Using the Potential Method to Color Near-bipartite Graphs

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WATERColor
24 September 2019
Introduction
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Prop: Theorems are equivalent.

Pf: "Fold away" all 4-faces.

Dream: Maybe we don't need planarity. Could sparsity be enough?

3-coloring $G$ also 3-colors each subgraph $H$, so also need $H$ sparse.

Prop: If $G$ is planar with no 3-cycle and no 4-cycle, then $\text{mad}(G) < \frac{10}{3}$, where $\text{mad}(G) := \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|}$.

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![Diagram of a graph with vertices v, w, x, y connected in a square configuration.](image-url)
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[Diagrams of planar graphs with no 3-cycles and no 4-faces]
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\[
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---

```
   v
 /  |
 y---w
 /  |
x
```

```
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![Graphs](image)

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Potential: a finer measure of edge density

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![Diagram of potential and necklace graphs]
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\[\begin{array}{ccc}
& & \\
& \text{circles} & \\
& \text{octagon} & \\
& \text{9-vertex} & \\
\end{array}\]

**Thm [Kostochka–Yancey ’12]:** If \( \text{pot}(G) \geq 3 \), then \( \chi(G) \leq 3 \).

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Contradiction.
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\text{Pf sketch:} & \quad \text{Note: } \text{pot}(G) > 0 \iff \text{mad}(G) < 10/3. \ G \text{ is min c/e, so } \delta(G) \geq 3. \ \text{WTS: Each 3-vertex has two } 4^+\text{-nbrs. Each vertex } v \text{ starts with } d(v) \text{ and each } 4^+\text{-vertex gives } 1/6 \text{ to each 3-nbrs.} \\
& \quad 3: \ 3 + 2(1/6) = 10/3. \quad 4^+: \ d(v) - d(v)/6 = 5d(v)/6 \geq 20/6. \ \text{Contradiction.} \\
\text{Problem:} & \quad \text{Need more power for reducibility.}
\end{align*}
\]
Using the Gap Lemma

**Gap Lemma:**
If $W \subseteq V(G)$ and $|W| \geq 2$, then $\rho(W) \geq 6$.

**Cor:**
For any $W \subseteq V(G)$ and $e/ \in G[W]$, $\chi(G[W] + e) \leq 3$.

**Pf:**
Let $G' = G[W] + e$. WTS $\rho(G') \geq 3$.

Fix $X \subseteq V(G')$.

If $|X| = 1$, then $\rho_{G'}(X) = \rho_G(X) = 5$.

If $|X| \geq 2$, then $\rho_{G'}(X) \geq \rho_G(X) - 3 \geq 6 - 3 = 3$.

**Cor:**
$G$ has no triangle with 2 or more 3-vertices.
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$G$
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![Diagram of graphs G and W]
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\[ \begin{figure}
\centering
\begin{tikzpicture}
  \node (G) at (0,0) {$G$};
  \node (W) at (-1,0) {$W$};
  \draw[->] (G) to (W);
\end{tikzpicture}
\end{figure} \]
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![Diagram of a graph G with a subset W and a possible triangle]
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**Cor:** For any \( W \subseteq V(G) \) and \( e \notin G[W] \), \( \chi(G[W] + e) \leq 3 \).

**Pf:** Let \( G' = G[W] + e \). WTS \( \text{pot}(G') \geq 3 \). Fix \( X \subseteq V(G') \).

If \( |X| = 1 \), then \( \rho_{G'}(X) = \rho_G(X) = 5 \). If \( |X| \geq 2 \), then \( \rho_{G'}(X) \geq \rho_G(X) - 3 \geq 6 - 3 = 3 \).

**Cor:** \( G \) has no triangle with 2 or more 3-vertices.
Proving the Gap Lemma

**Recall:** \( \rho(W) = 5|W| - 3|E(G[W])| \).
Proving the Gap Lemma

Recall: $\rho(W) = 5|W| - 3|E(G[W])|$. Obs: If $X, Y \subseteq V(G)$ and $X \cap Y \neq \emptyset$, then $\rho(X \cup Y) = \rho(X) + \rho(Y) - 3|E(X, Y)|$. 
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![Graph with sets G and R](image)

Easy to check when $|R| \leq 3$; assume $|R| \geq 4$. 
Proving the Gap Lemma

Recall: $\rho(W) = 5|W| - 3|E(G[W])|$. Obs: If $X, Y \subseteq V(G)$ and $X \cap Y \neq \emptyset$, then $\rho(X \cup Y) = \rho(X) + \rho(Y) - 3|E(X, Y)|$.

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Easy to check when $|R| \leq 3$; assume $|R| \geq 4$. 3-color $G[R]$; call it $\varphi$. 
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Recall: \( \rho(W) = 5|W| - 3|E(G[W])| \). Obs: If \( X, Y \subseteq V(G) \) and \( X \cap Y \neq \emptyset \), then \( \rho(X \cup Y) = \rho(X) + \rho(Y) - 3|E(X, Y)| \).

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Easy to check when \( |R| \leq 3 \); assume \( |R| \geq 4 \). 3-color \( G[R] \); call it \( \varphi \). Contract each color class to a single vertex to get \( G' \).
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![Diagram of gap lemma proof](image)

Easy to check when \( |R| \leq 3 \); assume \( |R| \geq 4 \). 3-color \( G[R] \); call it \( \varphi \). Contract each color class to a single vertex to get \( G' \). If \( \chi(G') \leq 3 \), then \( \chi(G) \leq 3 \). Some \( S \subseteq V(G') \) has \( \rho_{G'}(S) \leq 2 \).
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**Pf:** Choose \( R \varsubsetneq V(G) \) with \( |R| \geq 2 \) to minimize \( \rho(R) \).

\[
G \quad R \quad \varnothing \quad \rightarrow \quad G' \quad Z \quad S
\]

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If \( S \cap Z = \emptyset \), then \( 2 \geq \rho_{G'}(S) = \rho_G(S) \), a contradiction.
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\[
\begin{array}{c}
G \\
S \setminus Z \\
\varphi \\
R \\
\end{array}
\quad \leftrightarrow 
\quad
\begin{array}{c}
G' \\
S \\
Z \\
\end{array}
\]

Easy to check when $|R| \leq 3$; assume $|R| \geq 4$. 3-color $G[R]$; call it $\varphi$. Contract each color class to a single vertex to get $G'$. If $\chi(G') \leq 3$, then $\chi(G) \leq 3$. Some $S \subseteq V(G')$ has $\rho_{G'}(S) \leq 2$. If $S \cap Z = \emptyset$, then $2 \geq \rho_{G'}(S) = \rho_G(S)$, a contradiction. Instead

\[
\rho_G((S \setminus Z) \cup R) \leq \rho_{G'}(S) - \rho_{G'}(S \cap Z) + \rho_G(R)
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\leq 2 - 5 + 5 = 2,
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**Pf:** Choose \( R \not\subseteq V(G) \) with \( |R| \geq 2 \) to minimize \( \rho(R) \).

\[
\begin{align*}
G & \quad \quad \quad R \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad G' \\
S \setminus Z & \quad \quad \quad \varnothing \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad S \setminus Z \\
\end{align*}
\]

Easy to check when \( |R| \leq 3 \); assume \( |R| \geq 4 \). 3-color \( G[R] \); call it \( \varphi \). Contract each color class to a single vertex to get \( G' \).

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\[
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\]

Contradiction!
Near-bipartite Graphs

**Defn:** $G$ is near-bipartite (nb) if $V(G)$ has a partition $(I, F)$ with $I$ an independent set and $G[F]$ a forest.
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---

```
\begin{center}
\begin{tikzpicture}
  \begin{scope}
    \draw (0,0) -- (1,1) -- (0,2) -- (-1,1) -- cycle;
  \end{scope}
  \begin{scope}[xshift=2cm]
    \draw (0,0) -- (1,1) -- (0,2) -- (-1,1) -- cycle;
  \end{scope}
\end{tikzpicture}
\end{center}
```
Main Results: Near-bipartite Graphs

Defn: \( G \) is near-bipartite (nb) if \( V(G) \) has a partition \((I,F)\) with \( I \) an ind. set and \( G[F] \) a forest.

For multigraph \( G \) and \( W \subseteq V(G) \),
\[
\rho_m(W) := 3|W| - 2|E(G[W])|
\]
and
\[
pot_m(G) := \min_{W \subseteq V(G)} \rho(W)
\]

Thm: If \( G \) is a multigraph with \( pot_m(G) \geq -1 \) and \( G \) has no \( K_4 \) or Moser spindle, then \( G \) is nb. This is sharp infinitely often.

Defn: For a simple graph \( G \) and each \( W \subseteq V(G) \),
\[
\rho_s(W) := 8|W| - 5|E(G[W])|
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Thm: If \( G \) is a simple graph with \( pot_s(G) \geq -4 \) and \( G \) has no subgraph in a finite \( H \) then \( G \) is nb. This is sharp infinitely often.
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Complications

**Ques:** What is harder for us than in proof for 3-coloring?

- Colors $I$ and $F$ are “different”.

To prove gap lemma, color subgraph and contract. Specify which vertex is colored $I$ and which is colored $F$. Prove general result allowing precoloring.

To contract a subset $W$ with low potential, must ensure new graph $G'$ has no forbidden $H \in H$. Must really understand $H$.

Maybe $\text{mad}(G) > 16/5$, so discharging to get $16/5$ everywhere gives no contradiction. Show $G$ almost consists of independent set of 4-vertices and 3-vertices inducing a forest. Color $G$. 

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![Diagram showing the process of color subgraph and contract](image)
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$$
\begin{array}{c}
\text{W} & \text{I}_W & \text{F}_W \\
\end{array}
\rightarrow
\begin{array}{c}
\text{W} & \text{w}_i & \text{w}_f \\
\end{array}
$$

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$$\begin{array}{c}
W & l_w & F_w \\
\rightarrow & & \\
W & w_i & w_f
\end{array}$$

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\[ \begin{array}{c}
\text{To contract a subset } W \\
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![Diagram](image)

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![Diagram showing color subgraph and contract]

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![Diagram showing 4-vertices and 3-vertices]

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![Diagram showing contracted graph](image)

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Algorithms

Discharging Proof into Algorithm (Typical)

- Find reducible configuration $H$
- Recursively color $G - H$
- Extend coloring to $H$
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Our Proof into Algorithm

- Handle “easy” reducible configurations as above
Algorithms

Discharging Proof into Algorithm (Typical)

- Find reducible configuration $H$
- Recursively color $G - H$
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- May have many plausible reductions; need one with no $H \in \mathcal{H}$
  Finding right one takes time $O(n^{21})$; color recursively, extend
Thm [Goldberg '84]: Given arbitrary vertex and edge weights, we can find a set of minimum potential in polynomial time.
Summary

▶ Prove Grötzsch’s Theorem by edge density?
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- Fold 4-faces; need $\text{mad}(G) < 10/3 \implies \chi(G) \leq 3$. 

Necklaces are infinitely many counterexamples.

Better measure:

$$\rho(W) = 5 |W| - 3 |E(G[W])|.$$ 

$$\text{pot}(G) = \min \rho(W); \text{mad}(G) < 10/3 \text{ iff } \text{pot}(G) > 0.$$ 

For all necklaces, $\text{pot}(G) = 2$.

Thm [KY]: If $\text{pot}(G) \geq 3$, then $\chi(G) \leq 3$.

Pf: reducibility/discharging, gap lemma.

$2$-colorable $\subset$ near-bipartite (nb) $\subset$ $3$-colorable

$$\rho_s(W) = 8 |W| - 5 |E(G[W])|$$ and $\text{pot}_s(G) = \min \rho_s(W)$

If $\text{pot}_s(G) \geq -4$ and $G$ has no subgraph in $H$, then $G$ is nb.

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