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Pascal's Triangle: Not just a numerical array

ABSTRACT. Pascal's triangle is more than just an array of numbers. Within the triangle there exists a multitude of patterns and properties. Some are obvious, some are not, but all are worthy of recognition. Our purpose here is to introduce the triangle and to discuss and prove some of its most important properties. We will also connect the triangle to mathematical concepts such as the binomial expansions and basic combinatorics.

INTRODUCTION. Blaise Pascal (1623-62) did not "invent" the triangle that bears his name. In fact, the concept is accredited to the ancient Chinese; evidence suggests that the concept was alive in China even in the B. C. era (6). The prominent Italian mathematician Niccolo Tartaglia (1506-59) actually published a *rectangular* version of the triangle (known as *Tartaglia's rectangle*), though it did not appear until after his death. However, it was Pascal who, more than a century later, first documented the properties of the triangle and their relationships to various mathematical theories, including the relevance of the triangle to the solution of an important problem in probability. Pascal's triangle is not a triangle in the geometric sense, but is a triangular array of numbers. Though there are a few different methods of construction, all are based on the concept of Pascal's relations.

CONSTRUCTION. Imagine a sequence of numbers $a_0, a_1, a_2, \ldots, a_n$ for $n = 0, 1, 2, \ldots$ From this sequence, we can generate a new one $b_0, b_1, b_2, \ldots, b_{n+1}$, using the following rules:

(1) $b_0 = a_0$,

(2) $b_k = a_{k-1} + a_k \quad (1 \le k \le n)$,

(3) $b_{n+1} = a_n$.

Now, we start with the sequence consisting of the single number 1, which is known as *Pascal's zeroth sequence*, and use rules (1) – (3) to derive a new sequence $1, 1$, which is *Pascal's first sequence*. From this, we apply the rules again and get $1, 2, 1$, *Pascal's second sequence*. As we can see, a term is added each time we apply the rules to our new sequence. That is, in Pascal's n^{th} sequence there are $n+1$ terms. To create Pascal's triangle, we simply arrange our sequences in rows. Of course, there are infinitely many elements in Pascal's triangle. Shown below are the first nine rows (rows zero through eight) of the triangle.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

In general, each entry $\binom{n}{r}$ of Pascal's triangle, or the rth element of the nth row, is found by adding the two numbers in the $(n-1)^{th}$ row that are above and on either side of it. That is,

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

For example, $20 = \binom{6}{3} = \binom{5}{2} + \binom{5}{3}$. This result is often called *Pascal's formula*, and is fairly simple to prove using combinatorics.

Proof: Let S be a set of n objects. Distinguish an object, say x of S. The r-combinations of S can be divided into two classes A and B. In A, we put all those r-combinations of S which do not contain x. In B we put the others; that is, the r-combinations which do contain x. The number of r-combinations of S in A equals the number of r-combinations of the $(n-1)$-element set $S - \{x\}$, and this equals $\binom{n-1}{r}$. The number of r-combinations of S in B equals the number of $(r-1)$-combinations of the $(n-1)$-element set $S - \{x\}$, and this equals $\binom{n-1}{k-1}$. Thus, by the addition principle,

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

THE BINOMIAL COEFFICIENTS. Perhaps one of the most basic properties of the triangle is that any entry can be found using this simple formula from combinatorics: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$. In this case, we let n be the row number and r the location of an element in that row. Now, to find the rth element in the nth row, simply use the formula. For example, the sixth element of the tenth row in Pascal's Triangle is $\binom{10}{6} = \frac{10!}{6!(10-6)!} = 210.$

Another interesting property of the triangle involves the sum of the entries of the nth row.

As it turns out, the sum of the entries of the nth row of Pascal's Triangle is 2^n. That is, $\sum_{r=0}^{n}\binom{n}{r} = 2^n$.

For example, the sum of the entries of the 12th row of the triangle is $\sum_{r=0}^{12}\binom{12}{r} = 2^{12} = 4096$. To prove this result for any row n, we must first introduce and establish the reliability of the binomial theorem. The binomial theorem is:

$$(x+y)^n = y^n + \binom{n}{1}xy^{n-1} + \binom{n}{2}x^2y^{n-2} + \ldots + \binom{n}{n-1}x^{n-1}y + x^n = \sum_{r=0}^{n}\binom{n}{r}x^r y^{n-r}.$$

Proof: Write $(x+y)^n$ as a product of n factors each equal to $(x+y)$. Suppose we expand this product until no parentheses remain. Since for each factor $(x+y)$ we can either choose x or y, there are 2^n terms that result and each can be arranged in the form $x^r y^{n-r}$ for some $r = 0,1,\ldots,n.$

We obtain the term $x^r y^{n-r}$ by choosing x in r of the factors and y in the remaining $n-r$ factors. Thus, the number of times the term $x^r y^{n-r}$ occurs in the expanded product equals the number of r-combinations of the set of n factors, and this is $\binom{n}{r}$. Thus

$$(x+y)^n = \sum_{r=0}^{n}\binom{n}{r}x^r y^{n-r}. \qquad (1)$$

Now, if we let $x = y = 1$, then we have $(1+1)^n = \sum_{r=0}^{n} \binom{n}{r} 1^r 1^{n-r} = 2^n = \sum_{r=0}^{n} \binom{n}{r}$. Thus, the sum of the entries of the nth row of the triangle is 2^n.

Actually, whenever $y = 1$, equation (1) becomes $(x+1)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$. In fact, $\binom{n}{r}$ is the coefficient of the term x^r binomial expansion $(1+x)^n$. Here are the cases for $n = 2, 3, 4$:

$$(1+x)^2 = \binom{2}{0}x^0 + \binom{2}{1}x^1 + \binom{2}{2}x^2 = 1 + 2x + x^2$$

$$(1+x)^3 = \binom{3}{0}x^0 + \binom{3}{1}x^1 + \binom{3}{2}x^2 + \binom{3}{3}x^3 = 1 + 3x + 3x^2 + x^3$$

$$(1+x)^4 = \binom{4}{0}x^0 + \binom{4}{1}x^1 + \binom{4}{2}x^2 + \binom{4}{3}x^3 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

In general, $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{r}x^r + \ldots + \binom{n}{n}x^n$. This equation is commonly known as *Newton's binomial formula*, or more simply as *Newton's formula*.

TWO OTHER PROPERTIES. Another interesting property of the triangle is simply that any prime number p divides all elements of the pth row, excluding the ones (6). For example, consider row seven of the triangle: $1, 7, 21, 35, 35, 21, 7, 1$. According to this property, 7 divides itself, 21 and 35. Using mathematical notation, we let $\binom{p}{k}$ denote an element of the pth row. Then, we claim that p divides $\binom{p}{k}$. We can use induction to prove that this property is true for any prime number p and its corresponding row.

Proof: First, if $k = 1$ we claim that p divides $\binom{p}{1} = p$, so p divides p, which is true. Now, we assume that p divides $\binom{p}{k}$. We will show that p divides $\binom{p}{k+1}$. We know that

$$\binom{p}{k+1} = \frac{p!}{(k+1)!(p-k-1)!} = \frac{p!}{(k+1)!\,(p-k-1)!} \cdot \frac{(p-k)}{(p-k)} = \frac{p!\,(p-k)}{(k+1)k!(p-k)!} = \frac{p!}{k!(p-k)!} \cdot \frac{p-k}{k+1}$$

$= \binom{p}{k} \cdot \frac{p-k}{k+1}$. Now, we can write this last expression as $\left(\frac{p-k}{k+1} \cdot m\right)$, where m is the integer $\binom{p}{k}$. Also, now we know that $\left(\frac{p-k}{k+1} \cdot m\right) = \binom{p}{k+1}$ is an integer and a multiple of $\binom{p}{k}$. Thus, since p divides $\binom{p}{k}$, p must also divide $\binom{p}{k+1}$.

Another appealing pattern in the triangle is sometimes called the "Hockey Stick" pattern (6).

Pick any 1 along a side of the triangle (an element $\binom{n}{0}$ or $\binom{n}{n}$) and follow its diagonal as far as you like to a stopping point. For example, let's pick the element $\binom{4}{0}$ equal to 1, and follow its diagonal until we come to the entry $\binom{7}{3} = 35$. This means we have included $1, 5, 15,$ and 35 in our diagonal.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

Now, the sum of these numbers is equal to the entry that is in the row just below 35, and off this diagonal, which is 56. That is, $\binom{4}{0} + \binom{5}{1} + \binom{6}{2} + \binom{7}{3} = \binom{8}{3} = 56$. In general, the sum of the numbers that you have included in your diagonal up to and including your stopping point is equal to the element of the triangle in the row immediately

below your stopping point and off of its diagonal. That is,

$\binom{n}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\ldots+\binom{n+r}{r}=\binom{n+r+1}{r}$. In other words, we claim that

$\sum_{i=0}^{r}\binom{n+i}{i}=\binom{n+r+1}{r}$. We can prove this by induction:

Proof: First, let $r=0$. This gives us $\sum_{i=0}^{0}\binom{n+i}{i}=\binom{n+0}{0}=\binom{n+0+1}{0}=1$, which is true. Now, assume that the case $r=k$ is true. We will show that the case $r=k+1$ is also true. In other words, we assume that $\sum_{i=0}^{k}\binom{n+i}{i}=\binom{n+k+1}{k}$ is true, and show that $\sum_{i=0}^{k+1}\binom{n+i}{i}=\binom{n+k+2}{k+1}$. We know from Pascal's identity that $\binom{n+k+1}{k}+\binom{n+k+1}{k+1}=\binom{n+k+2}{k+1}$. Thus, by substitution we can write

$$\sum_{i=0}^{k+1}\binom{n+i}{i}=\sum_{i=0}^{k}\binom{n+i}{i}+\binom{n+k+1}{k+1}=\binom{n+k+2}{k+1}.$$

THE ORIGINAL TRIANGLE. We now move on to examine three properties of the triangle which were actually first noted by Pascal and proven by mathematical induction.

To consider the following properties, we must look at the triangle in the way that Pascal did; that is, let us rotate the previously introduced version of the triangle by 45 degrees (5). This gives the following effect:

1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	
1	3	6	10	15	21	28	36		
1	4	10	20	35	56	84			
1	5	15	35	70	126				
…									

This time, we will use the notation (n,r) to mean the element in the nth row and the rth column. Now, let us consider the first of three properties we will discuss that are related to this particular form of the triangle. Let us call

this Property 1: *Each element* (n,r) *in the triangle is equal to the sum of the entries in the* $(n-1)^{th}$ *row (the row just above it), from the leftmost entry to the one directly above* (n,r) (5). For example, let us consider the fourth and fifth rows of Pascal's Triangle as shown in the diagram above. We will refer to the row of ones at the very top of the diagram as row zero. Suppose we pick the element $(3,3)=20$. By the property just described, it must be true that 20 is the sum of all entries in row three from 1 through 10, since 10 is directly above 20. Upon examination, we see that this is true: $1+3+6+10=20$. Now, let's prove (4) this property for any entry in the triangle. In essence, what we will do is prove the assertion for any whole number r in two steps: 1) prove the assertion for $k=0$, and 2) assuming the assertion is true for $k=r$, where r is some whole number, show that it is also true for $k=r+1$ (5).

Proof: Let (n,k) denote the entry in the nth row and k^{th} column. The assertion is that (n,k) is the sum of the first $k+1$ entries of the $(n-1)^{th}$ row, or

$$(n,k)=(n-1,0)+(n-1,1)+...+(n-1,k). \qquad \textbf{(1)}$$

Step 1: If $k=0$, then equation **(1)** above becomes $(n,0)=(n-1,0)$. Since $(n,0)=(n-1,0)=1$, the assertion holds for $k=0$.

Step 2: Assuming **(1)** for $k=r$, and using the fact that each "interior" entry is the sum of the entry immediately preceding it in its column and the entry immediately preceding it in its row, we obtain

$$(n,r+1)=(n,r)+(n-1,r+1)=(n-1,0)+(n-1,1)+\ldots+(n-1,r)+(n-1,r+1) \qquad \textbf{(2)}$$

by using our assumption on r, that $(n,r)=(n-1,0)+(n-1,1)+\ldots+(n-1,r)$.

Since **(2)** is a restatement of **(1)** for $k=r+1$, we have shown that the truth of our assertion of property one for $k=r$ implies its truth for $k=r+1$. This completes the second step of the proof. Now, we will examine a similarly proven property of the triangle.

Property two, as we will call it, states the following: *Each entry* (n,k) *in Pascal's Triangle is the sum of the numbers in the column to the left, from the topmost to the entry directly left of* (n,k) (5). Obviously, this is an adaptation of property one, using columns instead of rows. For example, let us now consider columns four and five rather than rows four and five. Suppose we choose the element $(2,5)=21$. By Property two, 21 is the sum of

$1, 5,$ and 15 from column four (we stop at 15 since it is directly left of 21). Naturally, $1+5+15=21$. Now, the proof of property two is just the same as for property one, with the exception that the induction is performed on n rather than on k. To escape redundancy, we omit this and continue to the introduction of property three.

Of the three, the following property is perhaps the most interesting in that it involves a bit of geometry. Property three is stated as follows: *For each number* (n,k) *in the table,* $(n,k)-1$ *is the sum of all the numbers contained in the rectangle bounded by that column and that row whose intersection is the entry* (n,k) *(this row and column are not included in the rectangle)* (5). Let's examine this statement using an example. Suppose we choose $(3,4)=35$. By property three, our rectangle includes the following numbers: $1, 1, 1, 1, 1, 2, 3, 4, 1, 3, 6, 10$. We see that the sum of these numbers is 34, which is in fact equal to $(3,4)-1$. This property follows either from property one by induction on n or from property two by induction on k (4). At this time, we turn to a brief discussion of a fascinating and popular sequence in mathematics known as the Fibonacci sequence, and shall discuss its relationship to Pascal's Triangle.

THE FIBONACCI NUMBERS AND PASCAL'S TRIANGLE. Just as a reminder, the Fibonacci sequence is the following:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Each term of the sequence is equal to the sum of the previous two terms. It is interesting to note that this is very similar to the way Pascal's Triangle is formed. The Fibonacci sequence is applicable to the Golden Rectangle, to the lengths of the segments of a pentagram, and most interestingly, to things in nature (6). For example, "the regular arrangement of leaves or plant parts along the stem, apex, or flower of a plant captures the Fibonacci numbers in a succession of helices," (2). Let's reconsider the originally introduced (modern) version of Pascal's triangle. We see the Fibonacci sequence upon examination of the diagonals:

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & & 1 & & & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & & \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}
$$

$\ldots$ and so on.

Follow the diagonals according to color, adding the digits together. The successive sums will form the Fibonacci sequence. Since we show only eight rows of the triangle, we see only through the ninth term of the Fibonacci sequence, which is 34.

Using $\binom{n}{k}$ notation, the Fibonacci sequence is:

$$\binom{0}{0}, \binom{1}{0}, \binom{2}{0}+\binom{1}{1}, \binom{3}{0}+\binom{2}{1}, \binom{4}{0}+\binom{3}{1}+\binom{2}{2}, \ldots, \binom{n}{0}+\binom{n-1}{1}+\binom{n-2}{2}+\ldots+\binom{n-k}{k}.$$

More precisely, for $n \geq 0$ the nth Fibonacci number $f(n)$ satisfies

$$f(n)=\binom{n}{0}+\binom{n-1}{1}+\binom{n-2}{2}+\ldots+\binom{n-k}{k}, \text{ where } k=\frac{1}{2}n.$$

Proof: For $n \geq 0$, define

$$g(n)=\binom{n}{0}+\binom{n-1}{1}+\binom{n-2}{2}+\ldots+\binom{n-k}{k}, \text{ where } k=\frac{1}{2}n. \text{ Since } \binom{n}{p}=0 \text{ for an integer}$$

$p \succ n$, we can write

$$g(n)=\binom{n}{0}+\binom{n-1}{1}+\binom{n-2}{2}+\ldots+\binom{0}{n}.$$

We need to show that $f(n)=g(n)$ for all $n \geq 0$. Suppose we can verify that $f(0)=g(0)$ and $f(1)=g(1)$ and that $g(n)$ is a solution of the recurrence relation $F(n)=F(n-1)+F(n-2)$. Since the initial values

along with the recurrence relation uniquely determine the sequence of numbers, we can then conclude that $f(n) = g(n)$ for all $n \geq 0$. But

$$g(0) = \binom{0}{0} = 1 = f(0),$$

$$g(1) = \binom{1}{0} + \binom{0}{1} = 1 = f(1).$$

Using Pascal's formula, we see that for $n \geq 2$

$$g(n-1) + g(n-2) = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \ldots + \binom{0}{n-1} + \binom{n-2}{0} + \binom{n-3}{1} + \ldots + \binom{0}{n-2}$$

$$= \binom{n-1}{0} + \left[\binom{n-2}{1} + \binom{n-2}{0}\right] + \left[\binom{n-3}{2} + \binom{n-3}{1}\right] + \ldots + \left[\binom{0}{n-1} + \binom{0}{n-2}\right]$$

$$= \binom{n-1}{0} + \left[\binom{n-1}{1} + \binom{n-2}{2}\right] + \ldots + \binom{1}{n-1}$$

$$= \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \ldots + \binom{1}{n-1} + \binom{0}{n}$$

$$= g(n).$$

Thus, $g(n)$ is a solution of the recurrence relation $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$, and $f(n) = g(n)$ for all $n \geq 0$.

Next, we will examine the relationship between polygonal numbers and Pascal's triangle.

POLYGONAL NUMBERS. Polygonal numbers include triangular numbers, square numbers, pentagonal numbers, hexagonal numbers, and so on. The triangular numbers are $1, 3, 6, 10, 15, 21, 28, 34, \ldots$. The first term in the triangular number sequence is 1. To get the next term from it, add 2. To get the next term from this one, add 3. In general, if a_n is the nth triangular number, then $a_1 = 1$ and $a_{n+1} = a_n + n + 1$, for $n \geq 1$.. The triangular numbers appear in Pascal's Triangle beginning with the element $\binom{2}{0} = 1$, or with $\binom{2}{2} = 1$, depending on which diagonal one looks at.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

…

Suppose we examine the first few triangular numbers, assigning each with a "count." For example,

Triangular number	1	3	6	10	15	21	28	…
Count	1	2	3	4	5	6	7	…

It is not difficult to notice that every other triangular number (every red one) is divisible by its count. Stated differently, if the count of a triangular number is odd, then that odd number divides the triangular number. In fact, we can prove that a_{2n+1} (an arbitrary triangular number of odd count) is divisible by $2n+1$ for $n \geq 0$:

Proof: We know that $a_{2n+1} = 1+2+3+\ldots+(2n+1)$ from the way in which the triangular numbers are formed. We can write this sum as $\frac{(2n+1+1)(2n+1)}{2} = \frac{2(n+1)(2n+1)}{2} = (n+1)(2n+1)$. Also, since n is an integer, $n+1$ is also an integer. Now, we have that $a_{2n+1} = (n+1)(2n+1)$, which is a multiple of $2n+1$, so we conclude that a_{2n+1} is divisible by $2n+1$.

This brings us to the square numbers. Square numbers are just what their name suggests; they are squares of numbers. More specifically, they are squares of integers. The square numbers are:

$0, 1, 4, 9, 16, 25, 36, 49, 64, \ldots$. To see the square numbers in Pascal's triangle, we look at the the upward-

sloping diagonal containing the triangular numbers. We need to select the entries on the diagonal in pairs and list their sums to generate the square numbers:

				1				0
			1		1		0	
		1		2		1		
	1		3		3		1	
1		4		6		4		1
	1		5		10		10		5		1
		1		6		15		20		15		6		1
	1		7		21		35		35		21		7		1
1		8		28		56		70		56		28		8		1

To form the square numbers, we start with $0+0$, then compute $0+1$, $1+3$, $3+6$, $6+10$, and so on.

The square numbers are easy to represent. The square number sequence is simply $a_n = n^2$ for $n = 0, 1, \ldots$. We can see that the nth square number is equal to the nth triangular number plus the $(n-1)^{th}$ triangular number. To prove this, we represent the nth triangular number by $\binom{n}{2}$, since every triangular number is the second element of its row. It follows that $\binom{n+1}{2}$ is the next triangular number, so we can write $\binom{n}{2} + \binom{n+1}{2} = n^2$.

Proof:

$$\frac{n!}{(n-2)!\,2!} + \frac{(n+1)!}{2!\,(n+1-2)!} = \frac{(n+1-2)}{(n+1-2)} \cdot \frac{n!}{(n-2)!\,2!} + \frac{(n+1)!}{2!\,(n+1-2)!} = \frac{n!(n+1-2+n+1)}{2!\,(n+1-2)!} = \frac{n!\,2n}{2!\,(n+1-2)!}$$

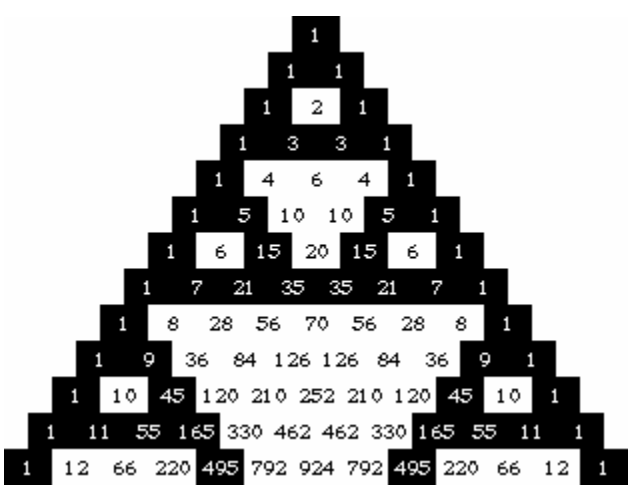
$$= \frac{n(n-1)!\,n}{(n-1)!} = n^2.$$

USING THE TRIANGLE. Consider the following situation. Suppose we have a set of ten videos and we wish to choose x of them to watch. How many different ways are there to do so? Most people who have taken an elementary statistics course have used the equation

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \quad \textbf{(3)}.$$

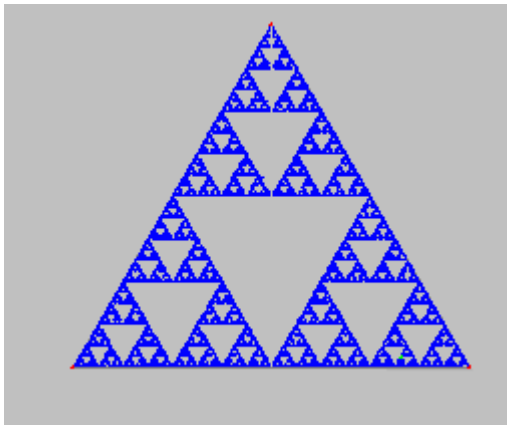
Of course, this is used to find the number of possible combinations of n things taken r at a time. Normally, to solve our video problem we would simply use this equation, letting r equal x (the number of videos we wish to choose from our ten), and of course $n = 10$. However, it is nice to know that we can also refer to Pascal's triangle for the answer. To find the number of combinations of n things taken r at a time, we simply find the rth element in the nth row of the triangle, which we have already discussed. Suppose we wish to choose six videos from our ten. To find the number of ways this can be done, we look at the sixth element of the tenth row of the triangle. The tenth row is $1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1$, so there are 210 ways to choose six videos from ten.

A GEOMETRIC CONNECTION. A particularly fun aspect of Pascal's triangle is its connection to the Sierpinski's Triangle, a fractal obtained by assigning a distinct color to the even numbers and to the odd numbers of Pascal's triangle. Geometrically, the Sierpinski triangle is constructed from an equilateral triangle. Its concept is easy. We simply connect the midpoints of each side of an equilateral triangle, creating four new triangles. Ignoring the new triangle in the center, we repeat the connecting of the midpoints with the remaining three triangles. This process continues infinitely (7). Now, to obtain a Sierpinski triangle from Pascal's triangle, we simply color the odd numbers and even numbers differently:



(Image courtesy of: http://www.zeuscat.com/andrew/chaos/sierpinski.html.)

It is slightly difficult to see the true pattern in the above image; here is one in which the pattern is clearer:



In this picture, the gray triangles represent the even numbers of Pascal's triangle, while the blue areas represent the odd numbers. In fact, there are many programs and internet sites available that allow the user to view various Sierpinski triangles as well as versions of Pascal's triangle obtained by coloring in particular multiples of some number (or all prime or composite numbers). Many fascinating patterns can be obtained in this sort of way. This brings us to the conclusion of our discussion.

CONCLUSION. Pascal's triangle is easily a fascinating mathematical element. The thought that we will never be able to see the triangle in its entirety, for it is by nature infinite and uncontainable, is awe-inspiring. Nonetheless, it is a wonderful object of study and a continuous reminder of the tremendous beauty of mathematics. We certainly encourage the interested reader to explore many other sites of information regarding Pascal's triangle and its patterns and relationships to various concepts in combinatorics, probability, and other branches of mathematics.

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