Linear (List) Coloring of Sparse Graphs

Daniel W. Cranston
Virginia Commonwealth University
dcranston@vcu.edu

Wake Forest AMS Meeting
September 24, 2011
Def: A linear coloring is a proper coloring where each pair of colors induces a linear forest (disjoint paths). The linear chromatic number, \( lc(G) \), is least \( k \) that allows a linear coloring with \( k \) colors.
**Def:** A linear coloring is a proper coloring where each pair of colors induces a linear forest (disjoint paths). The linear chromatic number, \( lc(G) \), is least \( k \) that allows a linear coloring with \( k \) colors.
Introduction

**Def:** A linear coloring is a proper coloring where each pair of colors induces a linear forest (disjoint paths). The linear chromatic number, \( \text{lc}(G) \), is least \( k \) that allows a linear coloring with \( k \) colors.
**Introduction**

**Def:** A **linear coloring** is a proper coloring where each pair of colors induces a linear forest (disjoint paths). The **linear chromatic number**, \( \text{lc}(G) \), is least \( k \) that allows a linear coloring with \( k \) colors.

**Obs:** For all \( G \), \( \lceil \Delta(G)^2 \rceil + 1 \leq \text{lc}(G) \leq \Delta(G)^2 + 1 \).

**Thm:** \([\text{Yuster '98}]\) \( \exists C_1 \) s.t. \( \forall G \) we have \( \text{lc}(G) \leq C_1 \left( \Delta(G) \right)^{3/2} \) and \( \exists C_2 \) s.t. \( \forall k \exists G \) s.t. \( \Delta(G) = k \) and \( \text{lc}(G) \geq C_2 \left( \Delta(G) \right)^{3/2} \).

**Big Question 1:** For which graphs does \( \text{lc}(G) = \lceil \Delta(G)^2 \rceil + 1 \)?

**Answer:** trees, but not cycles
Introduction

**Def:** A **linear coloring** is a proper coloring where each pair of colors induces a linear forest (disjoint paths). The **linear chromatic number**, $\text{lc}(G)$, is least $k$ that allows a linear coloring with $k$ colors.

**Obs:** For all $G$, $\left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \leq \text{lc}(G)$
**Def:** A linear coloring is a proper coloring where each pair of colors induces a linear forest (disjoint paths). The linear chromatic number, $\text{lc}(G)$, is least $k$ that allows a linear coloring with $k$ colors.

**Obs:** For all $G$, $\left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \leq \text{lc}(G) \leq \Delta(G^2) + 1$

---

**Big Question 1:** For which graphs does $\text{lc}(G) = \left\lceil \Delta(G) \right\rceil + 1$?

**Answer:** trees, but not cycles
**Def:** A **linear coloring** is a proper coloring where each pair of colors induces a linear forest (disjoint paths). The **linear chromatic number**, $\text{lc}(G)$, is least $k$ that allows a linear coloring with $k$ colors.

**Obs:** For all $G$, 
\[
\left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \leq \text{lc}(G) \leq \Delta(G^2) + 1 \leq (\Delta(G))^2 + 1.
\]
Def: A linear coloring is a proper coloring where each pair of colors induces a linear forest (disjoint paths). The linear chromatic number, \( \text{lc}(G) \), is least \( k \) that allows a linear coloring with \( k \) colors.

Obs: For all \( G \), \( \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \leq \text{lc}(G) \leq \Delta(G^2) + 1 \leq (\Delta(G))^2 + 1 \).

Thm:[Yuster '98] \exists C_1 \text{ s.t. } \forall G \text{ we have } \text{lc}(G) \leq C_1(\Delta(G))^{3/2}
**Introduction**

**Def:** A linear coloring is a proper coloring where each pair of colors induces a linear forest (disjoint paths). The linear chromatic number, $\text{lc}(G)$, is least $k$ that allows a linear coloring with $k$ colors.

**Obs:** For all $G$, $\left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \leq \text{lc}(G) \leq \Delta(G^2) + 1 \leq (\Delta(G))^2 + 1$.

**Thm:** [Yuster '98] $\exists C_1$ s.t. $\forall G$ we have $\text{lc}(G) \leq C_1(\Delta(G))^{3/2}$ and $\exists C_2$ s.t. $\forall k \exists G$ s.t. $\Delta(G) = k$ and $\text{lc}(G) \geq C_2(\Delta(G))^{3/2}$.

Big Question 1: For which graphs does $\text{lc}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$?

Answer: trees, but not cycles
**Introduction**

**Def:** A linear coloring is a proper coloring where each pair of colors induces a linear forest (disjoint paths). The linear chromatic number, \( lc(G) \), is least \( k \) that allows a linear coloring with \( k \) colors.

\[
\text{Obs: } \forall G \text{, } \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \leq lc(G) \leq \Delta(G^2) + 1 \leq (\Delta(G))^2 + 1.
\]

**Thm:** [Yuster ’98] \( \exists C_1 \text{ s.t. } \forall G \text{ we have } lc(G) \leq C_1(\Delta(G))^{3/2} \)
and \( \exists C_2 \text{ s.t. } \forall k \exists G \text{ s.t. } \Delta(G) = k \text{ and } lc(G) \geq C_2(\Delta(G))^{3/2} \).

**Big Question 1:** For which graphs does \( lc_\ell(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \)?
Introduction

**Def:** A linear coloring is a proper coloring where each pair of colors induces a linear forest (disjoint paths). The linear chromatic number, $\text{lc}(G)$, is least $k$ that allows a linear coloring with $k$ colors.

**Obs:** For all $G$, $\left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \leq \text{lc}(G) \leq \Delta(G^2) + 1 \leq (\Delta(G))^2 + 1$.

**Thm:** [Yuster ’98] $\exists C_1$ s.t. $\forall G$ we have $\text{lc}(G) \leq C_1(\Delta(G))^{3/2}$ and $\exists C_2$ s.t. $\forall k \exists G$ s.t. $\Delta(G) = k$ and $\text{lc}(G) \geq C_2(\Delta(G))^{3/2}$.

**Big Question 1:** For which graphs does $\text{lc}_\ell(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$?

**Answer:** trees
**Def:** A linear coloring is a proper coloring where each pair of colors induces a linear forest (disjoint paths). The linear chromatic number, \( lc(G) \), is least \( k \) that allows a linear coloring with \( k \) colors.

**Obs:** For all \( G \), \( \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \leq lc(G) \leq \Delta(G^2) + 1 \leq (\Delta(G))^2 + 1 \).

**Thm:** [Yuster '98] \( \exists C_1 \) s.t. \( \forall G \) we have \( lc(G) \leq C_1(\Delta(G))^{3/2} \) and \( \exists C_2 \) s.t. \( \forall k \exists G \) s.t. \( \Delta(G) = k \) and \( lc(G) \geq C_2(\Delta(G))^{3/2} \).

**Big Question 1:** For which graphs does \( lc_\ell(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \)?

**Answer:** trees, but not cycles
Sparse Graphs

**Big Question 2:** For which classes $\mathcal{G}$ of graphs does there exist $C$ such that for each $G \in \mathcal{G}$, we have

$$\ell(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + C?$$

*Answer:* Perhaps, graphs that are “close” to being trees.

**Def:** The maximum average degree, $\text{mad}(G)$, is defined as

$$\text{mad}(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|}.$$

A graph class $\mathcal{G}$ is sparse if $\exists k$ s.t. $\text{mad}(G) < k$ for all $G \in \mathcal{G}$.

1. For every tree $G$, we have $\text{mad}(G) < 2$.
2. For every planar $G$, we have $\text{mad}(G) < 6$.
3. For every planar $G$, we have $\text{mad}(G) < 2g - 2$. 


Sparse Graphs

**Big Question 2:** For which classes $\mathcal{G}$ of graphs does there exist $C$ such that for each $G \in \mathcal{G}$, we have

$$\ell_c(\ell(G)) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + C?$$

**Answer:** Perhaps, graphs that are “close” to being trees.
Sparse Graphs

**Big Question 2:** For which classes $\mathcal{G}$ of graphs does there exist $C$ such that for each $G \in \mathcal{G}$, we have

$$\text{lcl}(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + C?$$

**Answer:** Perhaps, graphs that are “close” to being trees.

**Def:** The maximum average degree, $\text{mad}(G)$, is defined as

$$\text{mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}.$$ 

A graph class $\mathcal{G}$ is *sparse* if $\exists k$ s.t. $\text{mad}(G) < k$ for all $G \in \mathcal{G}$.
Sparse Graphs

**Big Question 2**: For which classes $\mathcal{G}$ of graphs does there exist $C$ such that for each $G \in \mathcal{G}$, we have

$$1c_\ell(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + C?$$

**Answer**: Perhaps, graphs that are “close” to being trees.

**Def**: The maximum average degree, $\text{mad}(G)$, is defined as

$$\text{mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}.$$ 

A graph class $\mathcal{G}$ is **sparse** if $\exists k$ s.t. $\text{mad}(G) < k$ for all $G \in \mathcal{G}$.

1. For every tree $G$, we have $\text{mad}(G) < 2$. 
Sparse Graphs

**Big Question 2:** For which classes $\mathcal{G}$ of graphs does there exist $C$ such that for each $G \in \mathcal{G}$, we have

$$\text{lcl}(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + C?$$

**Answer:** Perhaps, graphs that are “close” to being trees.

**Def:** The maximum average degree, $\text{mad}(G)$, is defined as

$$\text{mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}.$$

A graph class $\mathcal{G}$ is **sparse** if $\exists k$ s.t. $\text{mad}(G) < k$ for all $G \in \mathcal{G}$.

1. For every tree $G$, we have $\text{mad}(G) < 2$.
2. For every planar $G$, we have $\text{mad}(G) < 6$. 
Sparse Graphs

**Big Question 2:** For which classes $\mathcal{G}$ of graphs does there exist $C$ such that for each $G \in \mathcal{G}$, we have

$$\ell c_\ell(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + C?$$

**Answer:** Perhaps, graphs that are “close” to being trees.

**Def:** The maximum average degree, $\text{mad}(G)$, is defined as

$$\text{mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}.$$ 

A graph class $\mathcal{G}$ is **sparse** if $\exists k$ s.t. $\text{mad}(G) < k$ for all $G \in \mathcal{G}$.

1. For every tree $G$, we have $\text{mad}(G) < 2$.
2. For every planar $G$, we have $\text{mad}(G) < 6$.
3. For every planar $G$, we have $\text{mad}(G) < \frac{2g}{g-2}$. 
Results

Main Theorem:

(a) If $G$ is planar and $g \geq 5$, then $\ell_c(G) \leq \lceil \Delta/2 \rceil + 4$.
Results

Main Theorem:

(a) If $G$ is planar and $g \geq 5$, then $\ell_G(G) \leq \lceil \Delta/2 \rceil + 4$.

(b) If $\text{mad}(G) < 3$ and $\Delta \geq 9$, then $\ell_G(G) \leq \lceil \Delta/2 \rceil + 2$.

Remarks

1. In (c), the bound $12/5$ is optimal, since $\text{mad}(K_2,3) = 12/5$ and $\ell_G(K_2,3) = 4 > \lceil \Delta(K_2,3)/2 \rceil + 1$.

2. In (b), the bound 3 cannot be raised in general, since $\text{mad}(K_3,3) = 3$ and $\ell_G(K_3,3) = 5 > \lceil \Delta(K_3,3)/2 \rceil + 2$.

3. Raspaud and Wang conjectured that $\ell_G(G) \leq \lceil \Delta(G)/2 \rceil + 2$ for all planar $G$ with $g \geq 6$. Part (b) proves their conjecture when $\Delta(G) \geq 9$. 
Results

Main Theorem:

(a) If $G$ is planar and $g \geq 5$, then $l_{c_{\ell}}(G) \leq \lceil \Delta/2 \rceil + 4$.
(b) If $\text{mad}(G) < 3$ and $\Delta \geq 9$, then $l_{c_{\ell}}(G) \leq \lceil \Delta/2 \rceil + 2$.
(c) If $\text{mad}(G) < 12/5$ and $\Delta \geq 3$, then $l_{c_{\ell}}(G) = \lceil \Delta/2 \rceil + 1$.
Results

Main Theorem:

(a) If $G$ is planar and $g \geq 5$, then $\ell_c(G) \leq \lceil\Delta/2\rceil + 4$.
(b) If $\text{mad}(G) < 3$ and $\Delta \geq 9$, then $\ell_c(G) \leq \lceil\Delta/2\rceil + 2$.
(c) If $\text{mad}(G) < 12/5$ and $\Delta \geq 3$, then $\ell_c(G) = \lceil\Delta/2\rceil + 1$.

Remarks

1. In (c), the bound $12/5$ is optimal, since $\text{mad}(K_{2,3}) = 12/5$ and $\ell_c(K_{2,3}) = 4 > \lceil\Delta(K_{2,3})/2\rceil + 1$. 
Results

Main Theorem:

(a) If $G$ is planar and $g \geq 5$, then $\ellc(G) \leq \lceil \Delta/2 \rceil + 4$.
(b) If $\text{mad}(G) < 3$ and $\Delta \geq 9$, then $\ellc(G) \leq \lceil \Delta/2 \rceil + 2$.
(c) If $\text{mad}(G) < 12/5$ and $\Delta \geq 3$, then $\ellc(G) = \lceil \Delta/2 \rceil + 1$.

Remarks

1. In (c), the bound $12/5$ is optimal, since $\text{mad}(K_{2,3}) = 12/5$ and $\ellc(K_{2,3}) = 4 > \lceil \Delta(K_{2,3})/2 \rceil + 1$.

2. In (b), the bound 3 cannot be raised in general, since $\text{mad}(K_{3,3}) = 3$ and $\ellc(K_{3,3}) = 5 > \lceil \Delta(K_{3,3})/2 \rceil + 2$. 
Results

Main Theorem:

(a) If $G$ is planar and $g \geq 5$, then $l_{c\ell}(G) \leq \lceil \Delta/2 \rceil + 4$.
(b) If $\text{mad}(G) < 3$ and $\Delta \geq 9$, then $l_{c\ell}(G) \leq \lceil \Delta/2 \rceil + 2$.
(c) If $\text{mad}(G) < 12/5$ and $\Delta \geq 3$, then $l_{c\ell}(G) = \lceil \Delta/2 \rceil + 1$.

Remarks

1. In (c), the bound $12/5$ is optimal, since $\text{mad}(K_{2,3}) = 12/5$ and $l_{c}(K_{2,3}) = 4 > \lceil \Delta(K_{2,3})/2 \rceil + 1$.

2. In (b), the bound 3 cannot be raised in general, since $\text{mad}(K_{3,3}) = 3$ and $l_{c}(K_{3,3}) = 5 > \lceil \Delta(K_{3,3})/2 \rceil + 2$.

3. Raspaud and Wang conjectured that $l_{c\ell}(G) \leq \lceil \Delta(G)/2 \rceil + 2$ for all planar $G$ with $g \geq 6$. 
Results

Main Theorem:
(a) If $G$ is planar and $g \geq 5$, then $\ellc(G) \leq \lceil \Delta/2 \rceil + 4$.
(b) If $\text{mad}(G) < 3$ and $\Delta \geq 9$, then $\ellc(G) \leq \lceil \Delta/2 \rceil + 2$.
(c) If $\text{mad}(G) < 12/5$ and $\Delta \geq 3$, then $\ellc(G) = \lceil \Delta/2 \rceil + 1$.

Remarks

1. In (c), the bound $12/5$ is optimal, since $\text{mad}(K_{2,3}) = 12/5$ and $\ellc(K_{2,3}) = 4 > \lceil \Delta(K_{2,3})/2 \rceil + 1$.
2. In (b), the bound 3 cannot be raised in general, since $\text{mad}(K_{3,3}) = 3$ and $\ellc(K_{3,3}) = 5 > \lceil \Delta(K_{3,3})/2 \rceil + 2$.
3. Raspaud and Wang conjectured that $\ellc(G) \leq \lceil \Delta(G)/2 \rceil + 2$ for all planar $G$ with $g \geq 6$.
   Part (b) proves their conjecture when $\Delta(G) \geq 9$. 
Proof of part (a): Outline

**Lemma:** If $G$ is planar with $\delta(G) \geq 2$ and $g \geq 5$, then $G$ contains one of the following two configurations:
Proof of part (a): Outline

**Lemma:** If $G$ is planar with $\delta(G) \geq 2$ and $g \geq 5$, then $G$ contains one of the following two configurations:

(C1) a 2-vertex adjacent to a 5-vertex,
Proof of part (a): Outline

**Lemma:** If $G$ is planar with $\delta(G) \geq 2$ and $g \geq 5$, then $G$ contains one of the following two configurations:

(C1) a 2-vertex adjacent to a 5-vertex,
(C2) a 5-face with four incident 3-vertices and the fifth incident vertex of degree at most 5.
Proof of part (a): Outline

**Lemma:** If $G$ is planar with $\delta(G) \geq 2$ and $g \geq 5$, then $G$ contains one of the following two configurations:

(C1) a 2-vertex adjacent to a 5−-vertex,
(C2) a 5-face with four incident 3-vertices and the fifth incident vertex of degree at most 5.

**Pf:** Each $v$ or $f$ gets $\mu(v) = 3d(v)/2 - 5$ or $\mu(f) = d(f) - 5$. 

\[ \sum_{x \in V \cup F} \mu(x) = \sum_{v \in V} (3d(v)/2 - 5) + \sum_{f \in F} (d(f) - 5) = (3|E| - 5|V|) + (2|E| - 5|F|) = -5(|F| - |E| + |V|) = -10. \]
Assume $G$ is a counterexample. If we can redistribute charge to get $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ everywhere, then $0 \leq \sum_{x \in V \cup F} \mu^*(x) = \sum_{x \in V \cup F} \mu(x) + 10$, contradiction!
Proof of part (a): Outline

**Lemma:** If $G$ is planar with $\delta(G) \geq 2$ and $g \geq 5$, then $G$ contains one of the following two configurations:

(C1) a 2-vertex adjacent to a 5-vertex,

(C2) a 5-face with four incident 3-vertices and the fifth incident vertex of degree at most 5.

**Pf:** Each $v$ or $f$ gets $\mu(v) = 3d(v)/2 - 5$ or $\mu(f) = d(f) - 5$.

\[
\sum_{x \in V \cup F} \mu(x) = \sum_{v \in V} (3d(v)/2 - 5) + \sum_{f \in F} (d(f) - 5)
\]
Proof of part (a): Outline

**Lemma:** If $G$ is planar with $\delta(G) \geq 2$ and $g \geq 5$, then $G$ contains one of the following two configurations:

(C1) a 2-vertex adjacent to a 5-vertex,
(C2) a 5-face with four incident 3-vertices and the fifth incident vertex of degree at most 5.

**Pf:** Each $v$ or $f$ gets $\mu(v) = 3d(v)/2 - 5$ or $\mu(f) = d(f) - 5$.

$$\sum_{x \in V \cup F} \mu(x) = \sum_{v \in V} (3d(v)/2 - 5) + \sum_{f \in F} (d(f) - 5)$$

$$= (3|E| - 5|V|) + (2|E| - 5|F|)$$
Proof of part (a): Outline

**Lemma:** If $G$ is planar with $\delta(G) \geq 2$ and $g \geq 5$, then $G$ contains one of the following two configurations:

(C1) a 2-vertex adjacent to a 5-vertex,
(C2) a 5-face with four incident 3-vertices and the fifth incident vertex of degree at most 5.

**Pf:** Each $v$ or $f$ gets $\mu(v) = 3d(v)/2 - 5$ or $\mu(f) = d(f) - 5$.

$$
\sum_{x \in V \cup F} \mu(x) = \sum_{v \in V} \left(\frac{3d(v)}{2} - 5\right) + \sum_{f \in F} (d(f) - 5)
$$

$$
= (3|E| - 5|V|) + (2|E| - 5|F|)
$$

$$
= -5(|F| - |E| + |V|) = -10.
$$
Proof of part (a): Outline

**Lemma:** If $G$ is planar with $\delta(G) \geq 2$ and $g \geq 5$, then $G$ contains one of the following two configurations:

(C1) a 2-vertex adjacent to a 5−-vertex,
(C2) a 5-face with four incident 3-vertices and the fifth incident vertex of degree at most 5.

**Pf:** Each $v$ or $f$ gets $\mu(v) = 3d(v)/2 - 5$ or $\mu(f) = d(f) - 5$.

$$\sum_{x \in V \cup F} \mu(x) = \sum_{v \in V} (3d(v)/2 - 5) + \sum_{f \in F} (d(f) - 5)$$

$$= (3|E| - 5|V|) + (2|E| - 5|F|)$$

$$= -5(|F| - |E| + |V|) = -10.$$ 

Assume $G$ is a counterexample.
Proof of part (a): Outline

**Lemma:** If $G$ is planar with $\delta(G) \geq 2$ and $g \geq 5$, then $G$ contains one of the following two configurations:

(C1) a 2-vertex adjacent to a 5-vertex,
(C2) a 5-face with four incident 3-vertices and the fifth incident vertex of degree at most 5.

**Pf:** Each $v$ or $f$ gets $\mu(v) = 3d(v)/2 - 5$ or $\mu(f) = d(f) - 5$.

$$\sum_{x \in V \cup F} \mu(x) = \sum_{v \in V} (3d(v)/2 - 5) + \sum_{f \in F} (d(f) - 5)$$

$$= (3|E| - 5|V|) + (2|E| - 5|F|)$$

$$= -5(|F| - |E| + |V|) = -10.$$

Assume $G$ is a counterexample. If we can redistribute charge to get $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ everywhere, then
Proof of part (a): Outline

**Lemma:** If $G$ is planar with $\delta(G) \geq 2$ and $g \geq 5$, then $G$ contains one of the following two configurations:

- (C1) a 2-vertex adjacent to a 5−-vertex,
- (C2) a 5-face with four incident 3-vertices and the fifth incident vertex of degree at most 5.

**Pf:** Each $v$ or $f$ gets $\mu(v) = 3d(v)/2 - 5$ or $\mu(f) = d(f) - 5$.

$$\sum_{x \in V \cup F} \mu(x) = \sum_{v \in V} (3d(v)/2 - 5) + \sum_{f \in F} (d(f) - 5)$$

$$= (3|E| - 5|V|) + (2|E| - 5|F|)$$

$$= -5(|F| - |E| + |V|) = -10.$$

Assume $G$ is a counterexample. If we can redistribute charge to get $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ everywhere, then

$$0 \leq \sum_{x \in V \cup F} \mu^*(x)$$
Proof of part (a): Outline

**Lemma:** If $G$ is planar with $\delta(G) \geq 2$ and $g \geq 5$, then $G$ contains one of the following two configurations:

(C1) a 2-vertex adjacent to a 5-vertex,
(C2) a 5-face with four incident 3-vertices and the fifth incident vertex of degree at most 5.

**Pf:** Each $v$ or $f$ gets $\mu(v) = 3d(v)/2 - 5$ or $\mu(f) = d(f) - 5$.

$$
\sum_{x \in V \cup F} \mu(x) = \sum_{v \in V} (3d(v)/2 - 5) + \sum_{f \in F} (d(f) - 5)

= (3|E| - 5|V|) + (2|E| - 5|F|)

= -5(|F| - |E| + |V|) = -10.
$$

Assume $G$ is a counterexample. If we can redistribute charge to get $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ everywhere, then

$$
0 \leq \sum_{x \in V \cup F} \mu^*(x) = \sum_{x \in V \cup F} \mu(x)
$$
Proof of part (a): Outline

**Lemma:** If $G$ is planar with $\delta(G) \geq 2$ and $g \geq 5$, then $G$ contains one of the following two configurations:

(C1) a 2-vertex adjacent to a $5^-$-vertex,
(C2) a 5-face with four incident 3-vertices and the fifth incident vertex of degree at most 5.

**Pf:** Each $v$ or $f$ gets $\mu(v) = 3d(v)/2 - 5$ or $\mu(f) = d(f) - 5$.

$$\sum_{x \in V \cup F} \mu(x) = \sum_{v \in V} (3d(v)/2 - 5) + \sum_{f \in F} (d(f) - 5)$$

$$= (3|E| - 5|V|) + (2|E| - 5|F|)$$

$$= -5(|F| - |E| + |V|) = -10.$$

Assume $G$ is a counterexample. If we can redistribute charge to get $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ everywhere, then

$$0 \leq \sum_{x \in V \cup F} \mu^*(x) = \sum_{x \in V \cup F} \mu(x) = -10$$
Proof of part (a): Outline

**Lemma:** If $G$ is planar with $\delta(G) \geq 2$ and $g \geq 5$, then $G$ contains one of the following two configurations:

(C1) a 2-vertex adjacent to a 5-vertex,
(C2) a 5-face with four incident 3-vertices and the fifth incident vertex of degree at most 5.

**Pf:** Each $v$ or $f$ gets $\mu(v) = 3d(v)/2 - 5$ or $\mu(f) = d(f) - 5$.

\[
\sum_{x \in V \cup F} \mu(x) = \sum_{v \in V} \left(\frac{3d(v)}{2} - 5\right) + \sum_{f \in F} (d(f) - 5)
= (3|E| - 5|V|) + (2|E| - 5|F|)
= -5(|F| - |E| + |V|) = -10.
\]

Assume $G$ is a counterexample. If we can redistribute charge to get $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ everywhere, then

\[
0 \leq \sum_{x \in V \cup F} \mu^*(x) = \sum_{x \in V \cup F} \mu(x) = -10 \quad \text{contradiction!}
\]
Proof of part (a): Structure

Start with $\mu(v) = \frac{3d(v)}{2} - 5$ and $\mu(f) = d(f) - 5$, and apply:
Proof of part (a): Structure

Start with $\mu(v) = \frac{3d(v)}{2} - 5$ and $\mu(f) = d(f) - 5$, and apply:

(R1) Each $4^+$-vertex $v$ gives charge $\frac{3d(v)/2 - 5}{d(v)}$ to each incident face

Claim: Each $f$ gives net at most $1/3$ to each adjacent pair on $f$.

If one is a 2-vert, then the other is a $6^+$-vert, so $1 - \frac{2}{3} = 1/3$.

Otherwise, each vert gets at most $1/6$ from $f$, so $2(1/6) = 1/3$.

So $\mu^*(f) \geq (d(f) - 5) - \frac{d(f)}{6}$ for $d(f) \geq 6$. 

$\Box$
Proof of part (a): Structure

Start with $\mu(v) = 3d(v)/2 - 5$ and $\mu(f) = d(f) - 5$, and apply:

(R1) Each $4^+$-vertex $v$ gives charge $\frac{3d(v)/2 - 5}{d(v)}$ to each incident face

(R2) Each face gives charge 1 to each incident 2-vertex and charge $1/6$ to each incident 3-vertex
Proof of part (a): Structure
Start with $\mu(v) = 3d(v)/2 - 5$ and $\mu(f) = d(f) - 5$, and apply:
(R1) Each $4^+$-vertex $v$ gives charge $\frac{3d(v)/2 - 5}{d(v)}$ to each incident face
(R2) Each face gives charge 1 to each incident 2-vertex and charge $1/6$ to each incident 3-vertex
Show $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ for all $v$ and $f$. 
Proof of part (a): Structure

Start with $\mu(v) = \frac{3d(v)}{2} - 5$ and $\mu(f) = d(f) - 5$, and apply:

(R1) Each $4^{+}$-vertex $v$ gives charge $\frac{3d(v)/2 - 5}{d(v)}$ to each incident face

(R2) Each face gives charge 1 to each incident 2-vertex
and charge $\frac{1}{6}$ to each incident 3-vertex

Show $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ for all $v$ and $f$.

d(v) = 2:
Proof of part (a): Structure

Start with $\mu(v) = 3d(v)/2 - 5$ and $\mu(f) = d(f) - 5$, and apply:

(R1) Each $4^+$-vertex $v$ gives charge $\frac{3d(v)/2-5}{d(v)}$ to each incident face

(R2) Each face gives charge 1 to each incident 2-vertex and charge $1/6$ to each incident 3-vertex

Show $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ for all $v$ and $f$.

$d(v) = 2$: $-2 + 2(1) = 0$
Proof of part (a): Structure

Start with \( \mu(v) = \frac{3d(v)}{2} - 5 \) and \( \mu(f) = d(f) - 5 \), and apply:

(R1) Each \( 4^+ \)-vertex \( v \) gives charge \( \frac{3d(v)/2-5}{d(v)} \) to each incident face

(R2) Each face gives charge 1 to each incident 2-vertex and charge \( \frac{1}{6} \) to each incident 3-vertex

Show \( \mu^*(v) \geq 0 \) and \( \mu^*(f) \geq 0 \) for all \( v \) and \( f \).

\[ d(v) = 2: \ -2 + 2(1) = 0 \]
Proof of part (a): Structure

Start with $\mu(v) = \frac{3d(v)}{2} - 5$ and $\mu(f) = d(f) - 5$, and apply:

(R1) Each $4^+$-vertex $v$ gives charge $\frac{3d(v)/2 - 5}{d(v)}$ to each incident face.

(R2) Each face gives charge 1 to each incident 2-vertex and charge $1/6$ to each incident 3-vertex.

Show $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ for all $v$ and $f$.

$d(v) = 2$: $-2 + 2(1) = 0 \checkmark$

$d(v) = 3$: $- \frac{1}{2} + 3 \cdot \frac{1}{6} = 0 \checkmark$
Proof of part (a): Structure

Start with $\mu(v) = \frac{3d(v)}{2} - 5$ and $\mu(f) = d(f) - 5$, and apply:

(R1) Each $4^+$-vertex $v$ gives charge $\frac{3d(v)/2-5}{d(v)}$ to each incident face

(R2) Each face gives charge 1 to each incident 2-vertex and charge $1/6$ to each incident 3-vertex

Show $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ for all $v$ and $f$.

d(v) = 2: \ -2 + 2(1) = 0 \ \checkmark \ \ d(v) = 3: \ -1/2 + 3(1/6) = 0
Proof of part (a): Structure

Start with $\mu(v) = 3d(v)/2 - 5$ and $\mu(f) = d(f) - 5$, and apply:

(R1) Each $4^+$-vertex $v$ gives charge $\frac{3d(v)/2 - 5}{d(v)}$ to each incident face

(R2) Each face gives charge 1 to each incident 2-vertex and charge $1/6$ to each incident 3-vertex

Show $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ for all $v$ and $f$.

$d(v) = 2$: $-2 + 2(1) = 0 \checkmark$

$d(v) = 3$: $-1/2 + 3(1/6) = 0 \checkmark$
Proof of part (a): Structure

Start with $\mu(v) = 3d(v)/2 - 5$ and $\mu(f) = d(f) - 5$, and apply:

(R1) Each $4^+$-vertex $v$ gives charge $\frac{3d(v)/2-5}{d(v)}$ to each incident face
(R2) Each face gives charge 1 to each incident 2-vertex and charge $1/6$ to each incident 3-vertex

Show $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ for all $v$ and $f$.

$\begin{align*}
d(v) = 2: & \quad -2 + 2(1) = 0 \checkmark \\
d(v) = 3: & \quad -1/2 + 3(1/6) = 0 \checkmark \\
d(v) \geq 4: & \quad
\end{align*}$
Proof of part (a): Structure

Start with $\mu(v) = 3d(v)/2 - 5$ and $\mu(f) = d(f) - 5$, and apply:

(R1) Each $4^+$-vertex $v$ gives charge $\frac{3d(v)/2-5}{d(v)}$ to each incident face

(R2) Each face gives charge 1 to each incident 2-vertex and charge $1/6$ to each incident 3-vertex

Show $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ for all $v$ and $f$.

$d(v) = 2$: $-2 + 2(1) = 0$ ✓
$d(v) = 3$: $-1/2 + 3(1/6) = 0$ ✓
$d(v) \geq 4$: $\mu(v) - d(v)(\mu(v)/d(v)) = 0$
Proof of part (a): Structure

Start with $\mu(v) = \frac{3d(v)}{2} - 5$ and $\mu(f) = d(f) - 5$, and apply:

(R1) Each $4^+$-vertex $v$ gives charge $\frac{3d(v)/2 - 5}{d(v)}$ to each incident face

(R2) Each face gives charge $1$ to each incident $2$-vertex and charge $1/6$ to each incident $3$-vertex

Show $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ for all $v$ and $f$.

$d(v) = 2$: $-2 + 2(1) = 0$ ✓

$d(v) = 3$: $-1/2 + 3(1/6) = 0$ ✓

$d(v) \geq 4$: $\mu(v) - d(v)(\mu(v)/d(v)) = 0$ ✓
Proof of part (a): Structure

Start with $\mu(v) = 3d(v)/2 - 5$ and $\mu(f) = d(f) - 5$, and apply:

(R1) Each $4^+$-vertex $v$ gives charge $\frac{3d(v)/2-5}{d(v)}$ to each incident face

(R2) Each face gives charge 1 to each incident 2-vertex and charge $1/6$ to each incident 3-vertex

Show $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ for all $v$ and $f$.

$d(v) = 2$: $-2 + 2(1) = 0$ ✓
$d(v) = 3$: $-1/2 + 3(1/6) = 0$ ✓
$d(v) \geq 4$: $\mu(v) - d(v)(\mu(v)/d(v)) = 0$ ✓

Claim: Each $f$ gives net at most $1/3$ to each adjacent pair on $f$. 
Proof of part (a): Structure

Start with $\mu(v) = \frac{3d(v)}{2} - 5$ and $\mu(f) = d(f) - 5$, and apply:

(R1) Each $4^+$-vertex $v$ gives charge $\frac{3d(v)/2 - 5}{d(v)}$ to each incident face

(R2) Each face gives charge 1 to each incident 2-vertex and charge $1/6$ to each incident 3-vertex

Show $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ for all $v$ and $f$.

$d(v) = 2$: $-2 + 2(1) = 0$ ✓ 
$d(v) = 3$: $-1/2 + 3(1/6) = 0$ ✓ 
$d(v) \geq 4$: $\mu(v) - d(v)(\mu(v)/d(v)) = 0$ ✓

Claim: Each $f$ gives net at most $1/3$ to each adjacent pair on $f$.

If one is a 2-vert, then the other is a $6^+$-vert, so $1 - 2/3 = 1/3$. 

Proof of part (a): Structure

Start with \( \mu(v) = 3d(v)/2 - 5 \) and \( \mu(f) = d(f) - 5 \), and apply:

(R1) Each 4\(^+\)-vertex \( v \) gives charge \( \frac{3d(v)/2 - 5}{d(v)} \) to each incident face

(R2) Each face gives charge 1 to each incident 2-vertex
and charge \( 1/6 \) to each incident 3-vertex

Show \( \mu^*(v) \geq 0 \) and \( \mu^*(f) \geq 0 \) for all \( v \) and \( f \).

\[
\begin{align*}
d(v) = 2: & \quad -2 + 2(1) = 0 \checkmark \\
d(v) = 3: & \quad -1/2 + 3(1/6) = 0 \checkmark \\
d(v) \geq 4: & \quad \mu(v) - d(v)(\mu(v)/d(v)) = 0 \checkmark
\end{align*}
\]

Claim: Each \( f \) gives net at most \( 1/3 \) to each adjacent pair on \( f \).
If one is a 2-vert, then the other is a 6\(^+\)-vert, so \( 1 - 2/3 = 1/3 \).
Otherwise, each vert gets at most \( 1/6 \) from \( f \), so \( 2(1/6) = 1/3 \).
Proof of part (a): Structure

Start with $\mu(v) = \frac{3d(v)}{2} - 5$ and $\mu(f) = d(f) - 5$, and apply:

(R1) Each $4^+$-vertex $v$ gives charge $\frac{3d(v)/2-5}{d(v)}$ to each incident face

(R2) Each face gives charge 1 to each incident 2-vertex and charge $1/6$ to each incident 3-vertex

Show $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ for all $v$ and $f$.

$d(v) = 2$: $-2 + 2(1) = 0 \checkmark$

$d(v) = 3$: $-1/2 + 3(1/6) = 0 \checkmark$

$d(v) \geq 4$: $\mu(v) - d(v)(\mu(v)/d(v)) = 0 \checkmark$

Claim: Each $f$ gives net at most $1/3$ to each adjacent pair on $f$.

If one is a 2-vert, then the other is a $6^+$-vert, so $1 - 2/3 = 1/3$.

Otherwise, each vert gets at most $1/6$ from $f$, so $2(1/6) = 1/3$.

So $\mu^*(f) \geq (d(f) - 5) - d(f)/6$
Proof of part (a): Structure

Start with $\mu(v) = 3d(v)/2 - 5$ and $\mu(f) = d(f) - 5$, and apply:

- (R1) Each $4^+$-vertex $v$ gives charge $\frac{3d(v)/2 - 5}{d(v)}$ to each incident face.
- (R2) Each face gives charge 1 to each incident 2-vertex and charge $1/6$ to each incident 3-vertex.

Show $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ for all $v$ and $f$.

- $d(v) = 2$: $-2 + 2(1) = 0$ ✓
- $d(v) = 3$: $-1/2 + 3(1/6) = 0$ ✓
- $d(v) \geq 4$: $\mu(v) - d(v)(\mu(v)/d(v)) = 0$ ✓

**Claim:** Each $f$ gives net at most $1/3$ to each adjacent pair on $f$.

If one is a 2-vert, then the other is a $6^+$-vert, so $1 - 2/3 = 1/3$.

Otherwise, each vert gets at most $1/6$ from $f$, so $2(1/6) = 1/3$.

So $\mu^*(f) \geq (d(f) - 5) - d(f)/6 \geq 0$ for $d(f) \geq 6$. 
Proof of part (a): Structure

Start with $\mu(v) = 3d(v)/2 - 5$ and $\mu(f) = d(f) - 5$, and apply:

(R1) Each 4$^+$-vertex $v$ gives charge $\frac{3d(v)/2 - 5}{d(v)}$ to each incident face
(R2) Each face gives charge 1 to each incident 2-vertex

and charge $1/6$ to each incident 3-vertex

Show $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ for all $v$ and $f$.

$d(v) = 2$: $-2 + 2(1) = 0 \checkmark$
$d(v) = 3$: $-1/2 + 3(1/6) = 0 \checkmark$
$d(v) \geq 4$: $\mu(v) - d(v)(\mu(v)/d(v)) = 0 \checkmark$

Claim: Each $f$ gives net at most $1/3$ to each adjacent pair on $f$.

If one is a 2-vert, then the other is a 6$^+$-vert, so $1 - 2/3 = 1/3$.

Otherwise, each vert gets at most $1/6$ from $f$, so $2(1/6) = 1/3$.

So $\mu^*(f) \geq (d(f) - 5) - d(f)/6 \geq 0$ for $d(f) \geq 6$. 
Proof of part (a): Structure

Start with \( \mu(v) = 3d(v)/2 - 5 \) and \( \mu(f) = d(f) - 5 \), and apply:

(R1) Each \( 4^+ \)-vertex \( v \) gives charge \( \frac{3d(v)/2 - 5}{d(v)} \) to each incident face

(R2) Each face gives charge 1 to each incident 2-vertex and charge \( 1/6 \) to each incident 3-vertex

Show \( \mu^*(v) \geq 0 \) and \( \mu^*(f) \geq 0 \) for all \( v \) and \( f \).

\[
\begin{align*}
d(v) = 2: & \quad -2 + 2(1) = 0 \checkmark \\
d(v) = 3: & \quad -1/2 + 3(1/6) = 0 \checkmark \\
d(v) \geq 4: & \quad \mu(v) - d(v)(\mu(v)/d(v)) = 0 \checkmark
\end{align*}
\]

Claim: Each \( f \) gives net at most \( 1/3 \) to each adjacent pair on \( f \).

If one is a 2-vert, then the other is a \( 6^+ \)-vert, so \( 1 - 2/3 = 1/3 \).

Otherwise, each vert gets at most \( 1/6 \) from \( f \), so \( 2(1/6) = 1/3 \).

So \( \mu^*(f) \geq (d(f) - 5) - d(f)/6 \geq 0 \) for \( d(f) \geq 6 \).
Proof of part (a): Structure

Start with \( \mu(v) = \frac{3d(v)}{2} - 5 \) and \( \mu(f) = d(f) - 5 \), and apply:

(R1) Each 4\(^+\)-vertex \( v \) gives charge \( \frac{3d(v)/2 - 5}{d(v)} \) to each incident face

(R2) Each face gives charge 1 to each incident 2-vertex
and charge \( \frac{1}{6} \) to each incident 3-vertex

Show \( \mu^*(v) \geq 0 \) and \( \mu^*(f) \geq 0 \) for all \( v \) and \( f \).

\[

d(v) = 2: \quad -2 + 2(1) = 0 \quad \checkmark \\

d(v) = 3: \quad -\frac{1}{2} + 3(\frac{1}{6}) = 0 \quad \checkmark \\

d(v) \geq 4: \quad \mu(v) - d(v)(\mu(v)/d(v)) = 0 \quad \checkmark
\]

Claim: Each \( f \) gives net at most 1/3 to each adjacent pair on \( f \).

If one is a 2-vert, then the other is a 6\(^+\)-vert, so \( 1 - 2/3 = 1/3 \).
Otherwise, each vert gets at most 1/6 from \( f \), so \( 2(1/6) = 1/3 \).

So \( \mu^*(f) \geq (d(f) - 5) - d(f)/6 \geq 0 \) for \( d(f) \geq 6 \).
Proof of part (a): Structure

Start with $\mu(v) = \frac{3d(v)}{2} - 5$ and $\mu(f) = d(f) - 5$, and apply:

(R1) Each $4^+$-vertex $v$ gives charge $\frac{3d(v)/2 - 5}{d(v)}$ to each incident face.

(R2) Each face gives charge $1$ to each incident $2$-vertex and charge $1/6$ to each incident $3$-vertex.

Show $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ for all $v$ and $f$.

- $d(v) = 2$: $-2 + 2(1) = 0 \checkmark$
- $d(v) = 3$: $-1/2 + 3(1/6) = 0 \checkmark$
- $d(v) \geq 4$: $\mu(v) - d(v)(\mu(v)/d(v)) = 0 \checkmark$

Claim: Each $f$ gives net at most $1/3$ to each adjacent pair on $f$.

If one is a $2$-vert, then the other is a $6^+$-vert, so $1 - 2/3 = 1/3$.

Otherwise, each vert gets at most $1/6$ from $f$, so $2(1/6) = 1/3$.

So $\mu^*(f) \geq (d(f) - 5) - d(f)/6 \geq 0$ for $d(f) \geq 6$. 

\[
\begin{align*}
\text{Left: } & 6^+ 2 6^+ 2 6^+ \\
\text{Right: } & 2 3^+ 6^+ 3^+ \\
\text{Net: } & -2 + 3(2/3) \quad -1 - 2/6 + 2(2/3)
\end{align*}
\]
Proof of part (a): Structure

Start with $\mu(v) = 3d(v)/2 - 5$ and $\mu(f) = d(f) - 5$, and apply:

(R1) Each $4^+$-vertex $v$ gives charge $\frac{3d(v)/2 - 5}{d(v)}$ to each incident face

(R2) Each face gives charge 1 to each incident 2-vertex and charge $1/6$ to each incident 3-vertex

Show $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ for all $v$ and $f$.

$d(v) = 2$: $-2 + 2(1) = 0$ ✓

$d(v) = 3$: $-1/2 + 3(1/6) = 0$ ✓

$d(v) \geq 4$: $\mu(v) - d(v)(\mu(v)/d(v)) = 0$ ✓

Claim: Each $f$ gives net at most $1/3$ to each adjacent pair on $f$.
If one is a 2-vert, then the other is a $6^+$-vert, so $1 - 2/3 = 1/3$.
Otherwise, each vert gets at most $1/6$ from $f$, so $2(1/6) = 1/3$.

So $\mu^*(f) \geq (d(f) - 5) - d(f)/6 \geq 0$ for $d(f) \geq 6$. 

![Diagram showing vertices and faces with charges]

$-2 + 3(2/3)$  $-1 - 2/6 + 2(2/3)$
Proof of part (a): Structure

Start with $\mu(v) = 3d(v)/2 - 5$ and $\mu(f) = d(f) - 5$, and apply:

(R1) Each $4^+$-vertex $v$ gives charge $\frac{3d(v)/2 - 5}{d(v)}$ to each incident face.
(R2) Each face gives charge 1 to each incident 2-vertex and charge $1/6$ to each incident 3-vertex.

Show $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ for all $v$ and $f$.

$d(v) = 2$: $-2 + 2(1) = 0$  
$d(v) = 3$: $-1/2 + 3(1/6) = 0$
$d(v) \geq 4$: $\mu(v) - d(v)(\mu(v)/d(v)) = 0$

Claim: Each $f$ gives net at most $1/3$ to each adjacent pair on $f$.
If one is a 2-vert, then the other is a $6^+$-vert, so $1 - 2/3 = 1/3$.
Otherwise, each vert gets at most $1/6$ from $f$, so $2(1/6) = 1/3$.

So $\mu^*(f) \geq (d(f) - 5) - d(f)/6 \geq 0$ for $d(f) \geq 6$. 

\[
\begin{align*}
\text{Vertex:} & \quad 6^+ & \quad 6^+ & \quad 6^+ \\
\text{Charge:} & \quad -2 + 3(2/3) & \quad -1 - 2/6 + 2(2/3) & \quad -4/6 + 2/3
\end{align*}
\]
Proof of part (a): Structure

Start with \( \mu(v) = 3d(v)/2 - 5 \) and \( \mu(f) = d(f) - 5 \), and apply:

(R1) Each 4\(^+\)-vertex \( v \) gives charge \( \frac{3d(v)/2 - 5}{d(v)} \) to each incident face

(R2) Each face gives charge 1 to each incident 2-vertex and charge 1/6 to each incident 3-vertex

Show \( \mu^*(v) \geq 0 \) and \( \mu^*(f) \geq 0 \) for all \( v \) and \( f \).

\[
\begin{align*}
d(v) = 2: & \quad -2 + 2(1) = 0 \quad \checkmark \\
d(v) = 3: & \quad -1/2 + 3(1/6) = 0 \quad \checkmark \\
d(v) \geq 4: & \quad \mu(v) - d(v)(\mu(v)/d(v)) = 0 \quad \checkmark \\
\end{align*}
\]

Claim: Each \( f \) gives net at most 1/3 to each adjacent pair on \( f \).

If one is a 2-vert, then the other is a 6\(^+\)-vert, so \( 1 - 2/3 = 1/3 \).

Otherwise, each vert gets at most 1/6 from \( f \), so \( 2(1/6) = 1/3 \).

So \( \mu^*(f) \geq (d(f) - 5) - d(f)/6 \geq 0 \) for \( d(f) \geq 6 \).
Proof of part (a): Structure

Start with $\mu(v) = 3d(v)/2 - 5$ and $\mu(f) = d(f) - 5$, and apply:

(R1) Each $4^+$-vertex $v$ gives charge $\frac{3d(v)/2 - 5}{d(v)}$ to each incident face

(R2) Each face gives charge 1 to each incident 2-vertex and charge $1/6$ to each incident 3-vertex

Show $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ for all $v$ and $f$.

\[ d(v) = 2: \quad -2 + 2(1) = 0 \checkmark \quad d(v) = 3: \quad -1/2 + 3(1/6) = 0 \checkmark \]

\[ d(v) \geq 4: \quad \mu(v) - d(v)(\mu(v)/d(v)) = 0 \checkmark \]

Claim: Each $f$ gives net at most $1/3$ to each adjacent pair on $f$.

If one is a 2-vert, then the other is a $6^+$-vert, so $1 - 2/3 = 1/3$.

Otherwise, each vert gets at most $1/6$ from $f$, so $2(1/6) = 1/3$.

So $\mu^*(f) \geq (d(f) - 5) - d(f)/6 \geq 0$ for $d(f) \geq 6$. 

\[ \begin{align*}
-2 + 3(2/3) \quad -1 - 2/6 + 2(2/3) \quad -4/6 + 2/3 \quad -3/6 + 2(1/4)
\end{align*} \]
Proof of part (a): Coloring

**Thm:** If $G$ is planar and $g \geq 5$, then $l_{c\ell}(G) \leq \lceil \Delta/2 \rceil + 4$. 
Proof of part (a): Coloring

**Thm:** If $G$ is planar and $g \geq 5$, then $l_{c\ell}(G) \leq \lceil \Delta/2 \rceil + 4$.

**Pf:** By the lemma, $G$ contains one of the 3 configurations:
Proof of part (a): Coloring

**Thm:** If $G$ is planar and $g \geq 5$, then $lc_\ell(G) \leq \lceil \Delta/2 \rceil + 4$.

**Pf:** By the lemma, $G$ contains one of the 3 configurations:

(C0) a 1-vertex,
Proof of part (a): Coloring

**Thm:** If $G$ is planar and $g \geq 5$, then $l_{c_\ell}(G) \leq \lceil \Delta/2 \rceil + 4$.

**Pf:** By the lemma, $G$ contains one of the 3 configurations:

(C0) a 1-vertex,

(C1) a 2-vertex adjacent to a 5-vertex, or
Proof of part (a): Coloring

**Thm:** If $G$ is planar and $g \geq 5$, then $\ell_1(G) \leq \lceil \Delta/2 \rceil + 4$.

**Pf:** By the lemma, $G$ contains one of the 3 configurations:

(C0) a 1-vertex,
(C1) a 2-vertex adjacent to a 5⁻-vertex, or
(C2) a 5-face with 4 incident 3-vertices and an incident 5⁻-vertex.
Proof of part (a): Coloring

**Thm:** If $G$ is planar and $g \geq 5$, then $l_{c_\ell}(G) \leq \lceil \Delta/2 \rceil + 4$.

**Pf:** By the lemma, $G$ contains one of the 3 configurations:

- **(C0)** a 1-vertex,
- **(C1)** a 2-vertex adjacent to a $5^-$-vertex, or
- **(C2)** a 5-face with 4 incident 3-vertices and an incident $5^-$-vertex.

But each configuration is forbidden in a minimal counterexample.
Proof of part (a): Coloring

**Thm:** If $G$ is planar and $g \geq 5$, then $\ellcl(G) \leq \lceil \Delta/2 \rceil + 4$.

**Pf:** By the lemma, $G$ contains one of the 3 configurations:

(C0) a 1-vertex,
(C1) a 2-vertex adjacent to a $5^-$-vertex, or
(C2) a 5-face with 4 incident 3-vertices and an incident $5^-$-vertex.

But each configuration is forbidden in a minimal counterexample.

(C0) \[ \begin{array}{cc} \bullet & \square \end{array} \]
Proof of part (a): Coloring

**Thm:** If $G$ is planar and $g \geq 5$, then $l_{c,\ell}(G) \leq \lceil \Delta/2 \rceil + 4$.

**Pf:** By the lemma, $G$ contains one of the 3 configurations:

(C0) a 1-vertex,
(C1) a 2-vertex adjacent to a 5-vertex, or
(C2) a 5-face with 4 incident 3-vertices and an incident 5-vertex.

But each configuration is forbidden in a minimal counterexample.

(C0) \[ \lceil \Delta/2 \rceil + 4 - \lceil (\Delta - 1)/2 \rceil - 1 \]
Proof of part (a): Coloring

**Thm:** If $G$ is planar and $g \geq 5$, then $l_{c, \ell}(G) \leq \lceil \Delta/2 \rceil + 4$.

**Pf:** By the lemma, $G$ contains one of the 3 configurations:

(C0) a 1-vertex,

(C1) a 2-vertex adjacent to a 5−-vertex, or

(C2) a 5-face with 4 incident 3-vertices and an incident 5−-vertex.

But each configuration is forbidden in a minimal counterexample.

(C0) \[ \lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1)/2 \rfloor - 1 \geq 4 \]
Proof of part (a): Coloring

**Thm:** If $G$ is planar and $g \geq 5$, then $l_{c\ell}(G) \leq \lceil \Delta/2 \rceil + 4$.

**Pf:** By the lemma, $G$ contains one of the 3 configurations:

(C0) a 1-vertex,

(C1) a 2-vertex adjacent to a 5$^-$-vertex, or

(C2) a 5-face with 4 incident 3-vertices and an incident 5$^-$-vertex.

But each configuration is forbidden in a minimal counterexample.

(C0) $\begin{array}{c}
\bullet \\
\square \\
\end{array}$  \[ \lceil \Delta/2 \rceil + 4 - \lceil (\Delta - 1)/2 \rceil - 1 \geq 4 \]

(C1) $\begin{array}{c}
\square \\
\bullet \\
\bullet \\
\end{array}$
Proof of part (a): Coloring

**Thm:** If $G$ is planar and $g \geq 5$, then $\ellcl(G) \leq \lceil \Delta/2 \rceil + 4$.

**Pf:** By the lemma, $G$ contains one of the 3 configurations:

(C0) a 1-vertex,
(C1) a 2-vertex adjacent to a $5^-$-vertex, or
(C2) a 5-face with 4 incident 3-vertices and an incident $5^-$-vertex.

But each configuration is forbidden in a minimal counterexample.

- (C0) $\Delta/2 + 4 - \lceil (\Delta - 1)/2 \rceil - 1 \geq 4$
- (C1) $\Delta/2 + 4 - \lceil ((\Delta - 1) + (5 - 1))/2 \rceil - 2$
Proof of part (a): Coloring

**Thm:** If $G$ is planar and $g \geq 5$, then $\ell c_\ell(G) \leq \lceil \Delta/2 \rceil + 4$.

**Pf:** By the lemma, $G$ contains one of the 3 configurations:

(C0) a 1-vertex,

(C1) a 2-vertex adjacent to a 5-valent-vertex, or

(C2) a 5-face with 4 incident 3-vertices and an incident 5-valent-vertex.

But each configuration is forbidden in a minimal counterexample.

(C0) \[ \lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1)/2 \rfloor - 1 \geq 4 \]

(C1) \[ \lceil \Delta/2 \rceil + 4 - \lfloor ((\Delta - 1) + (5 - 1))/2 \rfloor - 2 \geq 1 \]
Proof of part (a): Coloring

**Thm:** If $G$ is planar and $g \geq 5$, then $1c_{\ell}(G) \leq \lceil \Delta/2 \rceil + 4$.

**Pf:** By the lemma, $G$ contains one of the 3 configurations:

(C0) a 1-vertex,

(C1) a 2-vertex adjacent to a 5$^-$-vertex, or

(C2) a 5-face with 4 incident 3-vertices and an incident 5$^-$-vertex.

But each configuration is forbidden in a minimal counterexample.

(C0) $\lceil \Delta/2 \rceil + 4 - \lceil (\Delta - 1)/2 \rceil - 1 \geq 4$

(C1) $\lceil \Delta/2 \rceil + 4 - \lceil ((\Delta - 1) + (5 - 1))/2 \rceil - 2 \geq 1$

(C2)
Proof of part (a): Coloring

**Thm:** If $G$ is planar and $g \geq 5$, then $1c_\ell(G) \leq \lceil \Delta/2 \rceil + 4$.

**Pf:** By the lemma, $G$ contains one of the 3 configurations:

1. (C0) a 1-vertex,
2. (C1) a 2-vertex adjacent to a 5$^-$-vertex, or
3. (C2) a 5-face with 4 incident 3-vertices and an incident 5$^-$-vertex.

But each configuration is forbidden in a minimal counterexample.

- (C0) $\lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1)/2 \rfloor - 1 \geq 4$
- (C1) $\lceil \Delta/2 \rceil + 4 - \lfloor ((\Delta - 1) + (5 - 1))/2 \rfloor - 2 \geq 1$
- (C2) at $w$ and $x$: $\lceil \Delta/2 \rceil + 4 - \lfloor ((\Delta - 1) + 2 + 2)/2 \rfloor - 2$
Proof of part (a): Coloring

**Thm:** If $G$ is planar and $g \geq 5$, then $\ell c_\ell(G) \leq \lceil \Delta/2 \rceil + 4$.

**Pf:** By the lemma, $G$ contains one of the 3 configurations:

(C0) a 1-vertex,

(C1) a 2-vertex adjacent to a 5-vertex, or

(C2) a 5-face with 4 incident 3-vertices and an incident 5-vertex.

But each configuration is forbidden in a minimal counterexample.

(C0) \[ \lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1)/2 \rfloor - 1 \geq 4 \]

(C1) \[ \lceil \Delta/2 \rceil + 4 - \lfloor ((\Delta - 1) + (5 - 1))/2 \rfloor - 2 \geq 1 \]

(C2) at $w$ and $x$: \[ \lceil \Delta/2 \rceil + 4 - \lfloor ((\Delta - 1) + 2 + 2)/2 \rfloor - 2 \geq 1 \]
Proof of part (a): Coloring

**Thm:** If $G$ is planar and $g \geq 5$, then $1c_\ell(G) \leq \lceil \Delta/2 \rceil + 4$.

**Pf:** By the lemma, $G$ contains one of the 3 configurations:

(C0) a 1-vertex,
(C1) a 2-vertex adjacent to a 5$^-$-vertex, or
(C2) a 5-face with 4 incident 3-vertices and an incident 5$^-$-vertex.

But each configuration is forbidden in a minimal counterexample.

(C0) \[ \lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1)/2 \rfloor - 1 \geq 4 \]

(C1) \[ \lceil \Delta/2 \rceil + 4 - \lfloor ((\Delta - 1) + (5 - 1))/2 \rfloor - 2 \geq 1 \]

(C2) at $w$ and $x$: \[ \lceil \Delta/2 \rceil + 4 - \lfloor ((\Delta - 1) + 2 + 2)/2 \rfloor - 2 \geq 1 \]

If $c(y) = c(z)$:

![Diagram of a 5-face with 4 incident 3-vertices and an incident 5$^-$-vertex.]
Proof of part (a): Coloring

**Thm:** If $G$ is planar and $g \geq 5$, then $l_{c_e}(G) \leq \lceil \Delta/2 \rceil + 4$.

**Pf:** By the lemma, $G$ contains one of the 3 configurations:

(C0) a 1-vertex,
(C1) a 2-vertex adjacent to a $5^-$-vertex, or
(C2) a 5-face with 4 incident 3-vertices and an incident $5^-$-vertex.

But each configuration is forbidden in a minimal counterexample.

(C0) $\lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1)/2 \rfloor - 1 \geq 4$

(C1) $\lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1 + (5 - 1))/2 \rfloor - 2 \geq 1$

(C2) at $w$ and $x$: $\lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1 + 2 + 2)/2 \rfloor - 2 \geq 1$

If $c(y) = c(z)$: color $w$, then $x$.
Proof of part (a): Coloring

**Thm:** If $G$ is planar and $g \geq 5$, then $\ell_c(G) \leq \lceil \Delta/2 \rceil + 4$.

**Pf:** By the lemma, $G$ contains one of the 3 configurations:

(C0) a 1-vertex,
(C1) a 2-vertex adjacent to a 5-vertex, or
(C2) a 5-face with 4 incident 3-vertices and an incident 5-vertex.

But each configuration is forbidden in a minimal counterexample.

(C0) \[\lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1)/2 \rfloor - 1 \geq 4\]
(C1) \[\lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1 + (5 - 1))/2 \rfloor - 2 \geq 1\]
(C2) at $w$ and $x$: \[\lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1 + 2 + 2)/2 \rfloor - 2 \geq 1\]

If $c(y) = c(z)$: color $w$, then $x$.
If $c(y) \neq c(z)$:
Proof of part (a): Coloring

**Thm:** If $G$ is planar and $g \geq 5$, then $\ell_1(G) \leq \lceil \Delta/2 \rceil + 4$.

**Pf:** By the lemma, $G$ contains one of the 3 configurations:

(C0) a 1-vertex,

(C1) a 2-vertex adjacent to a $5^-$-vertex, or

(C2) a 5-face with 4 incident 3-vertices and an incident $5^-$-vertex.

But each configuration is forbidden in a minimal counterexample.

(C0) $\lceil \Delta/2 \rceil + 4 - \lceil (\Delta - 1)/2 \rceil - 1 \geq 4$

(C1) $\lceil \Delta/2 \rceil + 4 - \lceil (\Delta - 1 + (5 - 1))/2 \rceil - 2 \geq 1$

(C2) at $w$ and $x$: $\lceil \Delta/2 \rceil + 4 - \lceil (\Delta - 1 + 2 + 2)/2 \rceil - 2 \geq 1$

If $c(y) = c(z)$: color $w$, then $x$.

If $c(y) \neq c(z)$: color $x$, then $w$. 
Proof of part (a): Coloring

**Thm:** If $G$ is planar and $g \geq 5$, then $\ell_{c}(G) \leq \lceil \Delta/2 \rceil + 4$.

**Pf:** By the lemma, $G$ contains one of the 3 configurations:

(C0) a 1-vertex,

(C1) a 2-vertex adjacent to a 5−-vertex, or

(C2) a 5-face with 4 incident 3-vertices and an incident 5−-vertex.

But each configuration is forbidden in a minimal counterexample.

(C0) \[ \lceil \Delta/2 \rceil + 4 - \lfloor (\Delta - 1)/2 \rfloor - 1 \geq 4 \]

(C1) \[ \lceil \Delta/2 \rceil + 4 - \lfloor ((\Delta - 1) + (5 - 1))/2 \rfloor - 2 \geq 1 \]

(C2) at $w$ and $x$: \[ \lceil \Delta/2 \rceil + 4 - \lfloor ((\Delta - 1) + 2 + 2)/2 \rfloor - 2 \geq 1 \]

If $c(y) = c(z)$: color $w$, then $x$.

If $c(y) \neq c(z)$: color $x$, then $w$.

So the theorem is true.
Sketch of part (c)

**Thm:** If $\text{mad}(G) < 12/5$ and $\Delta \geq 3$, then $\text{lcl}(G) = \lceil \Delta/2 \rceil + 1$. 
Sketch of part (c)

**Thm:** If \( \text{mad}(G) < 12/5 \) and \( \Delta \geq 3 \), then \( l_{c\ell}(G) = \lceil \Delta/2 \rceil + 1 \).

**Pf Sketch:** Use discharging, with \( \mu(v) = d(v) - 12/5 \) to show:
**Thm:** If $\text{mad}(G) < 12/5$ and $\Delta \geq 3$, then $\ell(G) = \lceil \Delta/2 \rceil + 1$.

**Pf Sketch:** Use discharging, with $\mu(v) = d(v) - 12/5$ to show:

**Lemma:** If $\text{mad}(G) < 12/5$, $\delta(G) = 2$, and $\Delta(G) \geq 3$, then $G$ contains one of the following 4 configurations.
Sketch of part (c)

**Thm:** If $\text{mad}(G) < 12/5$ and $\Delta \geq 3$, then $l_{c\ell}(G) = \lceil \Delta/2 \rceil + 1$.

**Pf Sketch:** Use discharging, with $\mu(v) = d(v) - 12/5$ to show:

**Lemma:** If $\text{mad}(G) < 12/5$, $\delta(G) = 2$, and $\Delta(G) \geq 3$, then $G$ contains one of the following 4 configurations.

![Diagrams of configurations](image-url)
Thm: If $\text{mad}(G) < 12/5$ and $\Delta \geq 3$, then $l_{c\ell}(G) = \lceil \Delta/2 \rceil + 1$.

Pf Sketch: Use discharging, with $\mu(v) = d(v) - 12/5$ to show:

Lemma: If $\text{mad}(G) < 12/5$, $\delta(G) = 2$, and $\Delta(G) \geq 3$, then $G$ contains one of the following 4 configurations.

And each of these configurations is forbidden in a minimal counter.
**Thm:** If $\text{mad}(G) < 12/5$ and $\Delta \geq 3$, then $\ell_c(G) = \lceil \Delta/2 \rceil + 1$.

**Pf Sketch:** Use discharging, with $\mu(v) = d(v) - 12/5$ to show:

**Lemma:** If $\text{mad}(G) < 12/5$, $\delta(G) = 2$, and $\Delta(G) \geq 3$, then $G$ contains one of the following 4 configurations.

And each of these configurations is forbidden in a minimal counter.

**So the theorem is true!**