Euler’s Pentagonal Number Theorem

Dan Cranston

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Introduction
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Square Numbers: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
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![Triangular Numbers Diagram]

Square Numbers: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...

![Square Numbers Diagram]

Pentagonal Numbers: 1, 5, 12, 22, 35, 51, 70, 92, 117, 145, ...

![Pentagonal Numbers Diagram]
The $k$th pentagonal number, $P(k)$, is the $k$th partial sum of the arithmetic sequence $a_n = 1 + 3(n - 1) = 3n - 2$.

$$P(k) = k \sum_{n=1}^{k} (3n - 2) = k \sum_{n=1}^{k} 3n - k \sum_{n=1}^{k} 1 = 3\left(\frac{k(k+1)}{2}\right) - k\left(\frac{k}{2}\right) = \frac{3k^2 - k^2}{2} = \frac{k(k+1)}{2}.$$  

$P(8) = 92$, $P(500) = 374, 750$, etc., and $P(0) = 0$.

Extend domain, so $P(-8) = 100$, $P(-500) = 375, 250$, etc.

$\{P(0), P(1), P(-1), P(2), P(-2), \ldots\} = \{0, 1, 2, 5, 7, \ldots\}$ is an increasing sequence.
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Generalized Pentagonal Numbers

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\begin{itemize}
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  \item \( \{P(0), P(1), P(-1), P(2), P(-2), \ldots \} = \{0, 1, 2, 5, 7, \ldots \} \) is an increasing sequence.
\end{itemize}
Partition Numbers

A partition of a positive integer $n$ is a way of expressing $n$ as a sum of positive integers.

$3 = 2 + 1 = 1 + 1 + 1$, so $p(3) = 3$.

$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$, so $p(4) = 5$.

$5 = 4 + 1 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 3 + 2 = 1 + 1 + 1 + 1 + 1$, so $p(5) = 7$.

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Each summand in a certain partition is called a part. So $3$ has 1 part, $2 + 1$ has 2 parts, and $1 + 1 + 1$ has 3 parts.
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\text{6} & = 5+1 = 4+1+1 = 4+2 = 3+1+1+1 = 3+3 = 3+2+1 = 2+1+1+1+1 = 2+2+2 = 2+2+1+1 = 1+1+1+1+1+1, \\
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Three different partitions of 9:

- $5 + 3 + 1$
- $4 + 3 + 2$
- $4 + 3 + 1 + 1$
Special Partition Numbers

\[ p_d(n) = \text{number of partitions of } n \text{ into distinct parts} \]
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\[ p_e(n) = \text{number of partitions of } n \text{ into an even number of distinct parts} \]
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\[ p_e(n) = \text{number of partitions of } n \text{ into an even number of distinct parts}; \text{ similar for } p_o(n), \text{ so } p_e(n) + p_o(n) = p_d(n) \]
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\( p_e(n) = \) number of partitions of \( n \) into an even number of distinct parts; similar for \( p_o(n) \), so \( p_e(n) + p_o(n) = p_d(n) \)

\( \therefore 5 = 4+1 = 3+1+1 = 2+2+1 = 2+1+1+1 = 3+2 = 1+1+1+1+1, \text{ so } p_e(5) = 2. (p_o(5) = 1) \)
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Pentagonal Number Theorem

Main Theorem

\[
\prod_{m=1}^{\infty} (1 - x^m) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} + \ldots
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\[ = x^{P(0)} - x^{P(1)} - x^{P(-1)} + x^{P(2)} + x^{P(-2)} - \ldots \]
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Lemma 1

\[\prod_{m=1}^{\infty} (1 - x^m) = 1 + \sum_{n=1}^{\infty} (p_e(n) - p_o(n))x^n\]
Pentagonal Number Theorem

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Lemma 2

\[1 + \sum_{n=1}^{\infty} (p_e(n) - p_o(n))x^n = 1 - x - x^2 + x^5 + x^7 + \ldots\]
Proof of Lemma 1: The product as a sum

\[ \prod_{m=1}^{\infty} (1 - x^m) = (1 - x)(1 - x^2)(1 - x^3)(1 - x^4)(1 - x^5) \ldots \]
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- \( x^n \) occurs once for each partition of \( n \) into distinct parts.
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- \( x^n \) occurs once for each partition of \( n \) into distinct parts.
- Each partition of \( n \) into an even number of distinct parts contributes \(+1\) to the coefficient of \( x^n \), and each partition of \( n \) into an odd number of distinct parts contributes \(-1\).
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- Partitions of 5 into distinct parts: 5, 1+4, and 2+3.
Proof of Lemma 1: The product as a sum

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- So \( x^5 \) occurs in the expansion as
Proof of Lemma 1: The product as a sum

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- \( x^n \) occurs once for each partition of \( n \) into distinct parts.
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- Partitions of 5 into distinct parts: 5, 1+4, and 2+3.
- So \( x^5 \) occurs in the expansion as

\[ (-x^5) + (-x)(-x^4) + (-x^2)(-x^3) = \]
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\[ \prod_{m=1}^{\infty} (1 - x^m) = (1 - x)(1 - x^2)(1 - x^3)(1 - x^4)(1 - x^5) \ldots. \]

▶ \( x^n \) occurs once for each partition of \( n \) into distinct parts.
▶ Each partition of \( n \) into an even number of distinct parts contributes \( +1 \) to the coefficient of \( x^n \), and each partition of \( n \) into an odd number of distinct parts contributes \( -1 \).

▶ Partitions of 5 into distinct parts: 5, 1+4, and 2+3.
▶ So \( x^5 \) occurs in the expansion as

\[
(-x^5) + (-x)(-x^4) + (-x^2)(-x^3) = \\
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\]
Proof of Lemma 1: The product as a sum

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= 0.
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\]
\[ (p_e(6) - p_o(6))(x^6) = 0. \]
Pentagonal Number Theorem: Outline of Proof

Lemma 1:
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\prod_{m=1}^{\infty} (1 - x^m) = 1 + \sum_{n=1}^{\infty} (p_e(n) - p_o(n))x^n
\]

Lemma 2:
\[
1 + \sum_{n=1}^{\infty} (p_e(n) - p_o(n))x^n = 1 - x - x^2 + x^5 + x^7 + \ldots
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Pentagonal Number Theorem: Outline of Proof

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Proof Part 2: Cancellation of partition numbers

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Proof Part 2: Cancellation of partition numbers

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\[ 1 + \sum_{n=1}^{\infty} (p_e(n) - p_o(n))x^n = 1 - x - x^2 + x^5 + x^7 + \ldots \]

= \[ x^{P(0)} - x^{P(1)} - x^{P(-1)} + x^{P(2)} + \ldots \]
Proof Part 2: Cancellation of partition numbers

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\[ = \sum_{k=-\infty}^{\infty} (-1)^k x^{\frac{3k^2-k}{2}} \]
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\]

We must show:

1. That \(p_e(n) - p_o(n) = 0\) unless \(n\) is a pentagonal number.
2. If \(n\) is a pentagonal number (\(n = 3k^2 - k\)), then \(p_e(n) - p_o(n) = (-1)^k\).
Proof Part 2: Cancellation of partition numbers

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We must show:

- That \( p_e(n) - p_o(n) = 0 \) unless \( n \) is a pentagonal number.
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Proof Part 2: Cancellation of partition numbers

For any partition of $n$ in standard form, we define:

$s =$ number of dots along slope, and

$b =$ number of dots along base.

\[
\begin{array}{cccccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\]
Proof Part 2: Cancellation of partition numbers

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$n=29$, $b=3$, $s=2$;
Proof Part 2: Cancellation of partition numbers

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We want a bijection between $P_e$ and $P_o.$
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Given a partition of $n$, we either shift the slope down, or we shift the base up. This operation is self-inverse wherever it is defined.
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Proof Part 2: Cancellation of partition numbers

Consider an arbitrary partition of $n$ in standard form. If $b < s$, the operation is defined and self-inverse:
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If \( b < s \), the operation is defined and self-inverse:

If \( b > s + 1 \), the operation is defined and self-inverse:
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Note: This operation changes the parity of the number of parts.
Proof Part 2: Cancellation of partition numbers

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If $b < s$, the operation is defined and self-inverse:

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Note: This operation changes the parity of the number of parts.
Proof Part 2: Cancellation of partition numbers

Example: $n = 8$

The operation is a bijection between $P_e$ and $P_o$. 
Proof Part 2: Cancellation of partition numbers

What if our partition of \( n \) has \( b = s \) or \( b = s + 1 \)?
The problem occurs when the slope and base “intersect”.
Proof Part 2: Cancellation of partition numbers

What if our partition of $n$ has $b = s$ or $b = s + 1$? The problem occurs when the slope and base “intersect”.

Example 1:

$b = s$, no intersection
Proof Part 2: Cancellation of partition numbers

What if our partition of \( n \) has \( b = s \) or \( b = s + 1 \)?
The problem occurs when the slope and base “intersect”.

Example 1:

\[
\begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

\( b = s + 1 \), no intersection
Proof Part 2: Cancellation of partition numbers

What if our partition of $n$ has $b = s$ or $b = s + 1$?
The problem occurs when the slope and base “intersect”.

Example 2:

$b = s$, intersection
Proof Part 2: Cancellation of partition numbers

What if our partition of $n$ has $b = s$ or $b = s + 1$? The problem occurs when the slope and base "intersect".

Example 2:

not in standard form!
Proof Part 2: Cancellation of partition numbers

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Example 2:

not in standard form!
Proof Part 2: Cancellation of partition numbers

What if our partition of \( n \) has \( b = s \) or \( b = s + 1 \)? The problem occurs when the slope and base “intersect”.

Example 3:

\[
\begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet
\end{array}
\]

\( b = s + 1 \), intersection
Proof Part 2: Cancellation of partition numbers

What if our partition of $n$ has $b = s$ or $b = s + 1$? The problem occurs when the slope and base “intersect”.

Example 3:

not a valid partition!
Proof Part 2: Cancellation of partition numbers

What if our partition of $n$ has $b = s$ or $b = s + 1$?
The problem occurs when the slope and base “intersect”.

Example 3:

\[
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{example3}} \\
\end{array}
\]

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Proof Part 2: Cancellation of partition numbers

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Example 3:

\[ \begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array} \]

not in standard form!
Proof Part 2: Cancellation of partition numbers

When does $n$ have a problem partition?
When does $n$ have a problem partition?

Case 1: $b = s$

Note: The “parity” of this partition is the parity of $b$. 

\[ \text{Diagram: } \]
Proof Part 2: Cancellation of partition numbers

When does \( n \) have a problem partition?

Case 1: \( b = s \)

Note: The “parity” of this partition is the parity of \( b \).

\[
\begin{align*}
n &= b^2 + \sum_{i=1}^{b-1} i = \frac{2b^2 + b(b-1)}{2} = \frac{3b^2 - b}{2} = P(b)
\end{align*}
\]

For such \( n \), \( p_e(n) - p_o(n) = (-1)^b \).
Proof Part 2: Cancellation of partition numbers

When does $n$ have a problem partition?
Proof Part 2: Cancellation of partition numbers

When does $n$ have a problem partition?

Case 2: $b = s + 1$

Note: The “parity” of this partition is the parity of $b - 1$. 
Proof Part 2: Cancellation of partition numbers

When does \( n \) have a problem partition?
Case 2: \( b = s + 1 \)

Note: The “parity” of this partition is the parity of \( b - 1 \).

\[
\begin{align*}
    n &= (b - 1)^2 + \sum_{i=1}^{b-1} i \\
    &= \frac{2(b-1)^2 + b(b-1)}{2} \\
    &= \frac{2(b-1)^2 + b^2 - b}{2} \\
    &= \frac{3(b-1)^2 + (b-1)}{2} = P(-(b - 1))
\end{align*}
\]

For such \( n \), \( p_e(n) - p_o(n) = (-1)^{b-1} \).
Proof Part 2: Cancellation of partition numbers

Summary: When \( n \) is a pentagonal number, \( n \) has exactly one problem partition. We can tell whether the problem partition is even or odd by examining \( k \), where \( n = \frac{3k^2 - k}{2} \). Otherwise, \( n \) has no problem partitions, so we have a bijection between \( P_e \) and \( P_o \).
Proof Part 2: Cancellation of partition numbers

**Summary:** When \( n \) is a pentagonal number, \( n \) has exactly one problem partition. We can tell whether the problem partition is even or odd by examining \( k \), where \( n = \frac{3k^2 - k}{2} \). Otherwise, \( n \) has no problem partitions, so we have a bijection between \( P_e \) and \( P_o \).

**Example:** \( n = 7 \)
Pentagonal Number Theorem: Outline of Proof

Lemma 1:

\[ \prod_{m=1}^{\infty} (1 - x^m) = 1 + \sum_{n=1}^{\infty} (p_e(n) - p_o(n))x^n \]

Lemma 2:

\[ 1 + \sum_{n=1}^{\infty} (p_e(n) - p_o(n))x^n = 1 - x - x^2 + x^5 + x^7 + \ldots \]
Pentagonal Number Theorem: Outline of Proof

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Pentagonal Number Theorem: Outline of Proof

Lemma 1: \(\prod_{m=1}^{\infty} (1 - x^m) = 1 + \sum_{n=1}^{\infty} (p_e(n) - p_o(n))x^n\)

Lemma 2: \(1 + \sum_{n=1}^{\infty} (p_e(n) - p_o(n))x^n = 1 - x - x^2 + x^5 + x^7 + ...\)

We may now conclude that indeed,

\(\prod_{m=1}^{\infty} (1 - x^m) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} + ...\)
Pentagonal Number Theorem: Outline of Proof

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