MATH 195: Gödel, Escher, and Bach (Spring 2000)

Notes and Study Questions for Tuesday, March 6

Reading: Chapter VII: *The Propositional Calculus* (pp.188-197)
Homework: Problem Set 7: 2, 4, 5c, 5l, and 6c; Problem Set 8: 1e, 5a, 5b, 6a-c
For your effort, you'll be given up to 10 points on the upcoming exam.

Today's the day we do what Hofstadter wants us to do with this system: learn how to construct formal derivations of logical statements. There are two tricks that will help you a lot. One is to engage your powerful verbal intuition, which lets you see pretty well whether a statement is logically valid or not without any hocus pocus. The second is to just do it. You'll find that usually there's not a whole lot of choices available in how to derive a theorem, and just flailing about will often get you to where you want to go.

Justifying the Rules

Hofstadter tries to help you develop verbal intuition about the rules of deduction, but his examples might not be the best in the world, if you're not into Zen. Try making your own examples for each of the rules. They should make sense to you. For example, De Morgan's Rule can be translated into any number of real life situations. Here's one:

[Upon receipt of a phone call at dinner time from AT&T]

No, I am not interested in changing my long-distance carrier and, no, I will not confirm my address.

If we take **P** to be "*I am interested in changing my long-distance carrier*" and **Q** to be "*I will confirm my address*", then we get the statement:

<~P^~Q>

1. Apply De Morgan's Rule to this symbolic statement and translate the resulting statement back into English.

Perhaps the emotional force isn't as great, but the meaning is basically the same, no?

2. Illustrate all of the Rules of Inference with English sentences that make sense in your life.

Playing Around with the System and Semi-interpretation

3. Hofstadter says "*Now let us apply these rules to a previous theorem…* ***<PÉ~~P>*.**" Why is this statement described as a theorem? Can you derive it? [Hint: He said it was a "previous" theorem. Why?}
4. Follow each step in the derivation at the bottom of p.188. Does each step make sense?

5. Practice semi-interpretating the following statements and decide whether they sound like theorems:

   a. **< Q É <P^Q> >**   b. **< P É <QÉP> >**   c. **< <PvQ> v <Pv~Q> >**

Ganto's Ax

We finally come to a heavy-duty derivation. By it's end, Hofstadter is able to say, "*The power of the Propositional Calculus is shown in this example….*" Please realize that he is being a bit sarcastic here! We have to plow through line after line of mind-breaking logic to reach a conclusion that your verbal intuition should tell you is obvious. But plow we must, because by doing so we gain a machine that is able to produce unequivocally true statements, which our verbal intuition, for all its brilliance, cannot.

First of all, translate Ganto's axiom into English, or use the translation provided by the koan. Either way, feel the force of it -- what conclusion jumps out at you?

6. If someone approached you on the street and said "**< < P ÉQ> ^ <~PÉQ> >**" what logical conclusion would immediately pop into your head? (Resist the temptation to call for medical help) Semi-interpret the phrase. If that doesn't do it then translate it into English.

Skim through the derivation, not trying to understand it but just seeing what kind of manipulations are done.

7. What is the premise of the derivation? What is the goal of the derivation?

Now focus on the heart of the derivation, line (17).

8. Semi-interpret line (17) and then translate it into Ganto's English. Does it say anything different from the axiom, i.e., line (2)?
9. Suppose we could somehow derive line (17), which seems like a reasonable expectation, given how obvious it sounds. If so, then we could presume to be true that **< <Pv~P> É Q>**. What else would we need to show to be a theorem in order to derive **Q** from line (17)? If we had that something, what rule would be used to derive **Q**?

Lines (18) through (21) have as their goal deriving **<Pv~P>**. Notice how it's done. The Fantasy Rule only allows us <u>directly</u> to derive statements of the form **<*x* É *y*>**. Unfortunately, what we want is not of that form. How can we connect the two? A visit to the list of Rules of Inference[1] reveals that the forms **<*w* É *x*> and <*y* v *z*>** are related by the Switcheroo Rule.[2]

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[1] If you have not yet downloaded the Notes for Tuesday, Feb 27, you might want to do so right away. The last page gives you a summary of the Propositional Calculus that may prove helpful in answer these and other questions.

[2] Don't be confused by the choice of variables. We're free to choose whatever variable (e.g., *x*, *y*,[]) we like. The variables are not part of the system, just a means to enable us to talk about it.

10. What statement would produce **<Pv~P>** as the result of applying the Switcheroo Rule?

11. How does Hofstadter prove this theorem (the answer to SQ10) in lines (18) through (21)?

OK. So if you've understood matters thus far, then you can derive **Q** if you can derive **< <Pv~P> É Q>**. How can we accomplish that? This is the task for lines (1) through (17). You might view the proof as the result of flailing about in both directions -- forwards and backwards. First forwards:

12. Given the premise, **< < P ÉQ> ^ <~PÉQ> >**, what theorems are immediately apparent from the direct applications of appropriate Rules of Inference? For now, don't worry whether applying these Rules are moving you forward towards the end of the derivation. Notice that there are only two rules on the list that apply to strings of the form **<*x* ^ *y*>**. (This is like the now standard question of constructing a tree)

Note that the answer to SQ12 gives you Hofstadter's lines (3) and (5).

13. Given lines (3) and (5), what thereoms are immediately apparent from the direct applications of appropriate Rules of Inference? Notice that, again, there are only two rules that can apply.

The answer to SQ13 gives you Hofstadter's lines (4) and (6). Now let's flail from the opposite direction. We want to derive **< <Pv~P> É Q>**.

14. What strings can produce **< <Pv~P> É Q>**, using appropriate Rules of Inference? Note that there are only two possible rules. (This is like the backwards tree you made emanating from **MIU**)

We now have one string, **< ~Q É ~<Pv~P> >** that is of the form that can be produced by the fantasy rule and has something simple on the left hand side (it has a simple premise). So, the game now becomes, if we presume **~Q** to be true, can we derive **~<Pv~P>**, taking advantage of the various tools provided in lines (3) through (6)?

15. Which of the lines from (3) to (6) look like they may be useful in deriving **~<Pv~P>**, given **~Q**? To answer this question, notice which lines are of the form that might allow you to deduce something about **P** given **~Q**.

Now go through the entire derivation again. See if you can see the logic of each step and the role it plays in the derivation.

Do We Know the System is Consistent? The Carroll Dialogue Again

16. Whose argument are you most swayed by -- Prudence's or Imprudence's?

17. Hofstadter translates the core of *Two-Part Invention* into the symbols of the Propositional Calculus. Without going back to that dialogue, translate the symbols back into Tortoise/Achilles English.
