Detecting a Machine Failure in a Network, a.k.a. Vertex Identifying Codes

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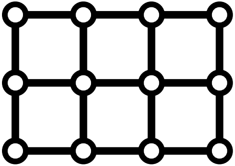
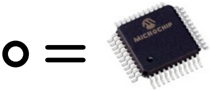
Joint with Gexin Yu

Applications of Graph Theory
Joint Math Meetings, San Francisco
13 January 2010

Definitions and Motivation





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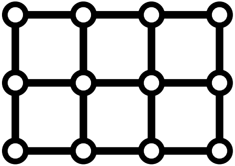

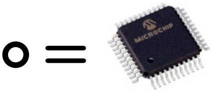

Goal: put sensors in the network to detect which machine failed

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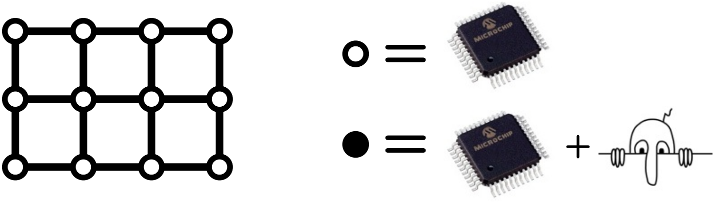

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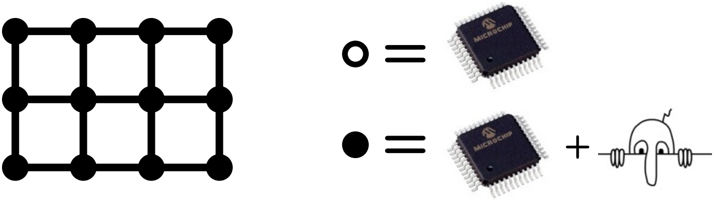

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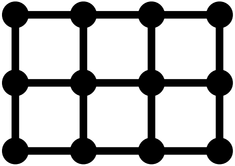

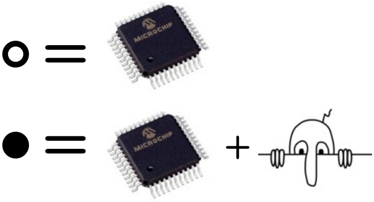

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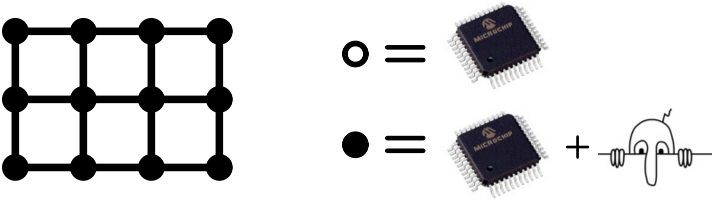

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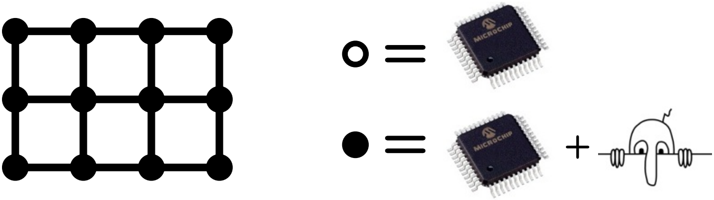

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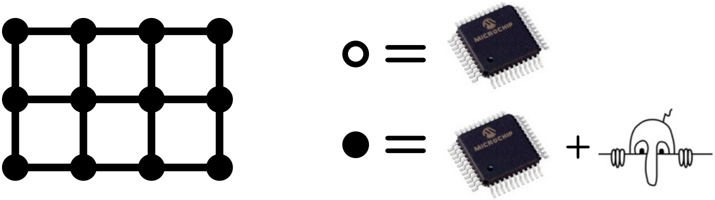

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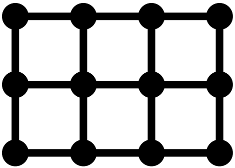

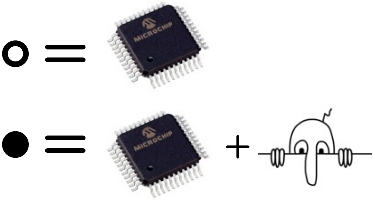

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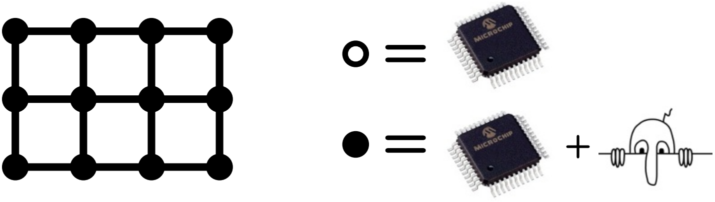

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Definition: We call such a set $\mathcal{C}$ a (vertex identifying) code.

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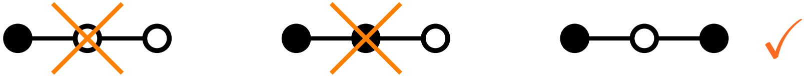

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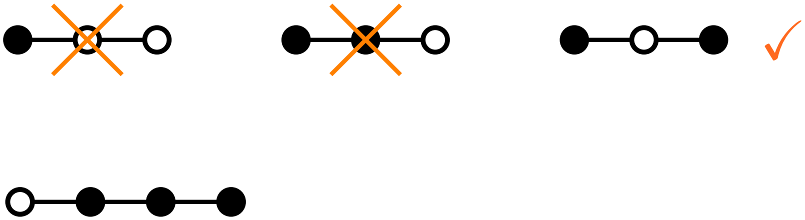

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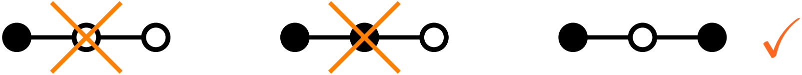

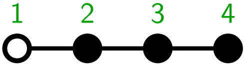

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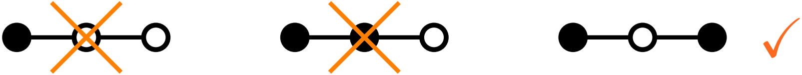

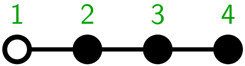

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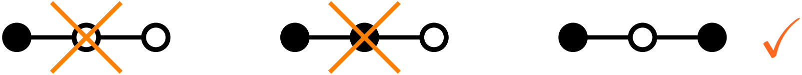

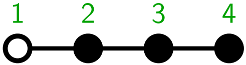

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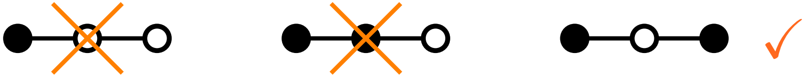

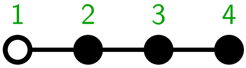

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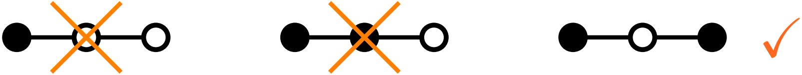

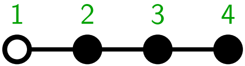

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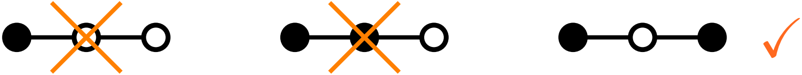

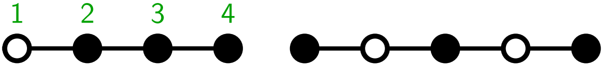

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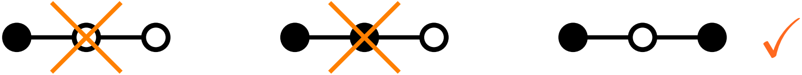

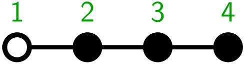

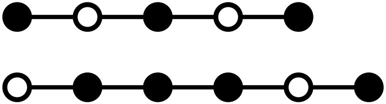

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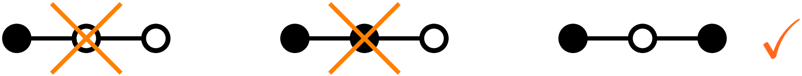

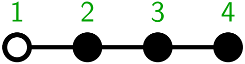

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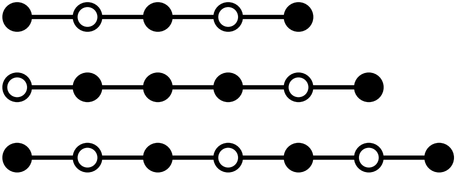

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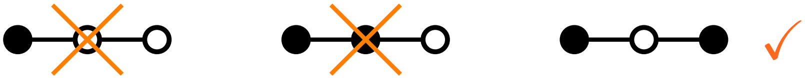

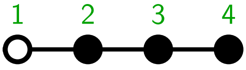

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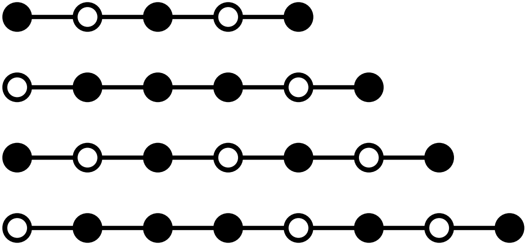

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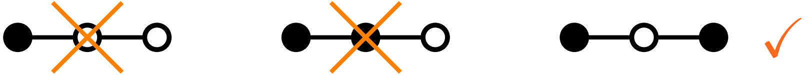

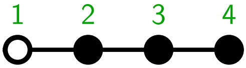

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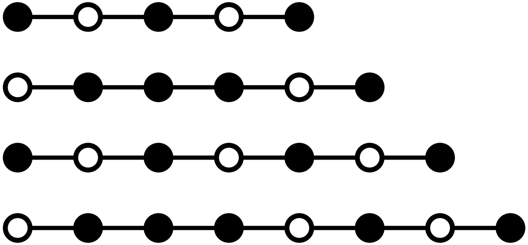
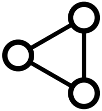


Observation: Every path has a code.

Finding the Right Problem



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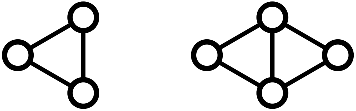

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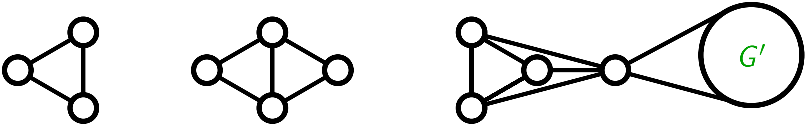

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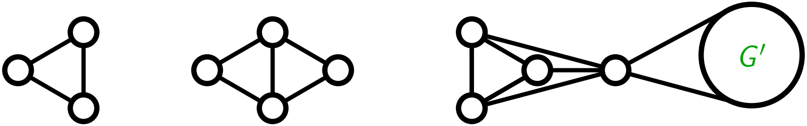

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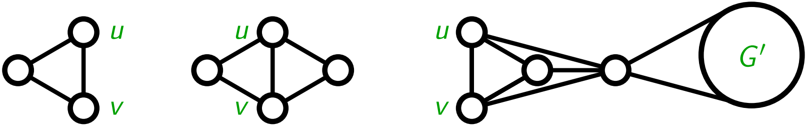

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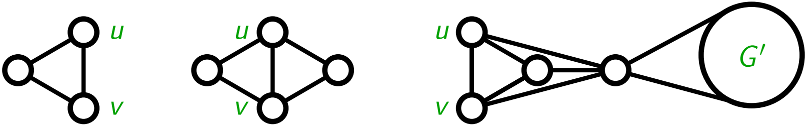

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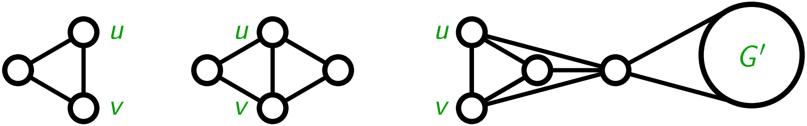

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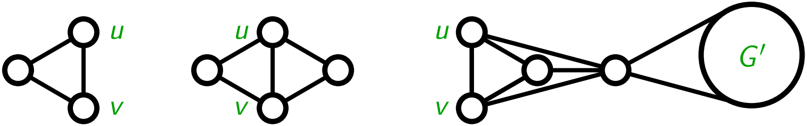

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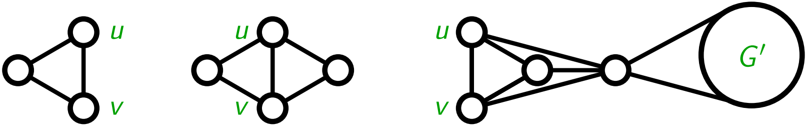

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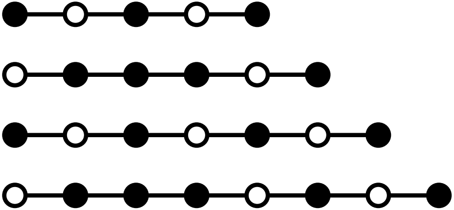

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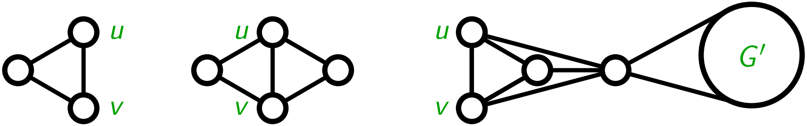

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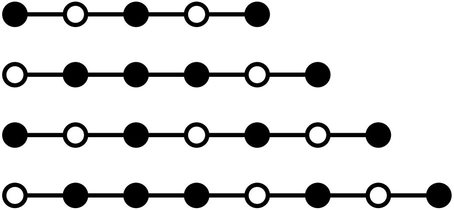

Exer. Show that min size of code for path on k nodes is $\lceil \frac{k+1}{2} \rceil$.

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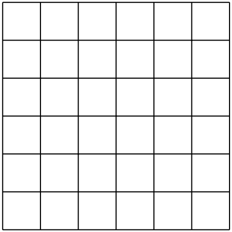

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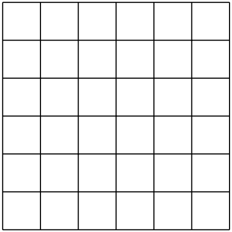

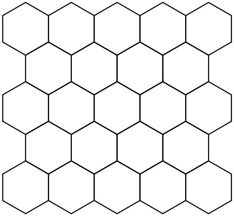

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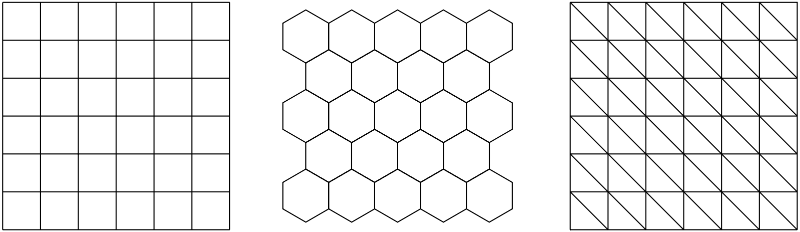

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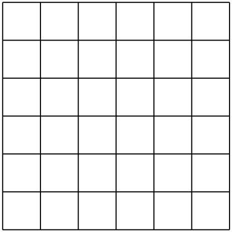

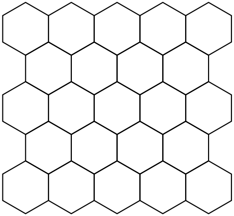

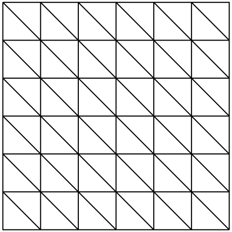

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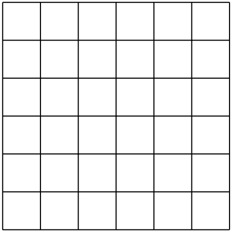

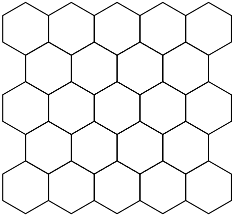

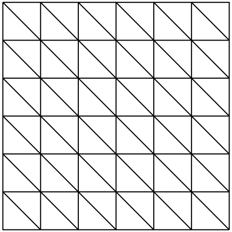

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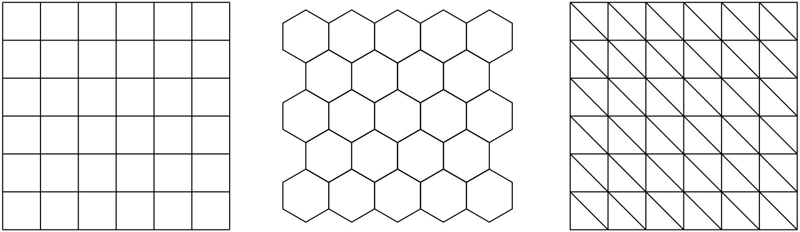

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Question: What is $\tau(G_{\mathbb{Z}})$?

Proving a Lower Bound (sketch)

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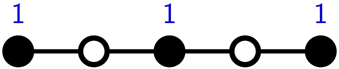

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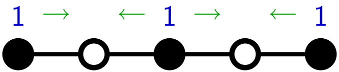

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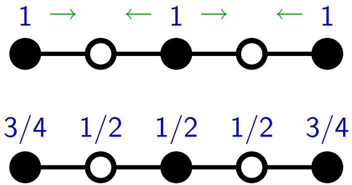

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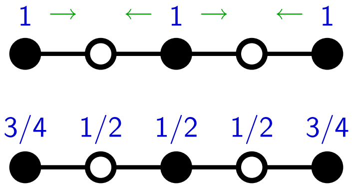

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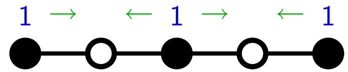

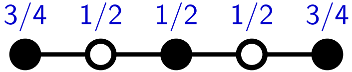

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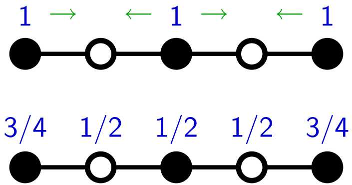

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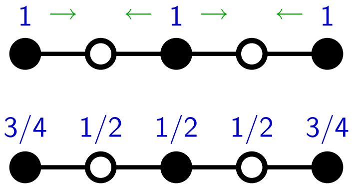

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Proof. Each vertex in $\mathcal{C}$ gives $\frac{2}{5k}$ cake to each neighbor not in $\mathcal{C}$ that has k neighbors in $\mathcal{C}$.

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Theorem: For the hex grid, $\tau \geq \frac{12}{29}$.

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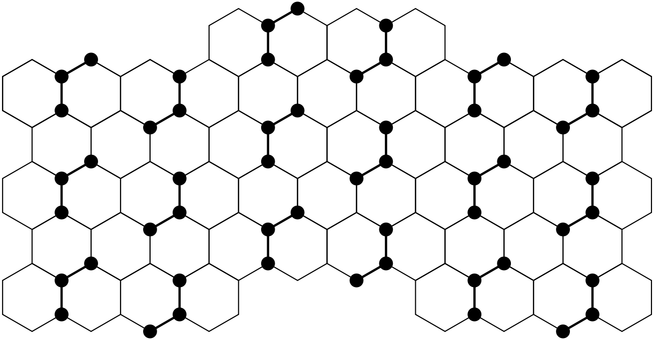

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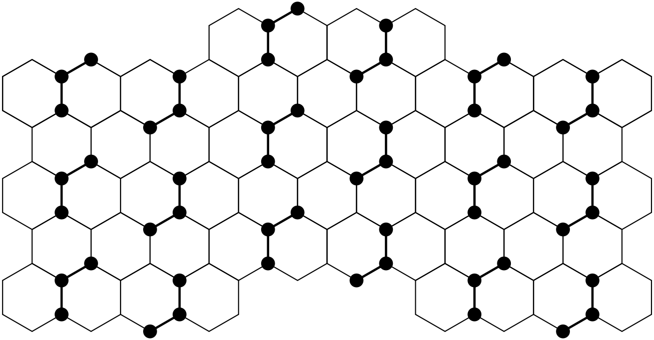

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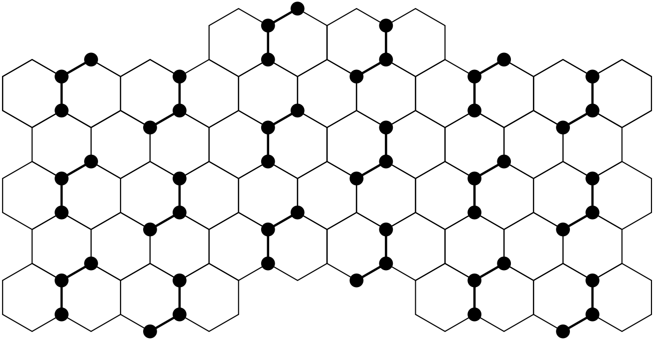

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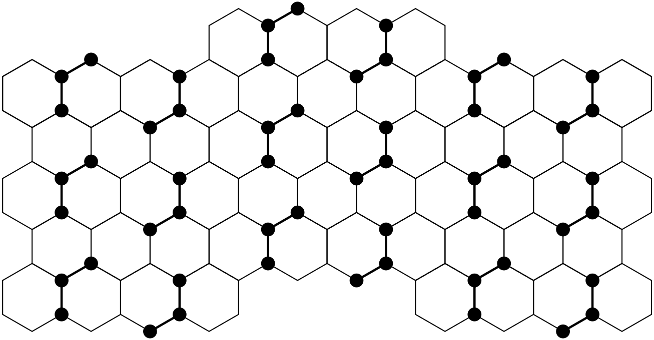

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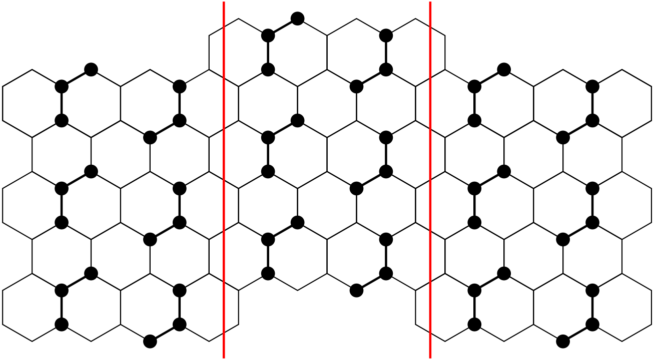

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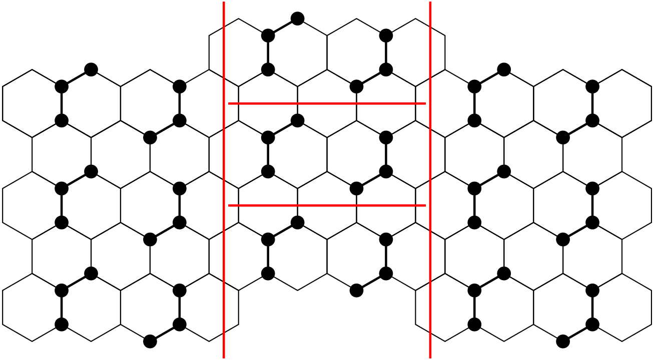

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