

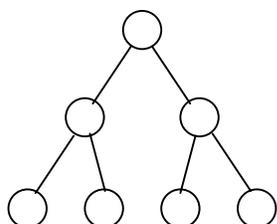
MATH 195: Gödel, Escher, and Bach (Spring 2001)

Problem Set 11: TNT and Rules of Derivation

1. Translate into English the statement given on p.215 and then derive it:

$$\langle S0=0 \vee \sim S0=0 \rangle$$

2. Suppose a new long-lasting disease is detected in the population. While non-lethal in the short term, it spreads with great rapidity, because each infected person infects, on average, two other people every week. Let's say, for simplicity, that each person infects EXACTLY two other people every week. You are a scientist at the Center for Disease Control trying to get a handle on this new outbreak. To aid your understanding, you doodle on a napkin the following picture:



Week	Infected that week	Total infected
1	1	1
2	2	3
3	4	7
?	?	?

You note that every week the number of people infected that week doubles. The number of TOTAL people in the population is a bit more complicated to understand, but you come up with the hypothesis that:

$$\text{Total infected} = 2^w - 1$$

where w is the number of the week. Plausible, but is it true? This is life and death stuff: you GOT to know how many infected people to expect during these critical next weeks of the infection (when it is unlikely that a person gets infected twice). Nothing much you can do to stop the spread of the disease over lunch, but you CAN at least prove that your hypothesis is true. Use the Rule of Induction.

- a. Is your rule true during week 1? (plug in $w = 1$)
- b. Suppose that the rule is true for some unknown week w , so it is given that for that week, the total infected is $2^w - 1$. How many new people were infected that week?
- c. How many new people will be infected the following week?
- d. What will be the new total infected at the end of the following week. Find this number by adding the newly infected next week to the total infected this week?
- e. Try to simplify the sum, noting that in general:

$$a^b \cdot a = a^{b+1} \quad \text{and} \quad a^b + a^b = 2 \cdot a^b$$
- f. Is it true, as predicted by your rule, that during week $w+1$ the total infected is indeed $2^{w+1} - 1$?
- g. Present an argument based on the Rule of Induction that your rule is true in general.

3. Translate each of the five axioms on p.216 into intelligible English.
4. Given that "*djinn*" is meant to represent the natural numbers (i.e. zero plus the positive integers), translate the five disguised Peano postulates into English, making up reasonable equivalences for "*Genie*" and "*meta*".
5. Make up a sentence in TNT with two variables governed by \forall . Start off $\forall \mathbf{a}: \exists \mathbf{b}: \dots$ and choose the sentence so that it happens to be true. Replace \mathbf{a} with the number "1". Is the sentence still true? Replace \mathbf{a} with the variable \mathbf{c} . Still true? Now, replace \mathbf{a} with the variable \mathbf{b} . Still true? It may or may not be. Find a sentence where the first two replacements preserve truth but the last one does not.
6. Translate the Rule of Generalization into English and provide an example within TNT that uses the rule.
7. Provide an example in English of the use of the Rule of Interchange.
8. Provide an example in English of the use of the Rule of Existence. You might do this by translating the TNT example provided with the rule.
9. Provide arithmetic formulas that illustrate the Rules of Equality and Successorship.
10. Go through the first derivation on p.219 (7 lines) and understand the interpreted meaning of each line and the justification for it.
11. What part of the Rule of Induction is fulfilled by the 9-line derivation on p.224? What part is not fulfilled?
12. If $X\{\mathbf{a}\}$ is the formula $\exists \mathbf{b}: (\mathbf{S}\mathbf{a} + \mathbf{2}) = (\mathbf{b} + \mathbf{1})$, then interpret $X\{\mathbf{S}\mathbf{a}/\mathbf{a}\}$.
13. Go through the formal description of the Rule of Induction on p.224 phrase by phrase translate it into plain English.