Linear (List) Coloring of Sparse Graphs

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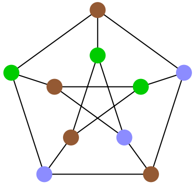

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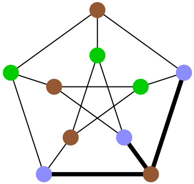

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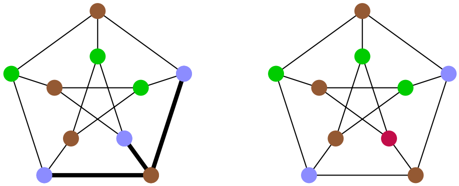

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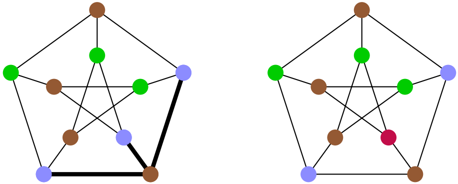

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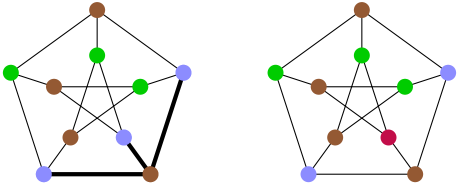

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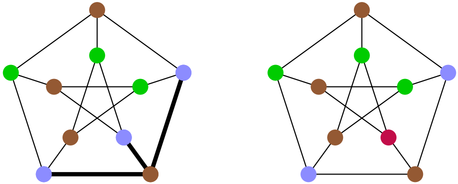

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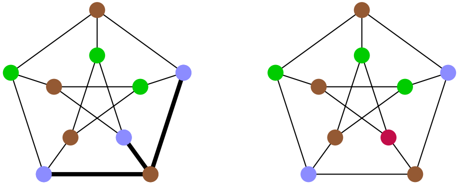

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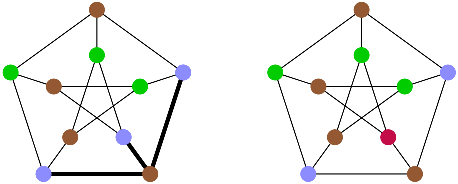

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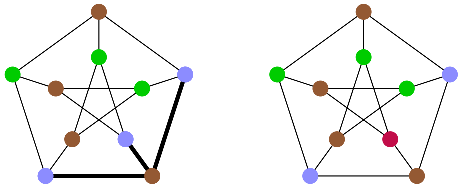

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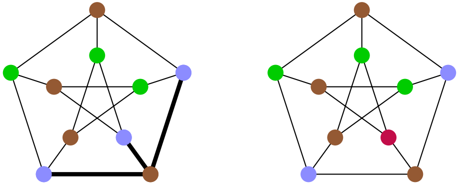

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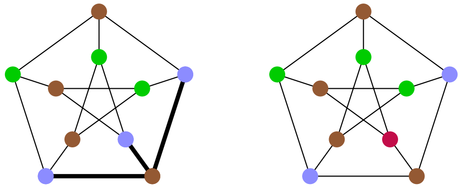

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Sparse Graphs

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2. For every planar G, we have $\mathrm{mad}(G) < 6$.
3. For every planar G, we have $\mathrm{mad}(G) < \frac{2g}{g-2}$.

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Main Theorem:

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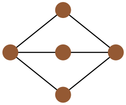

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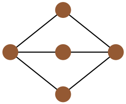
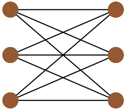


2. In (b), the bound 3 cannot be raised in general, since $\mathrm{mad}(K_{3,3}) = 3$ and $\mathrm{lc}(K_{3,3}) = 5 > \lceil \Delta(K_{3,3})/2 \rceil + 2$.



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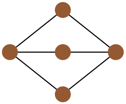

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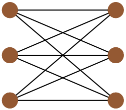

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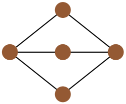

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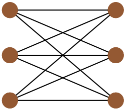

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   Part (b) proves their conjecture when $\Delta(G) \geq 9$.

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Lemma: If G is planar with $\delta(G) \geq 2$ and $g \geq 5$,
then G contains one of the following two configurations:

(C1) a 2-vertex adjacent to a 5^--vertex,

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Pf: Each v or f gets $\mu(v) = 3d(v)/2 - 5$ or $\mu(f) = d(f) - 5$.

$$\begin{aligned}\sum_{x \in V \cup F} \mu(x) &= \sum_{v \in V}(3d(v)/2 - 5) + \sum_{f \in F}(d(f) - 5) \\ &= (3|E| - 5|V|) + (2|E| - 5|F|) \\ &= -5(|F| - |E| + |V|) = -10.\end{aligned}$$

Assume G is a counterexample. If we can redistribute charge to get $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ everywhere, then

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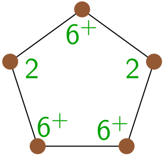

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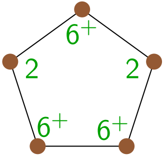

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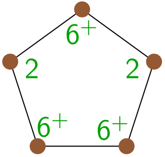

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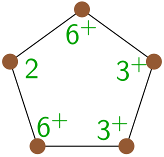

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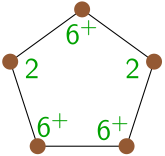

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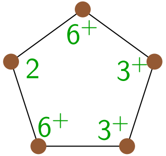

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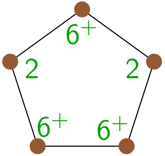

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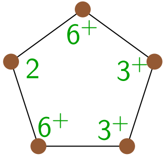

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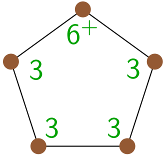

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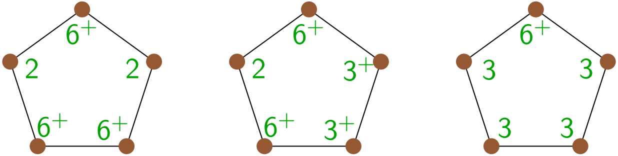

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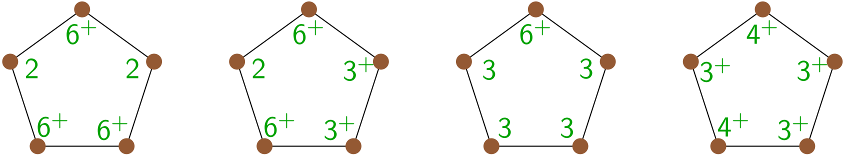

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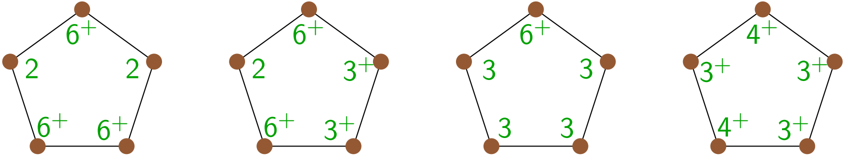

$-2 + 3(2/3)$ $\quad -1 - 2/6 + 2(2/3)$ $\quad -4/6 + 2/3$ $\quad -3/6 + 2(1/4)$

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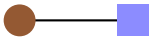
(C0) 

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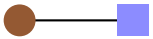
(C0)  $\lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1)/2 \rfloor - 1$

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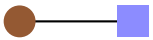
(C0)  $\lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1)/2 \rfloor - 1 \geq 4$

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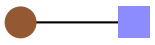
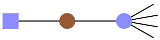
(C0)  $\lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1)/2 \rfloor - 1 \geq 4$

(C1) 

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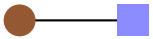
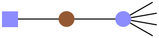
(C0)  $\lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1)/2 \rfloor - 1 \geq 4$

(C1)  $\lceil \Delta/2 \rceil + 4 - \lfloor ((\Delta - 1) + (5 - 1))/2 \rfloor - 2$

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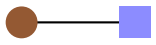
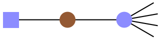
(C0)  $\lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1)/2 \rfloor - 1 \geq 4$

(C1)  $\lceil \Delta/2 \rceil + 4 - \lfloor ((\Delta - 1) + (5 - 1))/2 \rfloor - 2 \geq 1$

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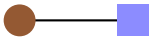
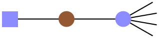
(C0)  $\lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1)/2 \rfloor - 1 \geq 4$

(C1)  $\lceil \Delta/2 \rceil + 4 - \lfloor ((\Delta - 1) + (5 - 1))/2 \rfloor - 2 \geq 1$

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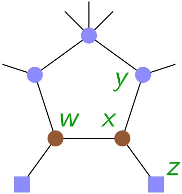

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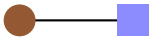
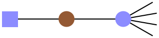
(C0)  $\lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1)/2 \rfloor - 1 \geq 4$

(C1)  $\lceil \Delta/2 \rceil + 4 - \lfloor ((\Delta - 1) + (5 - 1))/2 \rfloor - 2 \geq 1$

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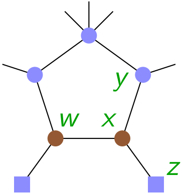

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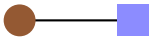
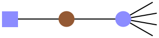
(C0)  $\lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1)/2 \rfloor - 1 \geq 4$

(C1)  $\lceil \Delta/2 \rceil + 4 - \lfloor ((\Delta - 1) + (5 - 1))/2 \rfloor - 2 \geq 1$

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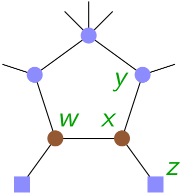

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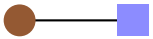
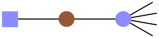
(C0)  $\lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1)/2 \rfloor - 1 \geq 4$

(C1)  $\lceil \Delta/2 \rceil + 4 - \lfloor ((\Delta - 1) + (5 - 1))/2 \rfloor - 2 \geq 1$

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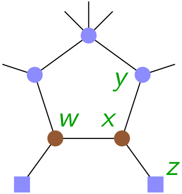

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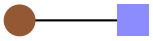
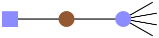
(C0)  $\lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1)/2 \rfloor - 1 \geq 4$

(C1)  $\lceil \Delta/2 \rceil + 4 - \lfloor ((\Delta - 1) + (5 - 1))/2 \rfloor - 2 \geq 1$

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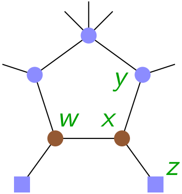

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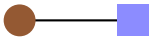
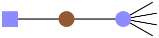
(C0)  $\lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1)/2 \rfloor - 1 \geq 4$

(C1)  $\lceil \Delta/2 \rceil + 4 - \lfloor ((\Delta - 1) + (5-1))/2 \rfloor - 2 \geq 1$

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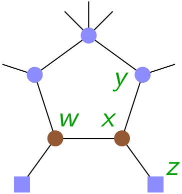

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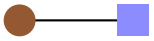
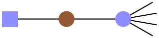
(C0)  $\lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1)/2 \rfloor - 1 \geq 4$

(C1)  $\lceil \Delta/2 \rceil + 4 - \lfloor ((\Delta - 1) + (5 - 1))/2 \rfloor - 2 \geq 1$

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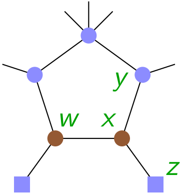

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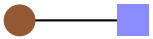
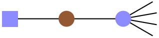
(C0)  $\lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1)/2 \rfloor - 1 \geq 4$

(C1)  $\lceil \Delta/2 \rceil + 4 - \lfloor ((\Delta - 1) + (5 - 1))/2 \rfloor - 2 \geq 1$

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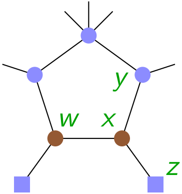

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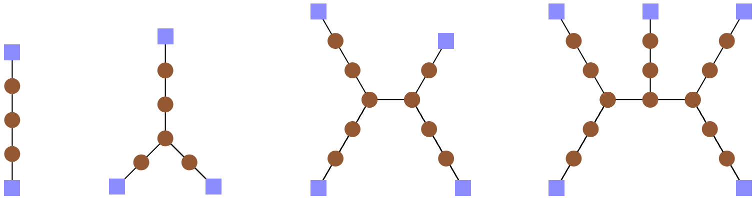

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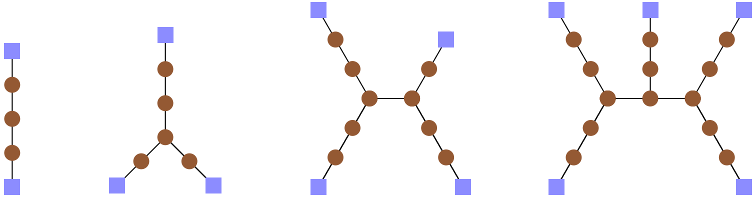

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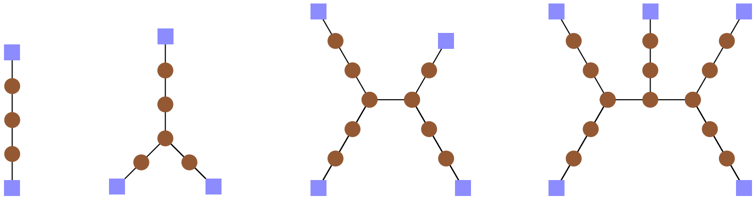
Pf Sketch: Use discharging, with $\mu(v) = d(v) - 12/5$ to show:

Lemma: If $\mathrm{mad}(G) < 12/5$, $\delta(G) = 2$, and $\Delta(G) \geq 3$, then G contains one of the following 4 configurations.



And each of these configurations is forbidden in a minimal counter.

So the theorem is true!