Bootstrap Percolation Thresholds in Plane Tilings using Regular Polygons

Daniel W. Cranston
Virginia Commonwealth University
dcranston@vcu.edu

Joint with Neal Bushaw
Slides available on my webpage

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Two Simple Examples

Bootstrap Percolation:
Some faces start infected.
Infected faces stay infected.
Uninfected faces with at least two infected neighbors become infected.
Does the whole graph become infected?
Ex:
Two Simple Examples

**Bootstrap Percolation:** Some faces start infected.
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**Ex:**

[Grid diagram with some faces marked with an 'x']
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Ex:

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Ex:

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**Ex:**

![Bootstrap Percolation Example](image)
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Ex:

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![Bootstrap Percolation Example](image)
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Ex:

Yes.  No.
The \( k \)-bootstrap Model

1. The initially infected faces are picked randomly.
2. Number of infected neighbors needed to infect a healthy face is \( k \).
3. We mainly consider infinite graphs.

\[ \text{The \( k \)-bootstrap Model:} \] Fix a plane graph \( G \), a \( p \)-random set \( I \) of initially infected faces, and an integer \( k \).

If a healthy face, \( f \), has at least \( k \) infected neighbors, then \( f \) becomes infected.

\[ \text{The percolation threshold of} \ G \ \text{is the largest} \ k \ \text{such that} \ G \ \text{eventually becomes completely infected with prob} \ \geq \ \frac{1}{2} \ \text{(since} \ I \ \text{is random).} \]

**Warmup:** In the 1-bootstrap model if \( I \neq \emptyset \), then \( I \) percolates.

**Pf:** Say that \( f_0 \in I \).

By induction, we show that each face within distance \( t \) of \( f_0 \) becomes infected (for all \( t \)).

So \( I \) percolates.
The \textit{k}-bootstrap Model

We make a few key changes to our game.

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The $k$-bootstrap Model: Fix a plane graph $G$, a $p$-random set $I$ of initially infected faces, and an integer $k$. 
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\textbf{Warmup: } In the 1-bootstrap model if \( I \neq \emptyset \), then \( I \) percolates.

\textbf{Pf: } Say that \( f_0 \in I \). By induction, we show that each face within distance \( t \) of \( f_0 \) becomes infected (for all \( t \)). So \( I \) percolates.
The Triangular Lattice

Lemma 2: Let $G$ be the triangular lattice and $I$ be a $p$-random set, with $p < 1$. In the 2-bootstrap model, $I$ percolates with prob. 0.

Corollary: The triangular lattice has percolation threshold 1, whenever $0 < p < 1$. 
Lem 1: Let $G$ be the triangular lattice and $I$ be a $p$-random set, with $0 < p$. In the 1-bootstrap model, $I$ percolates with prob. 1.
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Pf: Note that if any $\blacklozenge$ has no infected faces, then $I$ will not percolate.

Cor: The triangular lattice has percolation threshold 1, whenever $0 < p < 1$. 
Lem 2: Let $G$ be the triangular lattice and $I$ be a $p$-random set, with $p < 1$. In the 2-bootstrap model, $I$ percolates with prob. 0. 

Pf: Note that if any $\Box$ has no infected faces, then $I$ will not percolate. Let $B_1, B_2, \ldots$ denote face disjoint copies of $\Box$. 

Cor: The triangular lattice has percolation threshold $1$, whenever $0 < p < 1$. 
Lem 2: Let $G$ be the triangular lattice and $\mathcal{I}$ be a $p$-random set, with $p < 1$. In the 2-bootstrap model, $\mathcal{I}$ percolates with prob. 0.

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Cor: The triangular lattice has percolation threshold 1, whenever $0 < p < 1$. 
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**The Triangular Lattice**

**Lem 1:** Let $G$ be the triangular lattice and $\mathcal{I}$ be a $p$-random set, with $0 < p$. In the 1-bootstrap model, $\mathcal{I}$ percolates with prob. 1.

**Lem 2:** Let $G$ be the triangular lattice and $\mathcal{I}$ be a $p$-random set, with $p < 1$. In the 2-bootstrap model, $\mathcal{I}$ percolates with prob. 0.

**Cor:** The triangular lattice has percolation threshold 1, whenever $0 < p < 1$. 
The Hex Lattice
The Hex Lattice

**Lem 3:** Fix $p < 1$. For the hex lattice, in the 4-bootstrap model, a $p$-random set $I$ percolates with prob. 0.

**Pf:** Same as Lem 2, but with $\bigcirc$ in place of $\bigotimes$. ■
Lem 4: Fix \( p > 0 \). For the hex lattice, in the 3-bootstrap model, a \( p \)-random set \( I \) percolates with prob. \( > 0 \).
The Hex Lattice

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Lemma 4: Fix $p > 0$. For the hex lattice, in the $3$-bootstrap model, a $p$-random set $I$ percolates with prob. $> 0$.

Proof: Now $\Pr[\text{side with } t \text{ hexes is bad}] = (1 - p)^t > 0$, so $\Pr[\text{ring with } t \text{ hexes per side is bad}] \leq 6(1 - p)^t$. Sum for all rings: $S = \sum_{t=1}^{\infty} 6(1 - p)^t p < 1$ for big $j$. $\Pr[I \text{ percolates}] \geq (1 - S) \Pr[\text{all small rings good}] > 0$. 
The Hex Lattice

Lem 4: Fix $p > 0$. For the hex lattice, in the 3-bootstrap model, a $p$-random set $I$ percolates with prob. $> 0$.

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Lem 4: Fix $p > 0$. For the hex lattice, in the 3-bootstrap model, a $p$-random set $I$ percolates with prob. > 0.

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Sum for all rings: \( S = \sum_{t=j}^{\infty} 6(1 - p)^t = \frac{6(1-p)^j}{p} < 1 \) for big \( j \).
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$\Pr[\mathcal{I} \text{ percolates}] \geq (1 - S) \Pr[\text{all small rings good}] > 0$. ■
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**Corollary 6:** The hex lattice has threshold 3.
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Regular Lattices and Beyond

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- All faces are still regular polygons
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(3.12.12) 3
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Archimedean Lattices: Lower Bounds

[Diagram of Archimedean Lattice with labeled points 3, 1, 5, 7 and marked intersections]
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