup \mathbb{W}$

(where $\mathbb{Z}$ is the set of all integers)

$\mathbb{W} = \mathbb{N} \cup \{0\}$

(where $\mathbb{W}$ is the set of whole numbers)

$\mathbb{N} = \emptyset \cup \{1, 2, 3, \ldots\}$

(where $\emptyset$ is the empty set)

The concept of a proper subset within an infinite set will be addressed in the Cardinality of Infinite Sets section of this paper.

II. Bijective Functions

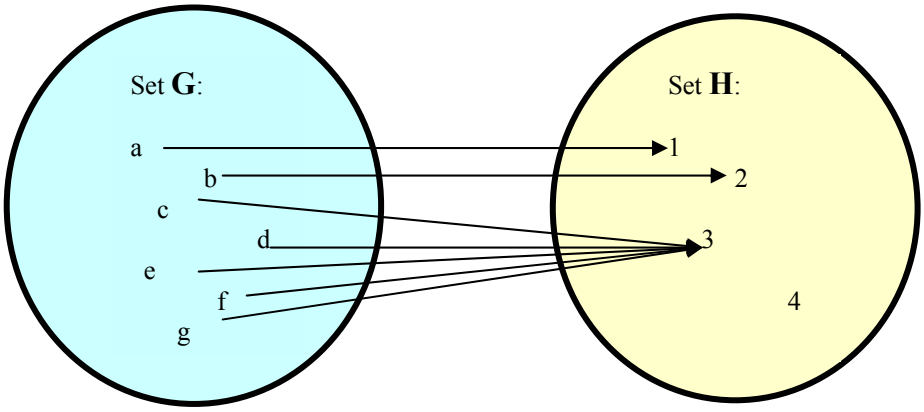
A "bijective function" imparts a "one to one correspondence" between two sets. For a function to be bijective it must be both a surjective and an injective function. A surjective function "f" maps all elements in one set, here we will call it **G,** onto another set, **H**, such that $f(\mathbf{G}) = \mathbf{H}$ or we say that for every h element in **H**, there exists a corresponding g element in **G**. Diagram #1 below illustrates a surjective function:

let $\mathbf{G} = \{a, b, c, d, e, f, g\}$; $\mathbf{H} = \{1, 2, 3, 4\}$, and function f: $\mathbf{G} \rightarrow \mathbf{H}$:

$f(a) = 1$ $f(b) = 2$ $f(c) = 3$ $f(d) = 3$ $f(e) = 3$

$f(f) = 3$ $f(g) = 3$ thus $f(\mathbf{G}) = \{1, 2, 3\}$.

Diagram #1:



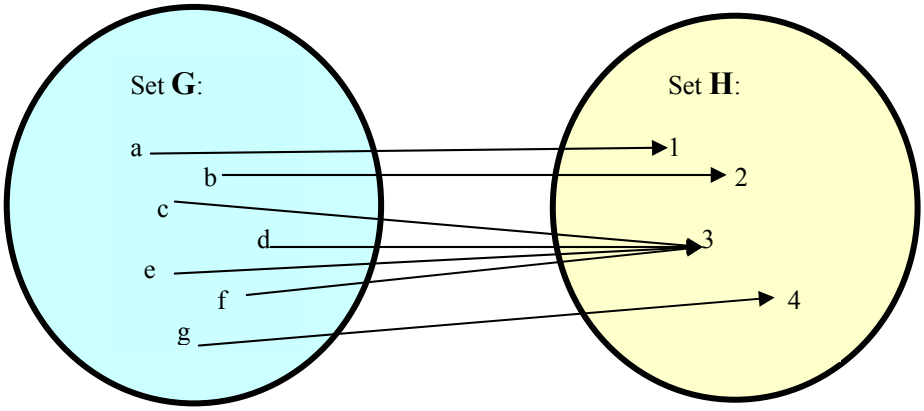
To be surjective $f(\mathbf{G})$ must equal **H**, however, it is obvious that $f(\mathbf{G}) = \{1, 2, 3\} \neq \mathbf{H}$, consequently this function is not surjective. However, Diagram #2 shows a surjective function.

let f: $\mathbf{G} \rightarrow \mathbf{H}$ was defined as:

$f(a) = 1$ $f(b) = 2$ $f(c) = 3$ $f(d) = 3$ $f(e) = 3$

$f(f) = 3$ $f(g) = 4$ then $f(\mathbf{G}) = \{1, 2, 3, 4\} = \mathbf{H}$

Diagram #2:



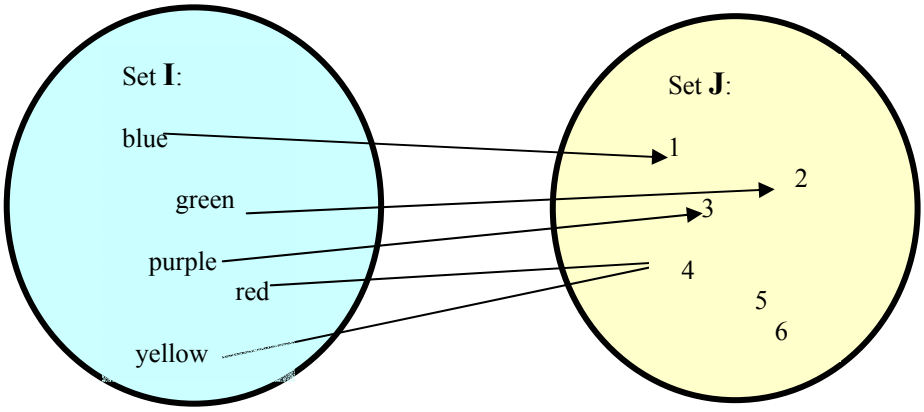
Here the range of **G** equals the domain of **H**, hence f is "onto" or surjective.

An injective function maps each element in one set to a unique element in another.

let **I** = {blue, green, purple, red, yellow}; **J** = {1, 2, 3, 4, 5, 6} and function f: **I**→**J** :

f(blue) = 1 f(green) = 2 f(purple) = 3 f(red) = 4 f(yellow) = 4

Diagram #3:

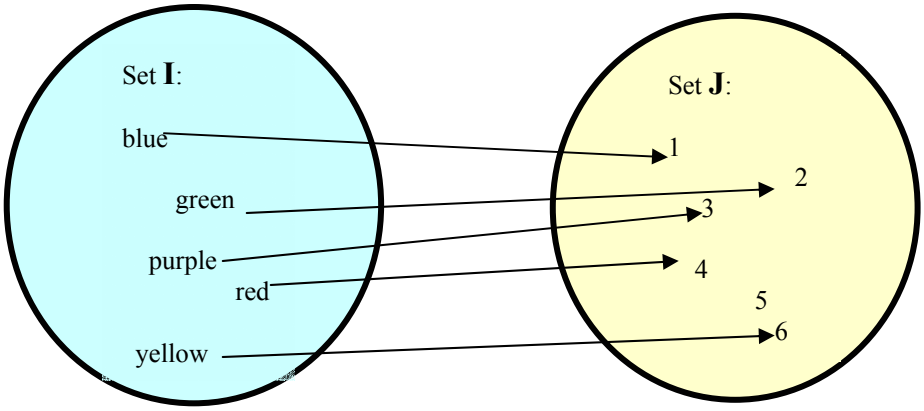


because f(red) = f(yellow) = 4, we observe that all elements in **I** do <u>not</u> map to a unique element in **J** – hence the function f is not injective. Alternately, Diagram #4 shows an injective function.

let f: **I**→**J** be defined as:

f(blue) = 1 f(green) = 2 f(purple) = 3 f(red) = 4 f(yellow) = 6

Diagram #4:



Note how each element in **I** is mapped uniquely to an element in **J** and therefore f is "one-to-one" or injective.

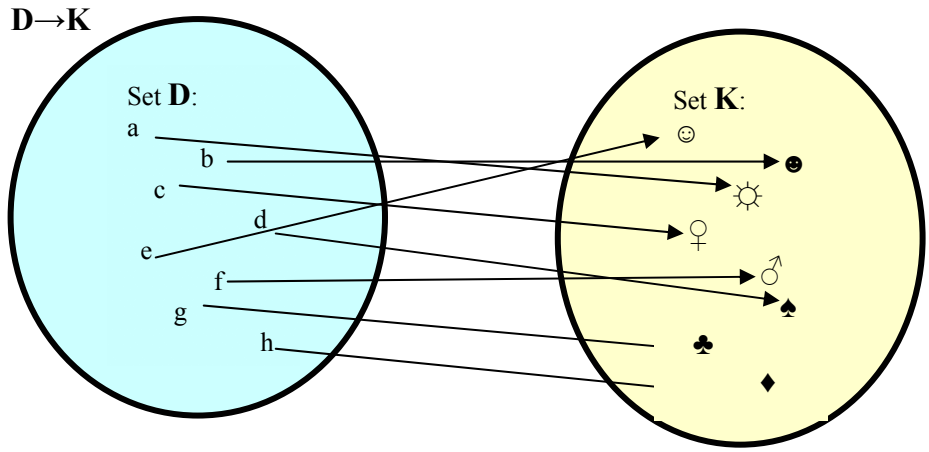
When a bijective function is both injective and surjective it is also said to be in one-to-one correspondence. We illustrate a bijective function below in Diagram #5 :

let **D** = {a , b , c , d , e , f , g , h}; **K** = {☺ , ☻ , ☼ , ♀ , ♂ , ♠ , ♣ , ♦}, and function f: **D**→**K** :

f(a) = ☼ f(b) = ☻ f(c) = ♀ f(d) = ♠ f(e) = ☺ f(f) = ♂

f(g) = ♣ f(h) = ♦ thus f(**D**) = {☺ , ☻ , ☼ , ♀ , ♂ , ♠ , ♣ , ♦}= **K**

Diagram #5: f: **D**→**K**



Observe that when a function is bijective the two finite sets have the same number of elements; **D** has eight elements and **K** has eight elements. The formal definition of cardinality can be stated as:

> *If set **A** and **B** are two sets contained in some universal set **U,** and **A** and **B** have the same cardinality and there exists a bijective function "f" that maps **A** to **B**. Then it can be said that **A** and **B** are numerically equivalent (equipotent).*

Note the terms equipotent, numerically equivalent, and equal cardinality have the same meaning.

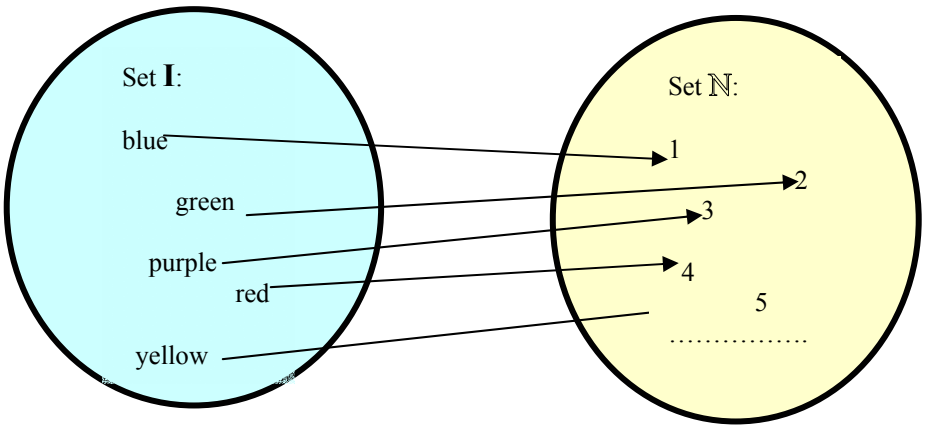
Cardinality

The number of elements in a set is called its cardinality (cardinal number) and is symbolically represented by putting bars on either side of the set, for example $|\mathbf{D}| = 8$. Then by definition, two sets are numerically equivalent if there exists a bijective function between them as shown above in Diagram #5 where $|\mathbf{D}| = |\mathbf{K}|$. The cardinality of a finite set is represented by the natural numbers $\mathbb{N}$ = {1, 2, 3, 4, …}, if we can map each element of some set **Y** to a natural number, **Y** is called a countable set, so a set of five colors would be countable and have a cardinality of 5.

let **I** = {blue, green, purple, red, yellow}; bijective function f: **I**→ℕ:

f(blue) = 1 f(green) = 2 f(purple) = 3 f(red) = 4 f(yellow) = 5

Diagram #6:



observe that the set **I** is countable because there exists a bijective function, f, that maps each element in **I** to ℕ and its cardinality is 5 because there are five elements. We can define countability accordingly:

> *A set of objects whose members can be put in one-to-one correspondence with the positive integers* (ℕ)*; a set whose members can be arranged in an infinite sequence in such a way that every member occurs in one and only one position.*

Note that this definition is not limited to finite sets, furthermore, by using the bijective relationship, we can determine the countability of infinite sets.

I. Cardinality of Infinite Sets

It is self evident that ℕ is a countable set because f: ℕ→ℕ is bijective. If we apply this rule to see if the infinite set of negative integers, $\mathbb{Z}^-$ is countable, then we need a bijective function that maps $\mathbb{Z}^-$ to the natural numbers, f: $\mathbb{Z}^- \to \mathbb{N}$**.** We can define this function thus: $f(x) = -x$

$f(-1) = 1$ $f(-2) = 2$ $f(-3) = 3$ …

Accordingly, every element in $\mathbb{Z}^-$ has a unique counterpart in ℕ**,** and the range of $f(\mathbb{Z}^-) = \mathbb{N}$, so this is a bijective function, hence the two sets have the same cardinality $|\mathbf{Z}^-| = |\mathbf{N}|$. We say the cardinality of an infinite set is a denumerable set or a countably denumerable set - the formal definition is stated:

A set of objects which either has an infinite number of members or can be put into one-to-one correspondence with the set of positive integers is denumerably infinite or countably infinite.

But how do we represent the number of elements in an infinite set if no number can be used that is not finite? Cantor resolved this by creating the set of transfinite cardinal numbers (infinite cardinal numbers), so the cardinality of the set $\mathbb{N}$ can be represented by "$\aleph_0$", pronounced, "a*leph-null*". The aleph symbol is the first letter in the Hebrew alphabet and as we have already proved, $\left|\mathbf{Z}^-\right| = \left|\mathbf{N}\right| = \aleph_0$".

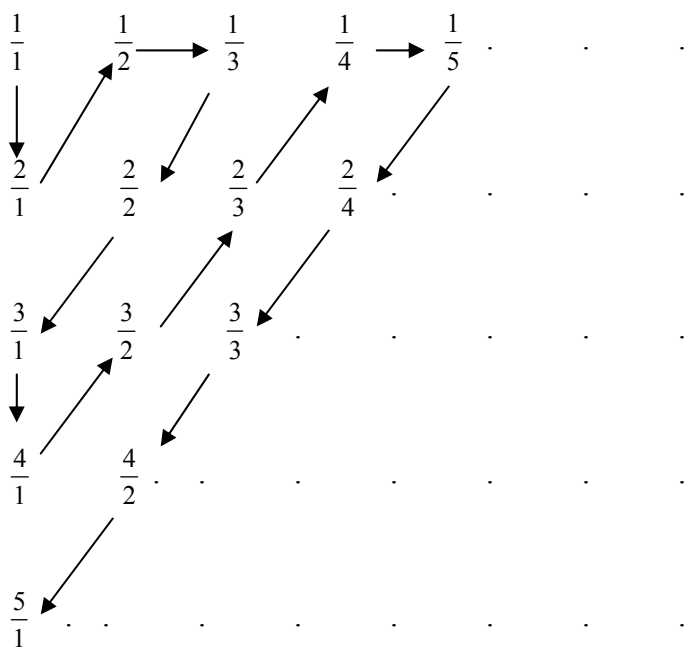
Let us now consider the set of rational numbers, $\mathbb{Q}$. We know that there are infinitely many rational numbers between 1 and 2, but is the set $\mathbb{Q}$ countable and if it is countable, what is its cardinality, noting of course, that its cardinality will be a transfinite cardinal number.

Let us define our function f according to the following pattern:

$$f(\tfrac{1}{1}) = 1 \qquad f(\tfrac{2}{1}) = 2 \qquad f(\tfrac{1}{2}) = 3.$$

$$f(\tfrac{1}{3}) = 4 \qquad f(\tfrac{2}{2}) = 5 \qquad f(\tfrac{3}{1}) = 6\ldots$$

Diagram #7:



Observe that $\mathbb{Q}$ and $\mathbb{N}$ are reciprocally unique and the range of $f(\mathbb{Q})$ = the domain of $\mathbb{N}$. Hence a bijective function exists, leading us to the conclusion that $\mathbb{Q}$ and $\mathbb{N}$ have the same cardinality:

$|\mathbf{Q}| = |\mathbf{N}| = \aleph_0$. This result is surprising considering $\mathbb{Q}$ is more densely populated than $\mathbb{N}$, and that $\mathbb{N} \subset \mathbb{Q}$. How do we explain the apparent contradiction when the "whole", in our case $\mathbb{Q}$, is equal to one of its "parts" or proper subsets: $\mathbb{N}$?

This was the focus of intensive study in the late 1800's and what must be understood is that infinite sets exhibit different behaviors from finite sets. For infinite sets, phrases such as "whole", "equal", "part", etc. reveal the limitations of conventional definitions. If we add one element to an infinite set its cardinality remains unchanged, an infinite set can be equal to one of its parts, a finite set cannot. Consequently, we find that the words "whole" and "part" are inappropriate when referencing infinite sets because they carry a finite inference.

Another illustration of this point is from Hans Hahn's article, "*Infinity*". Imagine we have a large orange-colored circle, and remove from it a one-centimeter square and place it alongside. If we want to describe the two pieces we could list shape, size, material, and color. Although the shape and size of the two pieces might be different, they both have the same color - orange. The property of color is independent from the properties of shape and volume. It is this analogy that applies to the cardinality of infinite sets and we see that the cardinality property of infinite sets is independent of its size. For example the set of even numbers $\mathbb{E}$, while a proper subset of the set of integers $\mathbb{Z}$, has the same cardinality as $\mathbb{Z}$: $\aleph_0$. The cardinality of finite and infinite sets likens to asking if $\aleph_0$ is either even or odd and is disparate, this was why Cantor needed to create the unique representation "$\aleph$" to represent the cardinality of infinite sets.

At this stage it is best to summarize what has been covered so far in this paper:

- A set is a collection of objects; these sets can be combined to form a new set using the union operation,
- We can see what sets are contained within another set by identifying its proper subsets,
- The number of elements within a set is called its cardinality, which is based on the set of natural numbers,
- If two sets are bijective, they have the same number of elements,

- The cardinality of a finite set can be found by creating a function that maps one set to the set of natural numbers,
- The cardinality of an infinite set is also found using a bijective function that maps it to the natural numbers, however its cardinality has different properties and is referred to as $\aleph_0$ "*aleph-null*".

II. Other Infinities

Now we will address the title of this paper, *Zero Measure, Infinite Count?* In 1874 Georg Cantor proved that there were other infinite sets that were not denumerable or uncountable - other transfinite cardinal numbers beyond aleph null. Cantor's first proof uses the set of real numbers between 0 and 1 to prove you can neither count nor create a bijective function from the real numbers $\mathbb{R}$ to the natural numbers $\mathbb{N}$.

<u>Proof A</u>

By using decimal numbers to represent real numbers we find we can list real numbers from 0 to 1 in the following fashion (note these are not in any order):

If:

$$\begin{aligned}
\mathbb{R}_1 &= 0.000000000000000001\ldots\ldots\ldots \\
\mathbb{R}_2 &= 0.000011111112222222\ldots\ldots\ldots \\
\mathbb{R}_3 &= 0.111112222223333333\ldots\ldots\ldots \\
\mathbb{R}_4 &= 0.567891011121314151\ldots\ldots\ldots \\
\mathbb{R}_5 &= 0.567891011121314152\ldots\ldots\ldots \\
\mathbb{R}_6 &= 0.5678910111213141523\ldots\ldots \\
\mathbb{R}_n &= \vdots
\end{aligned}$$

Assume these numbers continue indefinitely. If we wish to create a bijective function $f: \mathbb{R} \rightarrow \mathbb{N}$ then each element in $\mathbb{R}$ must have a unique corresponding number in $\mathbb{N}$:

$$f(\mathbb{R}_1) = 1 \qquad f(\mathbb{R}_2) = 2 \qquad f(\mathbb{R}_3) = 3, \ldots.$$

At this point the set of real numbers is well behaved and is in one to one correspondence with the natural numbers. However, if there exists one number that is not counted or matched to the set of natural numbers then it is not a bijective function because $f(\mathbb{R}) \neq \mathbb{N}$. As before, if the range of

our set $\mathbb{R}$ does not equal the domain of $\mathbb{N}$, then the function is not surjective. We may want to remember at this point the definition of countable sets:

> "… *a set whose members can be arranged in an infinite sequence in such a way that every member occurs in one and only one position.*"

If:

$\mathbb{R}_1 = 0.000000000000000001\ldots\ldots\ldots \quad f(\mathbb{R}_1) = 1$

$\mathbb{R}_2 = 0.000011111112222222\ldots\ldots\ldots \quad f(\mathbb{R}_2) = 2$

$\mathbb{R}_3 = 0.111112222223333333\ldots\ldots\ldots \quad f(\mathbb{R}_3) = 3$

$\mathbb{R}_4 = 0.567891011121314151\ldots\ldots\ldots \quad f(\mathbb{R}_4) = 4$

$\mathbb{R}_5 = 0.567891011121314152\ldots\ldots\ldots \quad f(\mathbb{R}_5) = 5$

$\mathbb{R}_6 = 0.567891011121314152 3\ldots\ldots \quad f(\mathbb{R}_6) = 6$

$\vdots$

$\mathbb{R}_n = 0.77909\text{xxx}\ \ldots\ldots\ldots\ldots \quad f(0.77909\text{xxx}\ \ldots) = ?$

We will be replacing the highlighted number with another number, if it is 0 replace it with a 7, if it is a 9 replace it with a 0, if it is any other number replace it with a 9. Combine all the new digits into a new number. Because this new number differs from all the previously listed numbers, it was uncounted by our function f. It is not surjective because the range of $f(\mathbb{R}) \neq \mathbb{N}$. This example illustrates that the real numbers are uncountable because they cannot be put in one-to-one correspondence with the totality of the natural numbers.

Cantor's second proof over the set of real numbers uses a visual representation of the real number line.

<u>Proof B</u>

Imagine we have a straight line representing the real numbers between 0 and 1 [0, 1], this is often referred to as the Standard Cantor Set illustrated below:

If we remove the middle third of the line segment we have the sets $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$:

Again if we remove the middle third we have the sets $[0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$:

Continuing this process ad infinitum, we find that the measure of the length of our line approaches 0 while the sum of the removed segments approaches 1. However all the points between 0 and 1 have not been eliminated because the intervals we removed were actually open

sets. Endpoints such as 0, $\frac{1}{9}$, $\frac{2}{9}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{7}{9}$, $\frac{8}{9}$, 1 etc. remain on our line and as we remove segments indefinitely, the set of numbers remaining on our line increases, consequently the Cantor Set is not empty.

To begin, let us first prove that the standard Cantor Set, after infinitum of cycles, has a length of zero. We know that the Cantor Set at the onset is the set of real numbers [0, 1] and has a length of 1. If we can show that the infinite cycles remove a total length of 1 from the Cantor Set, then we can say that the remaining Cantor Set has a length (or measure) of zero. We know from calculus that the sum of an infinite geometric series $S = 1 + r + r^2 + r^3 + \ldots + r^n$ can be found by the following process:

$$S = [1 + r + r^2 + r^3 + \ldots + r^n]$$

multiply both sides by r

$$r * S = [r + r^2 + r^3 + \ldots + r^n + r^{n+1}]$$

subtract $(r * S)$ *from* S

$$S - (r * S) = 1 - r^{n+1}$$

isolate S

$$S = \frac{1 - r^{n+1}}{1 - r}$$

$$\begin{cases} |r| < 1, \rightarrow & \lim_{n \to \infty} S = \frac{1}{1 - r}; \\ |r| > 1, \rightarrow & \lim_{n \to \infty} S \text{ diverges} \end{cases}$$

Since each time we perform a cycle on the Cantor Set we remove the middle $\frac{1}{3}$ of each segment, the set of the removed segments would be:

$$S = (\tfrac{1}{3}, \tfrac{2}{3}) \cup (\tfrac{1}{9}, \tfrac{2}{9}) \cup (\tfrac{7}{9}, \tfrac{8}{9}) \cup (\tfrac{1}{27}, \tfrac{2}{27}) \cup (\tfrac{7}{27}, \tfrac{8}{27}) \cup (\tfrac{19}{27}, \tfrac{20}{27}) \cup (\tfrac{25}{27}, \tfrac{26}{27}) \cup \ldots.$$

We can state these sets as lengths and the unions of these sets would be:

$$\text{Sum of } S = \tfrac{1}{3} + \tfrac{2}{3^2} + \tfrac{2^2}{3^3} + \tfrac{2^3}{3^4} + \ldots + \tfrac{2^n}{3^{n+1}} = \tfrac{1}{3} * \left[1 + \tfrac{2}{3} + \left(\tfrac{2}{3}\right)^2 + \left(\tfrac{2}{3}\right)^3 + \ldots + \left(\tfrac{2}{3}\right)^n \right]$$

Here $r = \frac{2}{3} < 1$ we can use the formula for infinite geometric series $S = \frac{1}{1-r}$:

$$Sum \quad of \quad S \;=\; \tfrac{1}{3} * [1 + \tfrac{2}{3} + (\tfrac{2}{3})^2 + (\tfrac{2}{3})^3 + \ldots + (\tfrac{2}{3})^n]$$

$$multiply \quad both \quad sides \quad by \quad \tfrac{2}{3}$$

$$\tfrac{2}{3} S \;=\; [\tfrac{2}{3} + (\tfrac{2}{3})^2 + (\tfrac{2}{3})^3 + (\tfrac{2}{3})^4 \ldots + (\tfrac{2}{3})^{n+1}]$$

$$subtract \quad (\tfrac{2}{3} * S) \quad from \quad S$$

$$S - (\tfrac{2}{3} S) \;=\; 1 - \tfrac{2}{3}$$

$$isolate \quad S$$

$$\tfrac{1}{3} S \;=\; \frac{1}{1 - \frac{2}{3}} \;=\; \frac{1}{\frac{1}{3}} \quad \ldots. \qquad \tfrac{1}{3} S \;=\; 3$$

$$Hence \quad Sum \quad of \quad S \;=\; 1$$

The limit of the length of segments removed as we approach an infinite number of cycles is 1, hence our Cantor Set is left with a measure of zero. To prove the numbers remaining in the Cantor Set are non-denumerable, we will replace our number system of base 10 with a system of base 3. We recognize that the decimal representations of these new numbers will use only 0, 1, and 2 and their values will not be the same in base 3 as they are in base 10, if we have the numbers:

100 which represents 10^2 in base 10, it symbolizes 3^2 in base 3, and equals 9

10 which represents 10^1 in base 10, it symbolizes 3^1 in base 3, and equals 3,

1 which represents 10^0 in base 10, it symbolizes 3^0 in base 3, and still equals 1,

0.1 which represents 10^{-1} in base 10, it symbolizes 3^{-1} in base 3, and equals $\tfrac{1}{3}$,

and **0.01** which represents 10^{-2} in base 10, it symbolizes 3^{-2} in base 3, and equals $\tfrac{1}{9}$,

Since the standard Cantor Set at point 1 is represented by 3^0 in base 3, all points on the line less than 1 would have an exponent no greater than -1. Hence the point $\tfrac{1}{3}$ would be $1 * 3^{-1}$ represented decimally as $0_{3}.1$, the point $\tfrac{2}{3}$ would be $2 * 3^{-1}$ represented decimally as $0_{3}.2$, hence the point $\tfrac{16}{27} = \tfrac{1}{3} + \tfrac{2}{9} + \tfrac{1}{27} = 0_{3}.121$. Any point on the line greater than zero but less than $\tfrac{1}{3}$ would be represented decimally as $0_{3}.0xxx$.

If we remove the middle third of the line segment we have the sets $[0,\ 0_{3}.1] \cup [0_{3}.2,\ 1]$:

Remove the middle thirds we have $[0,\ 0_{3}.01] \cup [0_{3}.02,\ 0_{3}.1] \cup [0_{3}.2,\ 0_{3}.21] \cup [0_{3}.22,\ 1]$:

After an infinitum of cycles, the set of removed segments becomes:

$$S = (0_{3}.1, 0_{3}.2) \cup (0_{3}.01, 0_{3}.02) \cup (0_{3}.21, 0_{3}.22) \cup (0_{3}.001, 0_{3}.002) \cup (0_{3}.021, 0_{3}.022) \cup (0_{3}.201, 0_{3}.202) \cup (0_{3}.221, 0_{3}.222)\ldots$$

By listing the numbers remaining in the Cantor Set in decimal form as we did with the set of real numbers, we find a collection of numbers, including infinitely long decimal numbers, with only 0's and 2's since the function always removes the middle $\frac{1}{3}$. The remaining numbers from the Cantor Set are:

$C_1 = 0_{3}.000000000000000001\ldots\ldots\ldots\ f(C_1) = 1$

$C_2 = 0_{3}.000020202020222222\ldots\ldots\ldots\ f(C_2) = 2$

$C_3 = 0_{3}.202002222220000002\ldots\ldots\ldots\ f(C_3) = 3$

$\vdots$

$C_n = 0_{3}.220\text{xxx}\ldots\ldots\ldots\ldots\ldots\ldots\ f(0_{3}.220\text{xxx}\ldots) = ?$

As before in our proof of non-denumerability for the set of real numbers [0, 1], we will be replacing the highlighted number with another number, if it is 0 replace it with a 2, if it is a 2 replace it with a 0. If we combine all the new digits into a new number we again find this new number differs from all the previously listed numbers and is uncounted by our function f. Illustrating it is not surjective because the range of $f(C) \neq \mathbb{N}$ and therefore is uncountable.

In fact, although the Cantor Set has a measure of zero, it contains an uncountable set of "endpoints" that has the same cardinality as the set of real numbers. Since the set of real numbers is uncountable it does not have a cardinality of $\aleph_0$. Instead, Cantor named the cardinality of the set of real numbers, "the power of the continuum (***c***)", $\aleph_1$ known as *aleph-one*.

The set of real numbers is the union of two sets, $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R}\text{-}\mathbb{Q})$. We proved earlier that the set $\mathbb{Q}$ is countable and the set of real numbers is uncountable. In fact, the reason that the set of real numbers is uncountable is because it contains the set of irrational numbers ($\mathbb{R}\text{-}\mathbb{Q}$). Cantor's proof for infinite non-repeating decimal numbers makes this point apparent. We can conclude then that the set of irrational numbers also has a cardinality of $\aleph_1$ "*aleph-one*".

Conclusion

In this paper we have discovered that when we deal with infinite sets we cannot always trust our intuition. We can use the bijective rule for determining cardinality of both finite and infinite sets, but when we speak of the cardinality of infinite sets, the finite inferences of terms such as "whole" and "part" are in conflict to the concept of infinity. We also have found that there are denumerable infinite sets and non-denumerable infinite sets with cardinality of $\aleph_0$ "*aleph-null*" and $\aleph_1$ "*aleph-one*" respectively and it was the numerical limitations of finite numbers that prompted Cantor's creation of the "*aleph*" number set. Moreover, there are an infinite number of "*aleph*" numbers. For example the set of power sets over the real numbers has a cardinality of $\aleph_2 = 2^{\aleph_1}$ "*aleph-two*". If one wanted to study infinity further there is ongoing work on the Continuum Hypothesis that states that the power of the continuum $\aleph_1$ is the next infinity after $\aleph_0$, however this hypothesis has yet to be proven in mathematics.