Overlap Number of Graphs

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Slides available on my preprint page
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Definitions

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Main Results

**Thm 1:** We have a linear-time algorithm to determine $\varphi(T)$ for every tree $T$. Corollary: $\varphi(T) \leq |T|$. 

**Thm 2:** If $G$ is a planar $n$-vertex graph and $n \geq 5$, then $\varphi(G) \leq 2n - 5$, which is sharp for $n = 8$ and $n \geq 10$.

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Preliminaries

**Decomposition Bound:** Let $\mathcal{F}$ be a decomposition of graph $G$ into cliques of order at most $k$, where $k \geq 2$. If $\delta(G) \geq k$, then $\Phi(G) \leq |\mathcal{F}|$. In particular, $\delta(G) \geq 2$ implies $\Phi(G) \leq |E(G)|$. 
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**Pf:** Give each clique in $\mathcal{F}$ its own label, and give each vertex all the labels of cliques that contain it.

**Prop:** If $G$ is triangle-free, then $\Phi(G) \geq |E(G)|$, and $\Phi(G) = |E(G)|$ when $\delta(G) \geq 2$.

**Pf:** We can’t do better than one label on each edge.

**Deletion Bound:** If $v$ is a vertex with $d(v) \leq 2$ in a graph $G$ with at least 3 vertices, then $\Phi(G) \leq \Phi(G - v) + 2$. If $d(v) \leq 1$, then $\phi(G) \leq \phi(G - v) + 2$.

**Pf:** Easy for $\Phi$, and not too hard for $\phi$. 
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**Lemma 1:** If $G$ is planar with $n \geq 5$ vertices, then $G$ decomposes into at most $2n - 5$ edges and facial triangles unless every face is a 4-cycle (then $G$ consists of $2n - 4$ edges).
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If $t = 0$, then Euler’s formula implies the claim. So suppose $t \geq 1$. 

- **Case 1:** $G'$ has a facial (non-4)-cycle. Now $|\mathcal{F}'| \leq 2(n + 1) - 5 = 2n - 3$, so $|\mathcal{F}| \leq (2n - 3) - 3 + 1 = 2n - 5$.
- **Case 2:** $G'$ has only facial 4-cycles. Now $|\mathcal{F}'| = 2(n + 1) - 4 = 2n - 2$, so $|\mathcal{F}| = (2n - 2) - 3 + 3 = 2n - 4$.
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\[
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Planar Graphs

**Lemma 1:** If $G$ is planar with $n \geq 5$ vertices, then $G$ decomposes into at most $2n - 5$ edges and facial triangles unless every face is a 4-cycle (then $G$ consists of $2n - 4$ edges).

**Pf:** Let $\mathcal{F}$ denote our decomposition of $G$ into edges and triangles. We induct on $t$, the number of facial triangles in $G$.

If $t = 0$, then Euler’s formula implies the claim. So suppose $t \geq 1$.

**Case 1:** $G'$ has a facial (non-4)-cycle. Now $|\mathcal{F}'| \leq 2(n + 1) - 5 = 2n - 3$,
so $|\mathcal{F}| \leq (2n - 3) - 3 + 1 = 2n - 5$.

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**What's missing?** Lot's of messy base cases.
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Bipartite Graphs

Lemma: Let $G$ be an $n$-vertex bipartite graph. If $n \geq 7$ and $\delta(G) \geq 2$, then $\varphi(G) \leq n^2/4 - n/2 - 1$. 

Pf: Since $\varphi(G) \leq |E(G)| - 1$, we have $|E(G)| > n^2/4 - n/2$. Let $X$ and $Y$ be the parts, with $k = |X| \leq |Y|$. If $G$ has a clone, we can delete it. So at most one vertex of $Y$ has degree $k$. Thus $|E(G)| \leq (k - 1)(n - k) + 1$, and $|X| = \lfloor n/2 \rfloor$ and $|Y| = \lceil n/2 \rceil$, and some $y \in Y$ has degree $k$ and all others have degree $k - 1$. Delete $y$ to form $G'$. Now $\Phi(G') \leq |E(G')| = \lfloor n^2/4 \rfloor - n/2 + 1 - \lfloor n/2 \rfloor = \lfloor n^2/4 - n/2 \rfloor + 1$. Let $f$ be a pure overlap labeling of $G'$ using one label per edge. Let $y' \in Y$ be a vertex of $G'$ and let $x'$ be its non-neighbor in $X$. Extend $f$ to $G$ as follows: let $f(y) = f(y') \cup a$ (where $a$ is a new label) and add $a$ to $f(x')$. So $\varphi(G) \leq \Phi(G') + 1 \leq \lfloor n^2/4 - n/2 \rfloor + 1$. 


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Delete $y$ to form $G'$. Now $\Phi(G') \leq |E(G')| = \lceil n^2/4 - n/2 + 1 \rceil - \lceil n/2 \rceil = \lceil n^2/4 - n + 1 \rceil$. 
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General $n$-vertex graphs

**Theorem:** If $G$ is an $n$-vertex graph, then $\varphi(G) \leq n^2/4 - n/2 - 1$. 

**Lemma:** If $G$ has a triangle $T$, then $\Phi(G) \leq \Phi(G-T) + n$.

**Lemma:** If $n \geq 7$, then $\Phi(G) \leq \left\lfloor n^2/4 \right\rfloor$.

**Pf sketch of theorem:**

▶ $G$ is bipartite
▶ $G$ is triangle-free, but not bipartite

Consider shortest odd cycle $C$, with length $2k + 1$

$|E(G)| \leq (2k + 1) + k(n - (2k + 1)) + (n - (2k + 1)) \leq 2n/4$

Edge bound is good enough unless $k = 2$, . . .

▶ $G$ has a triangle $T$
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▶ $G-T$ has a triangle $T'$

Now $\Phi(G-T-T') \leq \left\lfloor (n-6)^2/4 \right\rfloor$, so $\Phi(G) \leq \left\lfloor (n-6)^2/4 \right\rfloor + 2n - 3 \leq n^2/4 - n/2 - 1$.
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- \( G \) has a triangle \( T \)
  
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- $G$ is triangle-free, but not bipartite
  - Consider shortest odd cycle $C$, with length $2k + 1$
  - $|E(G)| \leq (2k + 1) + k(n - (2k + 1)) + (n - (2k + 1))^2 / 4$
  - Edge bound is good enough unless $k = 2, \ldots$
- $G$ has a triangle $T$
  - $G - T$ is bipartite
  - $G - T$ is triangle-free, but not bipartite
  - $G - T$ has a triangle $T'$
  - Now $\Phi(G - T - T') \leq \lfloor (n - 6)^2 / 4 \rfloor$, so $\Phi(G) \leq \lfloor (n - 6)^2 / 4 \rfloor + 2n - 3 \leq n^2 / 4 - n/2 - 1$