Sorting a Permutation by Cut-and-Paste Moves

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Definitions and an Example

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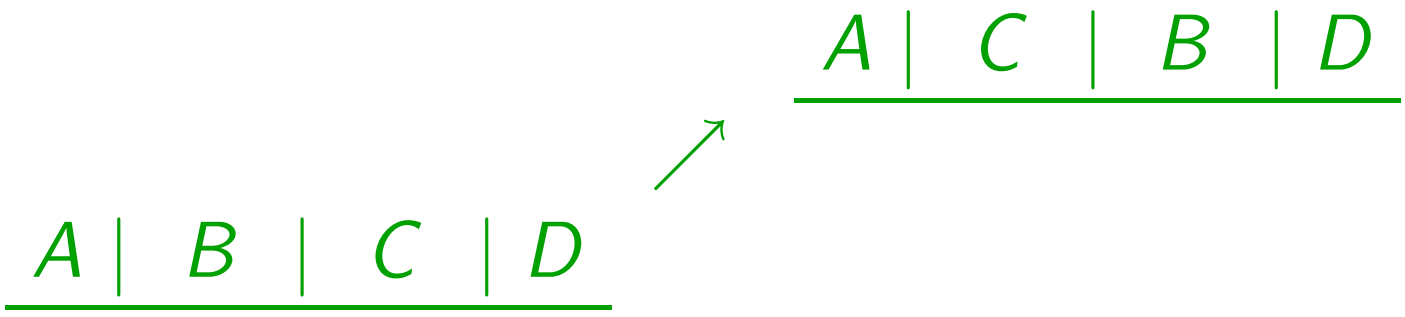

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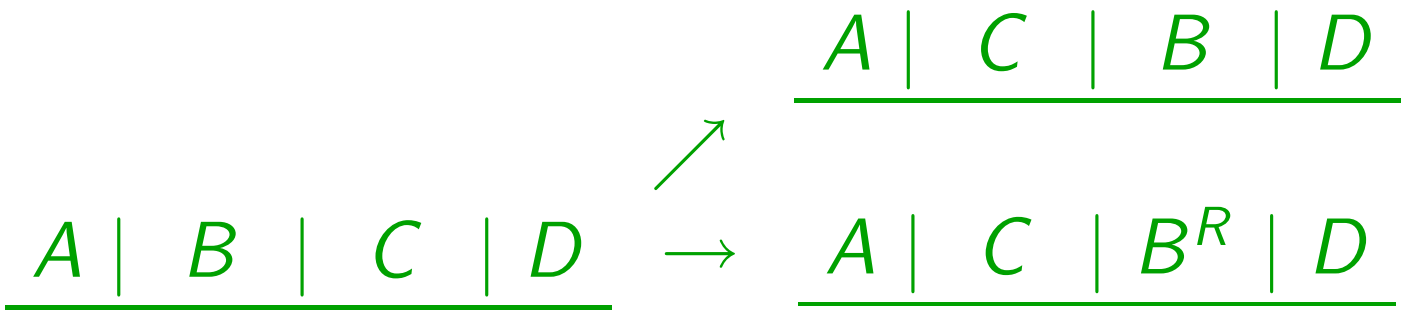

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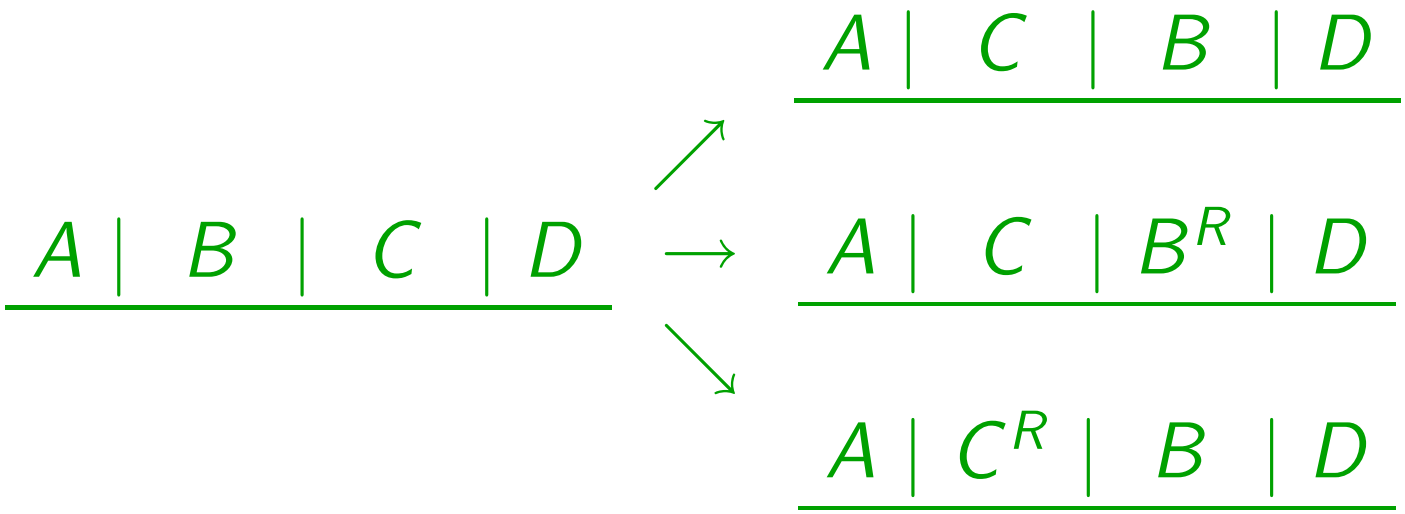

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1\ 2\ 13\ 7\ 4\ \underline{12\ 11\ 10\ 9}\ \overline{8}\ 6\ 5\ 3 \\
1\ 2\ \overline{8\ 9\ 10\ 11\ 12}\ 13\ \underline{7\ 4\ 6\ 5\ 3} \\
1\ 2\ \overline{3\ \underline{5\ 6}\ 4\ 7}\ 8\ 9\ 10\ 11\ 12\ 13 \\
1\ 2\ 3\ 4\ \overline{5\ 6}\ 7\ 8\ 9\ 10\ 11\ 12\ 13
\end{array}$$

Def. $g(n)$: max # of moves to sort a permuation of $\{1, 2, \ldots, n\}$

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1\ 2\ \overline{8\ 9\ 10\ 11\ 12}\ 13\ \underline{7\ 4\ 6\ 5\ 3} \\
1\ 2\ \overline{3\ \underline{5\ 6}\ 4\ 7}\ 8\ 9\ 10\ 11\ 12\ 13 \\
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Obs. $g(n) \leq n - 1$

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\end{array}$$

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Obs. $g(n) \leq n - 1$ and $g(n) \geq \lceil n/3 \rceil$

Proving the Lower Bound

Thm. $g(n) \geq \lfloor n/2 \rfloor$

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$$A\ 0|0\ B \mid C \mid D$$

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Pf.

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$$A\ 0|0\ B\ 1|1\ C\ 1|?\ D$$

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Note that $(0)12345\ldots n$ has n parity adjacencies.

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Note that $(0)12345\ldots n$ has n parity adjacencies.
And $(0)24\ldots531$ has only 1 parity adjacency.

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So $g(n) \geq \lceil (n-1)/2 \rceil = \lfloor n/2 \rfloor$. ■

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Def. block: maximal substring of adjacencies (size at least 2)

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Idea Always makes moves with gain at least 1.

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1 2 13 7 4 $\underline{10\ 9\ 5\ 3\ 6\ 8}$ $\underline{12\ 11}$

1 2 13 7 4 $\overline{12\ 11}$ 10 9 5 3 6 8 block

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		gain
1 2 13 7 4 10 9 5 3 6 8 12 11		
1 2 13 7 4 12 11 10 9 5 3 6 8	block	1

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1 2 13 7 4 10 9 5 3 6 8 12 11		gain
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1 2 13 7 4 10 9 5 3 6 8 12 11		
1 2 13 7 4 12 11 10 9 5 3 6 8	block	1
1 2 13 7 4 12 11 10 9 8 6 5 3	bonus	

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1 2 13 7 4 12 11 10 9 8 6 5 3	bonus	1
1 2 8 9 10 11 12 13 7 4 6 5 3	absorbing	

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1 2 8 9 10 11 12 13 7 4 6 5 3	absorbing	2/3

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Sorting Algorithm

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Sorting Algorithm

Put 1 2 at the left end

Proving the Upper Bound

Thm. $g(n) \leq \lceil 2n/3 \rceil + 1$. $w(\pi) =$ # blocks + 2/3 # singles

		gain
1 2 13 7 4 10 9 5 3 6 8 12 11		
1 2 13 7 4 12 11 10 9 5 3 6 8	block	1
1 2 13 7 4 12 11 10 9 8 6 5 3	bonus	1
1 2 8 9 10 11 12 13 7 4 6 5 3	absorbing	2/3
1 2 3 5 6 4 7 8 9 10 11 12 13	extrabonus	4/3

Sorting Algorithm

Put 1 2 at the left end

Repeat the following until sorted

Proving the Upper Bound

Thm. $g(n) \leq \lceil 2n/3 \rceil + 1$. $w(\pi) =$ # blocks + 2/3 # singles

		gain
1 2 13 7 4 10 9 5 3 6 8 12 11		
1 2 13 7 4 12 11 10 9 5 3 6 8	block	1
1 2 13 7 4 12 11 10 9 8 6 5 3	bonus	1
1 2 8 9 10 11 12 13 7 4 6 5 3	absorbing	2/3
1 2 3 5 6 4 7 8 9 10 11 12 13	extrabonus	4/3

Sorting Algorithm

Put 1 2 at the left end

Repeat the following until sorted

Case 1: make a block move

Proving the Upper Bound

Thm. $g(n) \leq \lceil 2n/3 \rceil + 1$. $w(\pi) =$ # blocks + 2/3 # singles

		gain
1 2 13 7 4 10 9 5 3 6 8 12 11		
1 2 13 7 4 12 11 10 9 5 3 6 8	block	1
1 2 13 7 4 12 11 10 9 8 6 5 3	bonus	1
1 2 8 9 10 11 12 13 7 4 6 5 3	absorbing	2/3
1 2 3 5 6 4 7 8 9 10 11 12 13	extrabonus	4/3

Sorting Algorithm

Put 1 2 at the left end

Repeat the following until sorted

- Case 1: make a block move
- Case 2: make a bonus move

Proving the Upper Bound

Thm. $g(n) \leq \lceil 2n/3 \rceil + 1$. $w(\pi) = \#$ blocks $+ 2/3 \#$ singles

		gain
1 2 13 7 4 10 9 5 3 6 8 12 11		
1 2 13 7 4 12 11 10 9 5 3 6 8	block	1
1 2 13 7 4 12 11 10 9 8 6 5 3	bonus	1
1 2 8 9 10 11 12 13 7 4 6 5 3	absorbing	2/3
1 2 3 5 6 4 7 8 9 10 11 12 13	extrabonus	4/3

Sorting Algorithm

Put 1 2 at the left end

Repeat the following until sorted

- Case 1: make a block move
- Case 2: make a bonus move
- Case 3: make an absorbing move, then an extra bonus move