Planar graphs are 9/2-colorable

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Joint with Landon Rabern
Slides available on my webpage

Connections in Discrete Math
Simon Fraser
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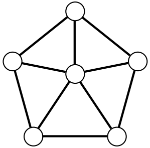

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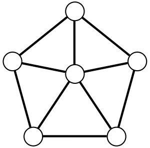

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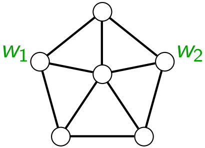

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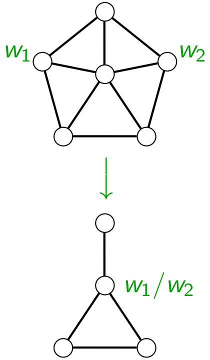

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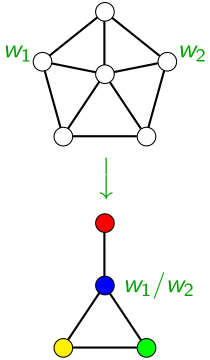

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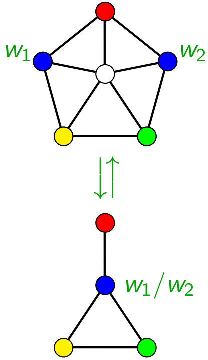

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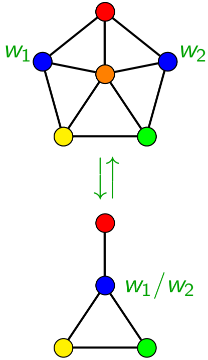

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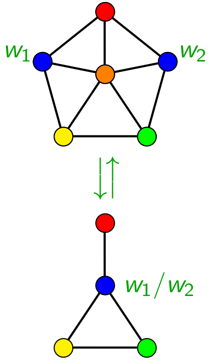

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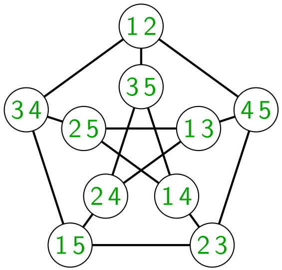

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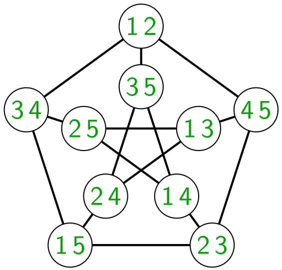

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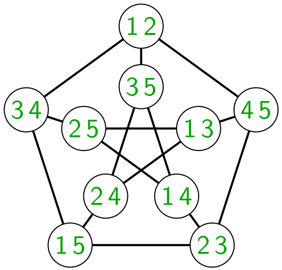

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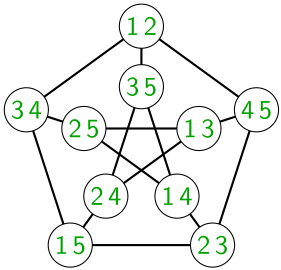

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G is t-colorable iff G has homomorphism to K_t.

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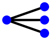
Key Fact: Denote the center vertex of  by v and the other vertices by u_1, u_2, u_3.

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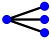
Key Fact: Denote the center vertex of  by v and the other vertices by u_1, u_2, u_3. If v has 5 allowable colors and each u_i has 3 allowable colors, then we can color each vertex with 2 colors, such that no color appears on both ends of an edge.

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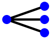
Key Fact: Denote the center vertex of  by v and the other vertices by u_1, u_2, u_3. If v has 5 allowable colors and each u_i has 3 allowable colors, then we can color each vertex with 2 colors, such that no color appears on both ends of an edge.

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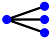
Key Fact: Denote the center vertex of  by v and the other vertices by u_1, u_2, u_3. If v has 5 allowable colors and each u_i has 3 allowable colors, then we can color each vertex with 2 colors, such that no color appears on both ends of an edge.

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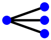
Key Fact: Denote the center vertex of  by v and the other vertices by u_1, u_2, u_3. If v has 5 allowable colors and each u_i has 3 allowable colors, then we can color each vertex with 2 colors, such that no color appears on both ends of an edge.

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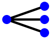
Key Fact: Denote the center vertex of  by v and the other vertices by u_1, u_2, u_3. If v has 5 allowable colors and each u_i has 3 allowable colors, then we can color each vertex with 2 colors, such that no color appears on both ends of an edge.

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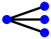
Key Fact: Denote the center vertex of  by v and the other vertices by u_1, u_2, u_3. If v has 5 allowable colors and each u_i has 3 allowable colors, then we can color each vertex with 2 colors, such that no color appears on both ends of an edge.

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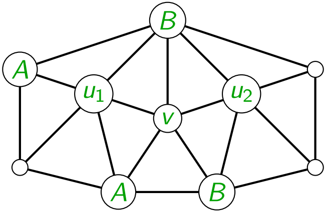

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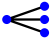
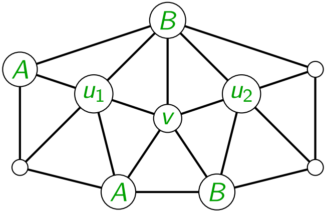
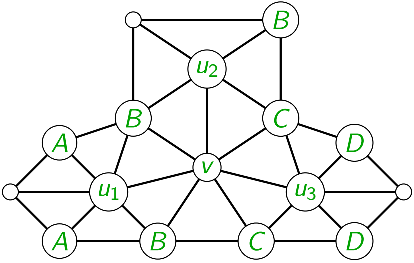
Key Fact: Denote the center vertex of  by v and the other vertices by u_1, u_2, u_3. If v has 5 allowable colors and each u_i has 3 allowable colors, then we can color each vertex with 2 colors, such that no color appears on both ends of an edge.

Pf: Give v a color available for at most one u_i, say u_1. $2(5) > 3(3)$ Now give v another color not available for u_1. Now color each u_i.





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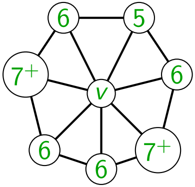

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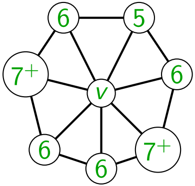

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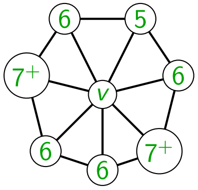

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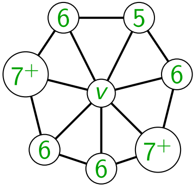

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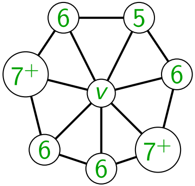

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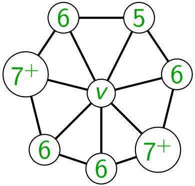

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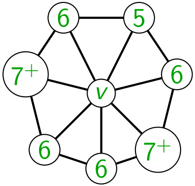

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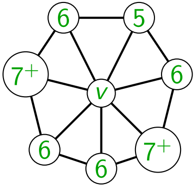

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Now show that $ch^*(v) \geq 0$ for all v.

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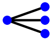
  - few 6^--verts near each other; Key Fact for coloring 

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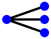
  - few 6^--verts near each other; Key Fact for coloring 
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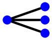
  - few 6^--verts near each other; Key Fact for coloring 
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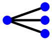
  - few 6^--verts near each other; Key Fact for coloring 

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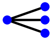
  - few 6^--verts near each other; Key Fact for coloring 

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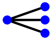
  - few 6^--verts near each other; Key Fact for coloring 

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