List-coloring the Square of a Subcubic Graph

Daniel Cranston and Seog-Jin Kim

dcransto@dimacs.rutgers.edu

DIMACS, Rutgers University and Bell Labs
Def. list assignment: $L(v)$ is the set of colors available at vertex $v$. 
Def. list assignment: $L(v)$ is the set of colors available at vertex $v$

Def. $L$-coloring: proper coloring where each vertex gets a color from its assigned list
**Def.** list assignment: $L(v)$ is the set of colors available at vertex $v$

**Def.** $L$-coloring: proper coloring where each vertex gets a color from its assigned list

**Def.** $k$-choosable: there exists an $L$-coloring whenever all $|L(v)| \geq k$
Def. list assignment: $L(v)$ is the set of colors available at vertex $v$

Def. $L$-coloring: proper coloring where each vertex gets a color from its assigned list

Def. $k$-choosable: there exists an $L$-coloring whenever all $|L(v)| \geq k$

Def. $\chi_l(G)$: minimum $k$ such that $G$ is $k$-choosable
Def. list assignment: $L(v)$ is the set of colors available at vertex $v$

Def. $L$-coloring: proper coloring where each vertex gets a color from its assigned list

Def. $k$-choosable: there exists an $L$-coloring whenever all $|L(v)| \geq k$

Def. $\chi_l(G)$: minimum $k$ such that $G$ is $k$-choosable
Def. list assignment: $L(v)$ is the set of colors available at vertex $v$

Def. $L$-coloring: proper coloring where each vertex gets a color from its assigned list

Def. $k$-choosable: there exists an $L$-coloring whenever all $|L(v)| \geq k$

Def. $\chi_l(G)$: minimum $k$ such that $G$ is $k$-choosable
Def. list assignment: $L(v)$ is the set of colors available at vertex $v$

Def. $L$-coloring: proper coloring where each vertex gets a color from its assigned list

Def. $k$-choosable: there exists an $L$-coloring whenever all $|L(v)| \geq k$

Def. $\chi_l(G)$: minimum $k$ such that $G$ is $k$-choosable
**Def.** list assignment: $L(v)$ is the set of colors available at vertex $v$

**Def.** $L$-coloring: proper coloring where each vertex gets a color from its assigned list

**Def.** $k$-choosable: there exists an $L$-coloring whenever all $|L(v)| \geq k$

**Def.** $\chi_l(G)$: minimum $k$ such that $G$ is $k$-choosable
**Def.** list assignment: $L(v)$ is the set of colors available at vertex $v$

**Def.** $L$-coloring: proper coloring where each vertex gets a color from its assigned list

**Def.** $k$-choosable: there exists an $L$-coloring whenever all $|L(v)| \geq k$

**Def.** $\chi_l(G)$: minimum $k$ such that $G$ is $k$-choosable
**Def.** list assignment: $L(v)$ is the set of colors available at vertex $v$

**Def.** $L$-coloring: proper coloring where each vertex gets a color from its assigned list

**Def.** $k$-choosable: there exists an $L$-coloring whenever all $|L(v)| \geq k$

**Def.** $\chi_l(G)$: minimum $k$ such that $G$ is $k$-choosable
Def. list assignment: \( L(v) \) is the set of colors available at vertex \( v \)

Def. \( L \)-coloring: proper coloring where each vertex gets a color from its assigned list

Def. \( k \)-choosable: there exists an \( L \)-coloring whenever all \( |L(v)| \geq k \)

Def. \( \chi_l(G) \): minimum \( k \) such that \( G \) is \( k \)-choosable
Def. list assignment: $L(v)$ is the set of colors available at vertex $v$

Def. $L$-coloring: proper coloring where each vertex gets a color from its assigned list

Def. $k$-choosable: there exists an $L$-coloring whenever all $|L(v)| \geq k$

Def. $\chi_l(G)$: minimum $k$ such that $G$ is $k$-choosable

Def. $G^2$ (square of $G$): formed from $G$ by adding edges between vertices at distance 2.
Thm. [Thomassen '08?] \( \chi(G^2) \leq 7 \) if \( G \) is planar and \( \Delta(G) = 3 \).
Results: Old and New

**Thm.** [Thomassen ’08?] \( \chi(G^2) \leq 7 \) if \( G \) is planar and \( \Delta(G) = 3 \).

**Conj.** [Kostochka & Woodall ’01] \( \chi_l(G^2) = \chi(G^2) \) for all \( G \).
Results: Old and New

Thm. [Thomassen ’08?] $\chi(G^2) \leq 7$ if $G$ is planar and $\Delta(G) = 3$.

Conj. [Kostochka & Woodall ’01] $\chi_l(G^2) = \chi(G^2)$ for all $G$.

Cor. $\chi_l(G^2) \leq 7$ if $G$ is planar and $\Delta(G) = 3$. 
Results: Old and New

Thm. [Thomassen ’08?] $\chi(G^2) \leq 7$ if $G$ is planar and $\Delta(G) = 3$.

Conj. [Kostochka & Woodall ’01] $\chi_l(G^2) = \chi(G^2)$ for all $G$.

Cor. $\chi_l(G^2) \leq 7$ if $G$ is planar and $\Delta(G) = 3$.

Thm. If $\Delta(G) = 3$ and $G$ is Petersen-free, then $\chi_l(G^2) \leq 8$. 
Results: Old and New

**Thm.** [Thomassen ’08?] $\chi(G^2) \leq 7$ if $G$ is planar and $\Delta(G) = 3$.

**Conj.** [Kostochka & Woodall ’01] $\chi_l(G^2) = \chi(G^2)$ for all $G$.

**Cor.** $\chi_l(G^2) \leq 7$ if $G$ is planar and $\Delta(G) = 3$.

**Thm.** If $\Delta(G) = 3$ and $G$ is Petersen-free, then $\chi_l(G^2) \leq 8$. 

---

A diagram is shown, which appears to be a graph representing a specific structure or concept related to the theorem and conjecture discussed.
Results: Old and New

**Thm.** [Thomassen ’08?] \( \chi(G^2) \leq 7 \) if \( G \) is planar and \( \Delta(G) = 3 \).

**Conj.** [Kostochka & Woodall ’01] \( \chi_l(G^2) = \chi(G^2) \) for all \( G \).

**Cor.** \( \chi_l(G^2) \leq 7 \) if \( G \) is planar and \( \Delta(G) = 3 \).

**Thm.** If \( \Delta(G) = 3 \) and \( G \) is Petersen-free, then \( \chi_l(G^2) \leq 8 \).
Results: Old and New

**Thm.** [Thomassen ’08?] $\chi(G^2) \leq 7$ if $G$ is planar and $\Delta(G) = 3$.

** Conj.** [Kostochka & Woodall ’01] $\chi_l(G^2) = \chi(G^2)$ for all $G$.

**Cor.** $\chi_l(G^2) \leq 7$ if $G$ is planar and $\Delta(G) = 3$.

**Thm.** If $\Delta(G) = 3$ and $G$ is Petersen-free, then $\chi_l(G^2) \leq 8$.

**Thm.** If $\Delta(G) = 3$, $G$ is planar, and girth $\geq 7$, then $\chi_l(G^2) \leq 7$. 

![Diagram](image-url)
Results: Old and New

**Thm.** [Thomassen ’08?] \( \chi(G^2) \leq 7 \) if \( G \) is planar and \( \Delta(G) = 3 \).

**Conj.** [Kostochka & Woodall ’01] \( \chi_l(G^2) = \chi(G^2) \) for all \( G \).

**Cor.** \( \chi_l(G^2) \leq 7 \) if \( G \) is planar and \( \Delta(G) = 3 \).

**Thm.** If \( \Delta(G) = 3 \) and \( G \) is Petersen-free, then \( \chi_l(G^2) \leq 8 \).

**Thm.** If \( \Delta(G) = 3 \), \( G \) is planar, and girth \( \geq 7 \), then \( \chi_l(G^2) \leq 7 \).

**Thm.** If \( \Delta(G) = 3 \), \( G \) is planar, and girth \( \geq 9 \), then \( \chi_l(G^2) \leq 6 \).
An Easy Lemma

**Lem.** For any edge $uv$ in $G$, we have $\chi_l(G^2 \setminus \{u, v\}) \leq 8$. 
An Easy Lemma

**Lem.** For any edge $uv$ in $G$, we have $\chi_l(G^2 \setminus \{u, v\}) \leq 8$.

**Pf.** Color the vertices greedily in order of decreasing distance from edge $uv$. 
An Easy Lemma

**Lem.** For any edge $uv$ in $G$, we have $\chi_l(G^2 \setminus \{u, v\}) \leq 8$.

**Pf.** Color the vertices greedily in order of decreasing distance from edge $uv$. 
An Easy Lemma

**Lem.** For any edge $uv$ in $G$, we have $\chi_l(G^2 \setminus \{u, v\}) \leq 8$.

**Pf.** Color the vertices greedily in order of decreasing distance from edge $uv$. 

![Diagram](image)
The Main Lemma

**Def.** $\text{ex}(v) = 1 + (\# \text{ colors free at } v) - (\# \text{ uncolored nbrs in } G^2)$
The Main Lemma

Def. \( \text{ex}(v) = 1 + (\# \text{ colors free at } v) - (\# \text{ uncolored nbrs in } G^2) \)
\[ \text{ex}(v) \geq 1 + 8 - 9 = 0 \]
The Main Lemma

**Def.** $\text{ex}(v) = 1 + (\# \text{ colors free at } v) - (\# \text{ uncolored nbrs in } G^2)$

\[
\text{ex}(v) \geq 1 + 8 - 9 = 0
\]

**Lem.** Suppose that $G$ has a partial coloring from its lists. Let $H$ be the subgraph induced by uncolored vertices. Suppose that $H$ is connected. If $H$ contains adjacent vertices $u$ and $v$ such that $\text{ex}(u) \geq 1$ and $\text{ex}(v) \geq 2$, then we can complete the coloring.
The Main Lemma

**Def.** $\text{ex}(v) = 1 + (\# \text{ colors free at } v) - (\# \text{ uncolored nbrs in } G^2)$

$\text{ex}(v) \geq 1 + 8 - 9 = 0$

**Lem.** Suppose that $G$ has a partial coloring from its lists. Let $H$ be the subgraph induced by uncolored vertices. Suppose that $H$ is connected. If $H$ contains adjacent vertices $u$ and $v$ such that $\text{ex}(u) \geq 1$ and $\text{ex}(v) \geq 2$, then we can complete the coloring.

**Pf.** Color greedily toward $uv$. 
The Main Lemma

**Def.** \( \text{ex}(v) = 1 + (\# \text{ colors free at } v) - (\# \text{ uncolored nbrs in } G^2) \)
\[ \text{ex}(v) \geq 1 + 8 - 9 = 0 \]

**Lem.** Suppose that \( G \) has a partial coloring from its lists. Let \( H \) be the subgraph induced by uncolored vertices. Suppose that \( H \) is connected. If \( H \) contains adjacent vertices \( u \) and \( v \) such that \( \text{ex}(u) \geq 1 \) and \( \text{ex}(v) \geq 2 \), then we can complete the coloring.

**Pf.** Color greedily toward \( uv \).

**Cor.** If \( G \) is Petersen-free and \( \delta(G) < 3 \), then \( \chi_l(G^2) \leq 8 \).
The Main Lemma

**Def.** \( ex(v) = 1 + (\# \text{ colors free at } v) - (\# \text{ uncolored nbrs in } G^2) \)

\[ ex(v) \geq 1 + 8 - 9 = 0 \]

**Lem.** Suppose that \( G \) has a partial coloring from its lists. Let \( H \) be the subgraph induced by uncolored vertices. Suppose that \( H \) is connected. If \( H \) contains adjacent vertices \( u \) and \( v \) such that \( ex(u) \geq 1 \) and \( ex(v) \geq 2 \), then we can complete the coloring.

**Pf.** Color greedily toward \( uv \).

**Cor.** If \( G \) is Petersen-free and \( \delta(G) < 3 \), then \( \chi_l(G^2) \leq 8 \).
**The Main Lemma**

**Def.** \( \text{ex}(v) = 1 + (\# \text{ colors free at } v) - (\# \text{ uncolored nbrs in } G^2) \)

\[ \text{ex}(v) \geq 1 + 8 - 9 = 0 \]

**Lem.** Suppose that \( G \) has a partial coloring from its lists. Let \( H \) be the subgraph induced by uncolored vertices. Suppose that \( H \) is connected. If \( H \) contains adjacent vertices \( u \) and \( v \) such that \( \text{ex}(u) \geq 1 \) and \( \text{ex}(v) \geq 2 \), then we can complete the coloring.

**Pf.** Color greedily toward \( uv \).

**Cor.** If \( G \) is Petersen-free and \( \delta(G) < 3 \), then \( \chi_l(G^2) \leq 8 \).

**Cor.** If \( G \) is Petersen-free and \( \text{girth}(G) = 3 \), then \( \chi_l(G^2) \leq 8 \).
Girth 4 to 6

**Lemma.** If $G$ is Petersen-free and $\text{girth}(G)=4$, then $\chi_l(G^2) \leq 8$. 
Girth 4 to 6

**Lem.** If $G$ is Petersen-free and $\text{girth}(G)=4$, then $\chi_l(G^2) \leq 8$.

**Pf.** Easy application of main lemma.
Girth 4 to 6

**Lem.** If $G$ is Petersen-free and $\text{girth}(G)=4$, then $\chi_l(G^2) \leq 8$.

**Pf.** Easy application of main lemma.

**Lem.** If $G$ is Petersen-free and $\text{girth}(G)=5$, then $\chi_l(G^2) \leq 8$. 
Girth 4 to 6

**Lem.** If $G$ is Petersen-free and $\text{girth}(G)=4$, then $\chi_l(G^2) \leq 8$.

**Pf.** Easy application of main lemma.

**Lem.** If $G$ is Petersen-free and $\text{girth}(G)=5$, then $\chi_l(G^2) \leq 8$.

**Pf.** Harder application of main lemma.
Girth 4 to 6

**Lem.** If $G$ is Petersen-free and $\text{girth}(G) = 4$, then $\chi_l(G^2) \leq 8$.

**Pf.** Easy application of main lemma.

**Lem.** If $G$ is Petersen-free and $\text{girth}(G) = 5$, then $\chi_l(G^2) \leq 8$.

**Pf.** Harder application of main lemma.

**Lem.** If $G$ is Petersen-free and $\text{girth}(G) = 6$, then $\chi_l(G^2) \leq 8$. 

Girth 4 to 6

**Lem.** If $G$ is Petersen-free and $\text{girth}(G) = 4$, then $\chi_l(G^2) \leq 8$.

**Pf.** Easy application of main lemma.

**Lem.** If $G$ is Petersen-free and $\text{girth}(G) = 5$, then $\chi_l(G^2) \leq 8$.

**Pf.** Harder application of main lemma.

**Lem.** If $G$ is Petersen-free and $\text{girth}(G) = 6$, then $\chi_l(G^2) \leq 8$.

**Pf.** Color all but a 6-cycle.
Girth 4 to 6

**Lemma.** If $G$ is Petersen-free and $\operatorname{girth}(G)=4$, then $\chi_l(G^2) \leq 8$.

**Proof.** Easy application of main lemma.

**Lemma.** If $G$ is Petersen-free and $\operatorname{girth}(G)=5$, then $\chi_l(G^2) \leq 8$.

**Proof.** Harder application of main lemma.

**Lemma.** If $G$ is Petersen-free and $\operatorname{girth}(G)=6$, then $\chi_l(G^2) \leq 8$.

**Proof.** Color all but a 6-cycle.

\[ H = \]
Girth 4 to 6

Lem. If $G$ is Petersen-free and $\text{girth}(G) = 4$, then $\chi_l(G^2) \leq 8$.

Pf. Easy application of main lemma.

Lem. If $G$ is Petersen-free and $\text{girth}(G) = 5$, then $\chi_l(G^2) \leq 8$.

Pf. Harder application of main lemma.

Lem. If $G$ is Petersen-free and $\text{girth}(G) = 6$, then $\chi_l(G^2) \leq 8$.

Pf. Color all but a 6-cycle.

$H^2 =$
Girth 4 to 6

**Lem.** If $G$ is Petersen-free and $\text{girth}(G) = 4$, then $\chi_l(G^2) \leq 8$.

**Pf.** Easy application of main lemma.

**Lem.** If $G$ is Petersen-free and $\text{girth}(G) = 5$, then $\chi_l(G^2) \leq 8$.

**Pf.** Harder application of main lemma.

**Lem.** If $G$ is Petersen-free and $\text{girth}(G) = 6$, then $\chi_l(G^2) \leq 8$.

**Pf.** Color all but a 6-cycle.

\[ H^2 = \]

\[
\chi_l(H^2) = 3
\]
Girth 4 to 6

**Lem.** If $G$ is Petersen-free and $\text{girth}(G) = 4$, then $\chi_l(G^2) \leq 8$.
**Pf.** Easy application of main lemma.

**Lem.** If $G$ is Petersen-free and $\text{girth}(G) = 5$, then $\chi_l(G^2) \leq 8$.
**Pf.** Harder application of main lemma.

**Lem.** If $G$ is Petersen-free and $\text{girth}(G) = 6$, then $\chi_l(G^2) \leq 8$.
**Pf.** Color all but a 6-cycle.

$H^2 = \begin{array}{c}
\begin{array}{c}
\text{Cycle + Triangle Thm} \\
[Fleischner, Steibitz '92]
\end{array}
\end{array}$
Girth 4 to 6

Lem. If $G$ is Petersen-free and $\text{girth}(G)=4$, then $\chi_l(G^2) \leq 8$.

Pf. Easy application of main lemma.

Lem. If $G$ is Petersen-free and $\text{girth}(G)=5$, then $\chi_l(G^2) \leq 8$.

Pf. Harder application of main lemma.

Lem. If $G$ is Petersen-free and $\text{girth}(G)=6$, then $\chi_l(G^2) \leq 8$.

Pf. Color all but a 6-cycle.

\[
\begin{align*}
\chi_l(H^2) &= 3 \\
\text{Cycle + Triangle Thm} &
\text{[Fleischner, Steibitz '92]} \\
\chi_l(C_{6k}^2) &= 3 \\
&\text{[Juvan, Mohar, Skrekovski '98]}
\end{align*}
\]
Large girth

**Obs.** If \( \text{girth}(G) \geq 7 \) and \( C \) is a shortest cycle in \( G \), then any two vertices that are each adjacent to the cycle are nonadjacent.
Large girth

**Obs.** If $\text{girth}(G) \geq 7$ and $C$ is a shortest cycle in $G$, then any two vertices that are each adjacent to the cycle are nonadjacent.
Large girth

**Obs.** If \( \text{girth}(G) \geq 7 \) and \( C \) is a shortest cycle in \( G \), then any two vertices that are each adjacent to the cycle are nonadjacent.

**Lem.** If \( \text{girth}(G) \geq 7 \), then \( \chi_l(G^2) \leq 8 \).
Large girth

**Obs.** If \( \text{girth}(G) \geq 7 \) and \( C \) is a shortest cycle in \( G \), then any two vertices that are each adjacent to the cycle are nonadjacent.

**Lem.** If \( \text{girth}(G) \geq 7 \), then \( \chi_l(G^2) \leq 8 \).

**Pf.** Let \( H \) be a shortest cycle and neighbors. Color \( G^2 \setminus V(H) \).
Large girth

**Obs.** If $\text{girth}(G) \geq 7$ and $C$ is a shortest cycle in $G$, then any two vertices that are each adjacent to the cycle are nonadjacent.

**Lem.** If $\text{girth}(G) \geq 7$, then $\chi_l(G^2) \leq 8$.

**Pf.** Let $H$ be a shortest cycle and neighbors. Color $G^2 \setminus V(H)$. Two cases depending on whether there exist $i \neq j$ s.t. $|i - j| \leq 2$ and $L(u_i) \cap L(u_j) \neq \emptyset$ or there exists $i$ s.t. $L(u_{i-1}) \cup L(u_i) \cup L(u_{i+1}) \not\subseteq L(v_i)$.
Large girth

**Obs.** If girth\( (G) \geq 7 \) and \( C \) is a shortest cycle in \( G \), then any two vertices that are each adjacent to the cycle are nonadjacent.

**Lem.** If girth\( (G) \geq 7 \), then \( \chi_l(G^2) \leq 8 \).

**Pf.** Let \( H \) be a shortest cycle and neighbors. Color \( G^2 \setminus V(H) \). Two cases depending on whether there exist \( i \neq j \) s.t. \(|i - j| \leq 2 \) and \( L(u_i) \cap L(u_j) \neq \emptyset \) or there exists \( i \) s.t. \( L(u_{i-1}) \cup L(u_i) \cup L(u_{i+1}) \not\subseteq L(v_i) \).

1) Suppose so:
Large girth

**Obs.** If \( \text{girth}(G) \geq 7 \) and \( C \) is a shortest cycle in \( G \), then any two vertices that are each adjacent to the cycle are nonadjacent.

**Lem.** If \( \text{girth}(G) \geq 7 \), then \( \chi_l(G^2) \leq 8 \).

**Pf.** Let \( H \) be a shortest cycle and neighbors. Color \( G^2 \setminus V(H) \). Two cases depending on whether there exist \( i \neq j \) s.t. \( |i - j| \leq 2 \) and \( L(u_i) \cap L(u_j) \neq \emptyset \) or there exists \( i \) s.t. \( L(u_{i-1}) \cup L(u_i) \cup L(u_{i+1}) \not\subseteq L(v_i) \).

1) **Suppose so:** We can color more vertices so that for some \( i \), \( \text{ex}(v_i) \geq 1 \) and \( \text{ex}(v_{i+1}) \geq 2 \). Then use our main lemma.
Large girth

Lem. If girth\((G) \geq 7\), then \(\chi_l(G^2) \leq 8\).

Pf. Let \(H\) be a shortest cycle and neighbors. Color \(G^2 \setminus V(H)\). Two cases depending on whether there exist \(i \neq j\) s.t. \(|i - j| \leq 2\) and \(L(u_i) \cap L(u_j) \neq \emptyset\) or there exists \(i\) s.t. \(L(u_{i-1}) \cup L(u_i) \cup L(u_{i+1}) \not\subseteq L(v_i)\).

1) Suppose so: We can color more vertices so that for some \(i\), \(ex(v_i) \geq 1\) and \(ex(v_{i+1}) \geq 2\). Then use our main lemma.

2) Suppose not:
Lem. If girth($G$) $\geq$ 7, then $\chi_l(G^2) \leq 8$.

Pf. Let $H$ be a shortest cycle and neighbors. Color $G^2 \setminus V(H)$. Two cases depending on whether there exist $i \neq j$ s.t. $|i - j| \leq 2$ and $L(u_i) \cap L(u_j) \neq \emptyset$ or there exists $i$ s.t. $L(u_{i-1}) \cup L(u_i) \cup L(u_{i+1}) \not\subseteq L(v_i)$

1) Suppose so: We can color more vertices so that for some $i$, $\text{ex}(v_i) \geq 1$ and $\text{ex}(v_{i+1}) \geq 2$. Then use our main lemma.

2) Suppose not: Choose $c(u_i)$ arbitrarily from $L(u_i)$. Choose $c(v_i)$ from $L(u_i) - c(u_i)$. 
Thank you!

Any Questions?