Reducibility and Discharging: An Introduction by Example

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Joint with Craig Timmons and André Kündgen
The 4-Color Theorem

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- Reproved in 1996 by Robertson, Sanders, Seymour, Thomas.
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![Graph diagram](image)

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Definitions and Examples

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Every planar $G$ has acyclic chromatic number, $\chi_a(G)$, at most 9.
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**Thm.** [Borodin 1979]
Every planar $G$ has acyclic chromatic number, $\chi_a(G)$, at most 5.
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**Thm.** [Fetin-Raspaud-Reed 2001]
Every planar $G$ has star chromatic number $\chi_s(G)$, at most 80.
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![Illustration of acyclic coloring](image1.png)

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![Illustration of star coloring](image2.png)

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![Diagram of two connected vertices]
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2. **Partition** $G - \{u, v, w\}$.
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   - Or put $u, v, w$ into $F$. 
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- **Partition $G - v$.**
  - Put $v$ into $F$.

- **Partition $G - \{u, v, w\}$.**
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- **Partition $G - H$.**
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   - Or put \( u, v, w \) into \( F \).

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   - Put \( w \) into \( I \) and others into \( F \).
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- Partition $G - H$. Put $w$ into $I$ and others into $F$. Or $v$ into $I$ and others into $F$. Or all into $F$. 

![Diagram](image-url)
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   Put $w$ into $I$ and others into $F$.
   Or $v$ into $I$ and others into $F$.
   Or all into $F$.

“nearby” 2-vertices
Discharging

Give charge $2l(f) - 28$ to each face $f$ and charge $12d(v) - 28$ to each vertex $v$. 
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Since girth $\geq 14$, each face has nonnegative charge.
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$$\sum_{v \in V} (12d(v) - 28) + \sum_{f \in F} (2l(f) - 28) = 28(|E| - |F| - |V|) = -56$$
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nonnegative
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- negative
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Discharging rule: each 2-vert receives 2 from each nearby 3\(^+\)-vert.
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Show each vertex has nonnegative charge.
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2-vert: $12(2) - 28 + 2(2) = 0$
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**Discharging rule:** each 2-vert receives 2 from each nearby 3\(^+-\)-vert.

Show each vertex has nonnegative charge.

2-vert: \(12(2) - 28 + 2(2) = 0\)

3-vert: \(12(3) - 28 - 4(2) = 0\)
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Since girth $\geq 14$, each face has nonnegative charge.

$$\sum_{v \in V} (12d(v) - 28) + \sum_{f \in F} (2l(f) - 28) = 28(|E| - |F| - |V|) = -56$$

**Discharging rule:** each 2-vert receives 2 from each nearby 3$^+$-vert.

Show each vertex has nonnegative charge.

2-vert: $12(2) - 28 + 2(2) = 0$

3-vert: $12(3) - 28 - 4(2) = 0$

4$^+$-vert: $12d(v) - 28 - 2d(v)2 = 8d(v) - 28 > 0$
Discharging

Give charge \(2l(f) - 28\) to each face \(f\) and charge \(12d(v) - 28\) to each vertex \(v\).

Since girth \(\geq 14\), each face has nonnegative charge.

\[
\sum_{v \in V}(12d(v) - 28) + \sum_{f \in F}(2l(f) - 28) = 28(|E| - |F| - |V|) = -56
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**Contradiction!** So \(G\) contains a reducible configuration.
An Efficient Coloring Algorithm
An Efficient Coloring Algorithm

Many discharging proofs translate into linear-time algorithms.
An Efficient Coloring Algorithm

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Generalization
An Efficient Coloring Algorithm

Many discharging proofs translate into linear-time algorithms.

Generalization

\[ \sum 12d(v) - 28 < 0 \]
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Many discharging proofs translate into linear-time algorithms.

Generalization

$$\sum 12d(v) - 28 < 0 \Rightarrow mad(G) < \frac{28}{12}$$
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Generalization

\[ \sum 12d(v) - 28 < 0 \Rightarrow mad(G) < \frac{28}{12} \]

**Thm.** If \( mad(G) < \frac{28}{12} \), then we can partition \( V(G) \) into sets \( I \) and \( F \) s.t. \( G[F] \) is a forest and \( I \) is a 2-independent set in \( G \).
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Generalization

\[ \sum 12d(v) - 28 < 0 \implies mad(G) < \frac{28}{12} \]

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Open Questions

- What is the minimum girth \( g \) s.t. \( G \) planar and girth \( \geq g \) implies an \( I,F \)-partition?
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Generalization

$$\sum 12d(v) - 28 < 0 \Rightarrow mad(G) < \frac{28}{12}$$

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- What is the minimum girth $g$ s.t. $G$ planar and girth $\geq g$ implies an $I, F$-partition?
  
  We know that $8 \leq g$
An Efficient Coloring Algorithm

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- What is the minimum girth \( g \) s.t. \( G \) planar and girth \( \geq g \) implies an \( I, F \)-partition?
  We know that \( 8 \leq g \leq 13 \)

- What is the minimum girth \( g \) s.t. \( G \) planar and girth \( \geq g \) implies \( \chi_s(G) \leq 4 \)?
An Efficient Coloring Algorithm

Many discharging proofs translate into linear-time algorithms.

Generalization

\[ \sum 12d(v) - 28 < 0 \Rightarrow \text{mad}(G) < \frac{28}{12} \]

**Thm.** If \( \text{mad}(G) < \frac{28}{12} \), then we can partition \( V(G) \) into sets \( I \) and \( F \) s.t. \( G[F] \) is a forest and \( I \) is a 2-independent set in \( G \).

Open Questions

- What is the minimum girth \( g \) s.t. \( G \) planar and girth \( \geq g \) implies an \( I,F \)-partition?
  
  We know that \( 8 \leq g \leq 13 \)

- What is the minimum girth \( g \) s.t. \( G \) planar and girth \( \geq g \) implies \( \chi_s(G) \leq 4 \)?

- For an arbitrary surface \( S \), what is the minimum \( \gamma_S \) s.t. girth \( \geq \gamma_S \) and \( G \) embedded in \( S \) implies an \( I,F \)-partition?