Chain-making games in grid-like posets

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Abstract

We study the Maker-Breaker game on the hypergraph of chains of fixed size in a poset. In a product of chains, the maximum size of a chain that Maker can guarantee building is $k - \lfloor r/2 \rfloor$, where k is the maximum size of a chain in the product, and r is the maximum size of a factor chain. We also study a variant in which Maker must follow the chain in order, called the *Walker-Blocker game*. In the poset consisting of the bottom k levels of the product of d arbitrarily long chains, Walker can guarantee a chain that hits all levels if $d \geq 14$; this result uses a solution to Conway's Angel-Devil game. When $d = 2$, the maximum that Walker can guarantee is only 2/3 of the levels, and 2/3 is asymptotically achievable in the product of two equal chains.

1 Introduction

The *Maker-Breaker* game on a hypergraph $\mathcal{H}$ is played by Maker and Breaker, who alternate turns (beginning with Maker). Each player moves by choosing a previously unchosen vertex of $\mathcal{H}$. Maker wins by acquiring all vertices of some edge of $\mathcal{H}$; Breaker wins if all vertices are chosen without Maker winning.

Maker-Breaker games have been studied for many hypergraphs, particularly when the vertices are the edges of an n-vertex complete graph. In that setting, when n is large enough Maker can build a spanning cycle, a complete subgraph with q vertices, a spanning

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k-connected subgraph, or various other structures. For an introduction to Maker-Breaker games (and more general positional games), see the surveys [1] and [2] or the book [3]. Recent work has concentrated on finding "efficient strategies", winning the game as quickly as possible (see [4, 7]).

In this paper, we study the *chain game* on posets, where the winning sets are the chains with a given size. For every poset there is a maximum size of chain that Maker can build against optimal play by Breaker; we seek this value. For special posets whose elements are integer d-tuples, we give efficient strategies for Maker that do not waste any move (every element that Maker selects is in the chain constructed).

Chains in posets are ordered from bottom to top, so a natural variant of the chain game is the *ordered chain game*, in which the chain must be built from bottom to top. To distinguish this from the (unordered) chain game, we call its players *Walker* and *Blocker*.

The *product* of d chains with sizes $r_1, \ldots, r_d$, written $\prod_{i=1}^{d} \mathbf{r}_i$, is the set of d-tuples x such that $0 \leq x_i < r_i$ for $1 \leq i \leq d$, ordered by $x \preceq y$ if and only if $x_i \leq y_i$ for $1 \leq i \leq d$. The *d-dimensional k-wedge*, written $\mathcal{W}_k^d$, is the subposet of $\mathbf{k}^d$ consisting of the nonnegative-integer d-tuples with sum less than k, under the coordinate-wise order.

For the chain game and ordered chain game on these posets, we prove two main results.

Theorem 1.1. *Let P be a product of d chains, with r being the maximum size among these chains and k being the maximum size of a chain in P. In the chain game on P, Maker can build a chain of size $k - \lfloor r/2 \rfloor$, and Breaker can prevent Maker from building a larger chain.*

Theorem 1.2. *If $d \geq 14$, and $k \in \mathbb{N}$, then in the ordered chain game on $\mathcal{W}_k^d$, Walker can build a chain of size k (hitting all levels).*

In small dimensions, Walker cannot guarantee hitting all levels. In particular, when $d = 2$ Walker can only get two-thirds of the levels. Somewhere between dimension 2 and dimension 14 there is a "phase transition" after which Walker can hit all levels. We do not know the value where this occurs, but we conjecture that Walker wins already when $d = 3$.

We begin in Section 2 with the Maker-Breaker chain game on products of chains. The remainder of the paper addresses the Walker-Blocker game. In Section 3 we study the 2-dimensional case for both wedges and grids (products of equal chains). For $\mathcal{W}_k^2$, the maximum size of a chain that Walker can guarantee building is $\lceil 2k/3 \rceil$. In the product of two chains of equal size, which is contained in a 2-dimensional wedge, Walker can still guarantee asymptotically 2/3 of the levels; we prove this using a "potential function" argument. Section 4 relates the Walker-Blocker game to Conway's famous Angel-Devil game. We apply this relationship in Section 5 to prove Theorem 1.2. We conjecture that the conclusion of Theorem 1.2 holds in fact for $d \geq 3$. Finally, Section 6 addresses the biased game in which Blocker makes b moves after each move by Walker.

2 Maker-Breaker on Chain-products

In a product of chains, we use *level* ℓ to denote the set of elements whose entries sum to ℓ. A *successor* of a d-tuple x is a d-tuple y such that $x \prec y$. To evoke familiar terminology from games on physical boards, we refer to an element chosen at a particular time as a *move* and say that the player *plays* that move at that time.

In order to solve the Maker-Breaker game on products of finite chains, we first solve the Walker-Blocker game on products of 2-element chains. We then apply this lemma to build an optimal strategy for Maker in the unordered chain game on arbitrary finite chain-products. Let $[d] = \{1, \ldots, d\}$. There is a natural isomorphism from $\mathbf{2}^d$ to the lattice of subsets of $[d]$ in which each binary d-tuple x is mapped to $\{i\colon\ x_i = 1\}$.

Lemma 2.1. *For $d \geq 2$, let P' be the poset obtained from $\mathbf{2}^d$ by deleting the top element and the bottom element. The maximum size of a chain in P' is $d-1$, and Walker can build a chain of size $d-1$ in P', even if Blocker moves first.*

Proof. We refer to the elements of P' by the corresponding subsets in $\{1, \ldots, d\}$. On his kth turn, Walker plays a move S at level k such that (i) S is above all his previous moves and (ii) Blocker has played no successor of S. Successfully executing this strategy for $d-1$ turns builds a chain of size $d-1$.

Let S be the previous move by Walker. If Blocker responded with a move not a successor of S, then Walker can add any element to S. Otherwise, since the highest level of P' is $d-1$, the move by Blocker omits some $e \in [d]$. Walker now plays $S \cup \{e\}$ and restores the property that no successor of the current move has been played. $\square$

Since Walker can build a chain hitting all levels in P', we conclude also that Maker can build a chain hitting all levels in the unordered game. The latter statement, along with the freedom to let Breaker move first, is what we need to analyze arbitrary chain-products. Since Maker need not take the elements of a chain in order, Maker can build chains independently in different copies of the poset P' in Lemma 2.1; they will combine to form a large chain.

To show optimality of the resulting strategy for Maker, we present a strategy for Breaker. Since every chain in a chain-product is contained in a longest chain, it suffices to give a pairing strategy for Breaker that guarantees blocking enough of every longest chain.

Theorem 2.2. *Let $P = \prod_{i=1}^{d} \mathbf{r}_i$. In the unordered chain game on P with $r = \max_i r_i$, Maker can guarantee building a chain of size $k - \lfloor r/2 \rfloor$, where k is the maximum size of a chain in P, and Breaker can keep Maker from building any larger chain.*

Proof. The elements of P are the d-tuples x such that $0 \leq x_i < r_i$ for $1 \leq i \leq d$. By symmetry, we may assume that $r = r_d = \max_i r_i$. For $0 \leq j < r$, let z_j be the element of P

whose ith coordinate is $\min(j, r_i - 1)$. For $1 \le j < r$, let A_j be the subposet of P consisting of all x such that $z_{j-1} \prec x \prec z_j$. Note that $z_0 \prec \cdots \prec z_{r-1}$, that $A_1, \ldots, A_{r-1}$ are pairwise disjoint, and that each A_j is isomorphic to the poset P' of Lemma 2.1 for some dimension (as j increases beyond some r_i, the dimension decreases). Fig. 1 illustrates $A_1, \ldots, A_{r-1}$.

The key to Maker's strategy is that chains in $A_1, \ldots, A_{r-1}$ combine to form a longer chain in P. Let $Z = \{z_0, \ldots, z_{r-1}\}$; these are the bold elements in Fig. 1. Maker begins by playing an element in Z. When Breaker plays an element in Z, Maker responds by playing another element in Z if one is available. Maker treats each subposet A_j as an instance of the poset P' of Lemma 2.1; when Breaker plays in A_j, Maker responds using the strategy of Lemma 2.1. When Breaker plays any other move, Maker plays to increase the chain in some A_j or Z.

By Lemma 2.1, Maker obtains in each A_j a chain hitting all levels. These combine with Z to form a long chain in P. The only levels that Maker misses are those containing an element of Z played by Breaker. Maker's strategy ensures that Breaker plays at most $\lfloor |Z|/2 \rfloor$ such moves. Since $|Z| = r$, the bound follows.

To prove optimality, Breaker uses a pairing strategy. For all j with $0 \le j < \lfloor r/2 \rfloor$ and all $(x_1, \ldots, x_{d-1})$, Breaker pairs the element $(x_1, \ldots, x_{d-1}, 2j)$ with the element $(x_1, \ldots, x_{d-1}, 2j+1)$. When Maker plays a paired element, Breaker plays its mate; when Maker plays an unpaired element, Breaker responds arbitrarily. We show that Maker misses at least $\lfloor r/2 \rfloor$ elements from every maximal chain.

Given a maximal chain X, let X_j be the subchain of X consisting of all elements whose last coordinate has value j, for $0 \le j \le r-1$. Let x be the last element of X in X_{2m}, and let y be the first element of X in X_{2m+1} (note that X cannot skip any X_j). Since Breaker has paired x with y, Maker misses at least one of these. Because there are $\lfloor r/2 \rfloor$ even integers that are at least 0 and less than $r-1$, the theorem follows. $\square$

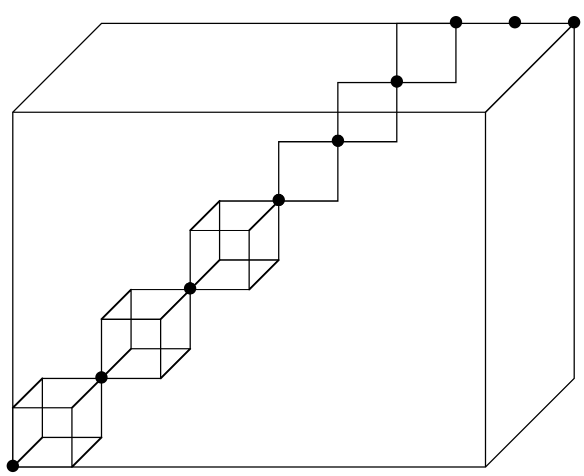

Figure 1: Maker strategy for chain-products

3 Walker-Blocker in Two Dimensions

For the Walker-Blocker game, wedges are easier to analyze than chain-products, because the poset formed by the successors of any element is isomorphic to a truncation of the same wedge by discarding the highest levels. This allows Walker to define a strategy locally. In a chain-product, when the top of a growing chain reaches the maximum in a given coordinate, no further moves in that direction are possible. Breaker may be able to take advantage of Walker being "trapped in a corner". To overcome this, Walker may need to look farther ahead to plan a strategy.

In this section, we first give an exact solution for the Walker-Blocker game on $\mathcal{W}_k^2$. We then express Walker's strategy in a more general way using a "potential function" to show that asymptotically as big a chain can be built in the game on the product of two $(k+1)$-element chains as on the wedge $\mathcal{W}_{2k+1}^2$ that contains it.

Theorem 3.1. *In the ordered chain game on $\mathcal{W}_k^2$, Walker can build a chain of size $\lceil 2k/3 \rceil$ in $\lceil 2k/3 \rceil$ moves. Also, Blocker can prevent Walker from building a larger chain.*

Proof. We present a strategy in which Walker follows a single chain, with no wasted moves. Walker first plays $(0,0)$. For each subsequent move, Walker plays a successor of his previous move; among all unchosen successors, he plays one at the lowest level. A level containing a move by Walker is "won by Walker". After a move (a,b), Walker next plays at level $a+b+1$ unless Blocker has already played both $(a+1,b)$ and $(a,b+1)$. We then say that Blocker wins level $a+b+1$.

Since Blocker spends at least two moves in a level to win it, while Walker spends only one move per level won, the number of levels that have been won by Walker is always at least twice the number won by Blocker. At the end of the game all k levels have been won by one player or the other; hence Walker has won at least $\lceil 2k/3 \rceil$ levels.

For the upper bound, we present a strategy for Blocker to keep Walker from building a larger chain. If Walker's previous move was at (a,b) and exactly one of $(a+1,b)$ and $(a,b+1)$ have been played, then Blocker plays the other. If neither of them has been played, then Blocker plays $(a+1,b+1)$, if available. Otherwise, Blocker plays an arbitrary move.

Once the game has finished, let (a,b) be an element on a largest chain C that was occupied by Walker in order. If when Walker played (a,b) one of $(a+1,b)$ and $(a,b+1)$ had already been played, then C has no element from level $a+b+1$. If C has an element x from level $a+b+1$, then before x was played the element $(a+1,b+1)$ was played by one of the players. Blocker next ensures that the other successor of x at level $a+b+2$ is occupied, thus preventing C from having an element from level $a+b+2$.

We have shown that Blocker's strategy prevents Walker from building a chain in order that hits three consecutive levels. Hence Walker wins at most $\lceil 2k/3 \rceil$ levels. ∎

In efficient strategies, where all moves by Walker form a chain, we refer to the most recent move by Walker as the *head* of the chain. A more global view of the strategy for Walker uses a potential function to measure the difficulty that Walker faces in the levels above the head.

We define a potential function to measure the future levels that Walker may need to skip. Thus Walker wants to keep the potential small. When Walker skips a level, the potential will decrease by 1. Other moves by Walker will not increase the potential. A move by Blocker will increase the potential by at most 1/2. We design such a potential and strategy for $\mathcal{W}^2_{2k+1}$ and use it to show that even when the game is restricted to the subposet $\mathbf{(k+1)}^2$, Walker can still win asymptotically 2/3 of the levels.

Blocker's move at a position (c, d) makes it harder for Walker to win level $c + d$. To measure this difficulty when the head is at (a, b), we define the *influence* of (c, d) on (a, b), where $a \le c$ and $b \le d$, to be $\binom{c'+d'}{d'}2^{-(c'+d')}$, where $(c', d') = (c, d) - (a, b)$ (the influence is 0 if $a > c$ or $b > d$). We write $f_{a,b}(c, d)$ for the influence of (c, d) on (a, b). Define the *potential* $\Phi_{a,b}$ to be the total influence on (a, b) of the moves Blocker has played. Large potential is good for Blocker.

To motivate these definitions, note that the influence of (c, d) on (a, b) equals the probability that a random walk from (a, b) to level $c+d$ will end at position (c, d), where the walk iteratively increases a randomly chosen coordinate by 1. Let (a, b) be the current head of the chain, and let (c, d) be another position. The average of the influences of (c, d) on $(a + 1, b)$ and $(a, b + 1)$ equals the influence of (c, d) on (a, b). Walker will want to choose the option that produces the smaller potential.

Theorem 3.2. *In the ordered chain game on* $\mathbf{(k+1)}^2$*, Walker can build a chain hitting more than* $\frac{2}{3}(2k + 1) - 4\sqrt{k \ln k}$ *of the* $2k + 1$ *levels, and this is asymptotically sharp.*

Proof. Since Blocker can limit Walker to winning $\lceil(2/3)(2k + 1)\rceil$ levels in $\mathcal{W}^2_{2k+1}$, Walker can do no better on the subposet $\mathbf{(k+1)}^2$. Hence it suffices to prove the lower bound.

Consider the game on $\mathcal{W}^2_{2k+1}$. At a given time, let $S(a, b)$ denote the set of elements at or above (a, b) that Blocker has played. Recall that the potential $\Phi_{a,b}$ at a point (a, b) is $\sum_{(c,d)\in S(a,b)} f_{a,b}(c, d)$.

We have noted that always $f_{a,b}(c, d) = \frac{1}{2}\left[f_{a+1,b}(c, d) + f_{a,b+1}(c, d)\right]$. When the head of the chain is at (a, b), we have $(a, b) \notin S(a, b)$, and hence summing over $S(a, b)$ yields $\Phi_{(a,b)} = \frac{1}{2}(\Phi_{(a+1,b)} + \Phi_{(a,b+1)})$. To keep the potential small, Walker wants to move to whichever of $(a + 1, b)$ and $(a, b + 1)$ has smaller potential.

If this strategy directs Walker to play a move (a', b') that Blocker already played (that is, $(a', b') \in S(a, b)$), then Walker simply computes the choice the strategy would make from (a', b') instead. The influence of (a', b') on $\Phi_{a',b'}$ is 1, and by skipping (a', b') this influence is lost. Thus when Walker chooses a successor of (a', b'), the potential decreases by (at least)

1. Walker may skip several levels before the preferred option is available, decreasing the potential by 1 for each level skipped.

Since Blocker cannot play where Walker just played, the increase in potential from Blocker's move is at most 1/2. Since the potential is 0 at the start and the end of the game, Blocker must make at least two moves for every level skipped by Walker. Walker wins a level for each move played, so Walker wins at least twice as many levels as are skipped.

In order to restrict play to $\mathbf{(k+1)}^2$, which is a subposet of $\mathcal{W}^2_{2k+1}$, we grant Blocker initially all moves that are outside $\mathbf{(k+1)}^2$. The key observation, which we formalize below, is that all of these free moves for Blocker increase the potential only by $o(k)$. Since again the potential decreases to 0 at the end, Walker loses at most $o(k)$ more levels than in the original game on $\mathcal{W}^2_{2k+1}$, and hence Walker gets at least the fraction $2/3 - o(1)$ of the $2k+1$ levels. (Here $o(g(k))$ denotes any function of k whose ratio to $g(k)$ tends to 0 as $k \to \infty$.)

To bound the initial influence of the forbidden moves, recall that $f(c,d)$ is the probability that a random walk from $(0,0)$ to level $c+d$ ends at (c,d). The distribution of the endpoint is the standard binomial distribution for $c+d$ trials. Let X_n be the binomial random variable counting the heads in n flips of a fair coin. The initial value of the potential function is $\sum_{n=k+1}^{2k} \Pr(|X_n - n/2| > k - n/2)$. Let $k' = \left\lceil \sqrt{2k \ln k} \right\rceil$ and $n_0 = 2k - k'$. For each n greater than n_0, the probability is at most 1. For $n \le n_0$, we use the well-known Chernoff bound.

The Chernoff bound states that $\Pr(|X_n - n/2| > nt) \le 2e^{-2nt^2}$; we apply it with $t = k/n - 1/2$. Since $2nt^2$ increases as n decreases, we may assume $n = n_0$ and use $2e^{-2n_0t^2}$ as a bound on the contribution from these terms. We have $n_0 = 2k(1-x)$, where $x = k'/2k > \sqrt{\ln k/(2k)}$. Also, $2t = \frac{1}{1-x} - 1 = \frac{x}{1-x}$. We compute

$$2n_0t^2 = \frac{n_0}{2}(2t)^2 = k(1-x)\frac{x^2}{(1-x)^2} > kx^2 > \frac{1}{2}\ln k.$$

Thus $2e^{-2n_0t^2} < 2k^{-1/2}$, bounding the total contribution from these terms by $2\sqrt{k}$. From the k' terms with largest n, the bound on the total is $\left\lceil \sqrt{2k \ln k} \right\rceil$. Hence the initial potential is less than $4\sqrt{k \ln k}$. As a result, Walker misses fewer than $4\sqrt{k \ln k}$ levels in addition to the $(1/3)(2k+1)$ levels missed by the earlier argument. $\square$

4 Angel-Devil Games on Digraphs

In this section, we define a slightly more general version of Conway's famous Angel-Devil game [6]. We show in the next section that this game is closely related to the Walker-Blocker game on $\mathcal{W}^d_k$, and we will translate known results about the Angel-Devil game to apply there.

A *rooted digraph* is a digraph G, possibly with loops, with one vertex designated as the *root* or *start vertex*. The *Angel-Devil game* is played on an infinite rooted digraph G by two players, Angel and Devil. Angel and Devil alternate turns, with Angel moving first. Each vertex of G is either *burned* or *unburned*, with all vertices initially unburned. Angel starts at the root of G. At each turn, Angel moves to an unburned out-neighbor of his current position. At each turn, Devil *burns* at most one vertex, forever denying Angel its use.

Devil wins if Angel is ever unable to move (when all out-neighbors of his current position are burned). Angel wins by having a strategy to move forever. When every vertex of G has finite outdegree, an equivalent statement of the victory condition for Angel is that for every natural number n, Angel has a strategy to guarantee moving for n turns.

A poset is *graded* if all maximal chains joining any two elements have the same length; like those we have discussed, all graded posets have well-defined levels. A *rooted poset* is a graded poset with a unique minimal element x_0 called the *root*. A *k-prefix* is a chain of size k consisting of elements $x_0, \dots, x_{k-1}$ such that x_0 is the root and x_i covers x_{i-1} for $1 \le i \le k$. The top element of a prefix is its *head*; *climbing* a prefix means following it in order.

The *k-prefix game* on a rooted poset P is the Walker-Blocker game in which Walker must climb a k-prefix to win. If Walker wins the k-prefix game on P, then Walker also wins the ordered k-chain game, since a k-prefix is a k-chain with the additional requirements of skipping no levels and starting at the bottom. We say that Blocker wins the *prefix game* on P (without a specified parameter) if Blocker wins the k-prefix game on P for some k.

We prove that the prefix game on a rooted poset P is equivalent to the Angel-Devil game on the rooted cover digraph of P. This is not immediately obvious because, unlike Angel, Walker can backtrack and take an alternative climbing route when blocked. Thus Walker is more powerful than Angel in the corresponding game, and it is easy to obtain a winning strategy for Walker from a winning strategy for Angel.

Theorem 4.1. *Let P be a rooted poset with minimal element x_0, and assume that every element of P is covered by finitely many elements. Let G be the rooted digraph with start vertex x_0 that is the cover digraph of P. Blocker wins the prefix game on P if and only if Devil wins the Angel-Devil game on G.*

Proof. Walker can copy a winning Angel strategy. Both begin at x_0 and thereafter remain at corresponding vertices. Walker can treat a move by Blocker as a move by Devil in the Angel-Devil game, using Angel's response as a Walker move in the prefix game. This keeps Walker at the same vertex as Angel, so the process continues.

To prove the converse, we obtain a winning strategy for Blocker from a winning strategy for Devil. Imagine Walker as starting with an infinite stack of Angels at x_0. Walker will maintain having an infinite stack of Angels at each element of each prefix that has been climbed. When Walker extends a prefix, he splits the stack at the previous head into two infinite stacks and sends one to the new head. Blocker examines the history of where the

Angels in the new stack have been and responds as Devil would if a lone Angel had followed that path. Since the stack moved along that path, all vertices Devil needed to defend against that Angel have been burned, and hence Blocker has played them all. Hence Blocker always has all the moves needed to block all prefixes started by Walker.

However, Walker may play a move that extends more than one prefix. The coalescing stacks have different histories. Different moves may be needed to maintain blocking those different Angels, but Blocker can play only one of them. It suffices for Blocker to pick any one of the Angel histories that reach the new head and copy the Devil's move to block that Angel, absorbing the coalescing stacks into that one stack. The reason is that all those Angels are now in the same position and move together. All moves needed to block any one of them via the Devil strategy are in place, so Devil/Blocker can continue blocking the Angel with the chosen history wherever it goes. The moves that have been played to block Angels on the other paths reaching this position are bonuses for Blocker and can be ignored.

This strategy may direct Blocker to play a move that has already been played by Blocker or Walker; in either case, Blocker can play arbitrarily. When Blocker is directed to play a position x already played by Walker, the moves are already in place to block all prefixes using x, so Blocker has no need for x.

Since Devil has a winning strategy and Blocker can employ it against Angels sitting at all heads of prefixes, simultaneously, Walker cannot play arbitrarily long prefixes. $\square$

It is difficult to devise winning strategies in Angel-Devil games. To benefit from the few explicit strategies that are known, we seek ways to transfer these strategies between games. We define a type of map from one rooted digraph to another that facilitates such a transfer.

Definition 4.2. Let G and H be digraphs with roots g_0 and h_0. A *robust map* from G to H is a map $\phi\colon V(G) \to V(H)$ with $\phi(g_0) = h_0$ such that whenever there is an edge from $\phi(v)$ to w in H, there is also some vertex z in G such that $\phi(z) = w$ and $vz \in E(G)$.

Informally, a map is robust if, whenever the image $\phi(P)$ in H of a path P in G can be extended, P can also be extended to P' in G so that $\phi(P')$ is the extended path in H.

Theorem 4.3. *Let G and H be two rooted digraphs, and let $\phi\colon G \to H$ be a robust map from G to H. If Angel wins the Angel-Devil game in H, then Angel also wins in G.*

Proof. Given a winning strategy for Angel in H, we define a winning strategy in G. We play an imaginary game in H to track and simulate the actual game in G. The G-Angel will maintain a position in G that maps under ϕ to the current position of the H-Angel in H. This holds initially, since they both start at the root.

At some time later, let v be the location of the G-Angel, so the imagined H-Angel is at $\phi(v)$. The G-Devil moves by burning some vertex y in G. The imagined H-Devil burns the corresponding vertex $\phi(y)$. The imagined H-Angel has a winning response w for this move.

Since w must be an out-neighbor of the current vertex $\phi(v)$ in H, the robustness of ϕ guarantees a vertex z in G such that $vz \in E(G)$ and $\phi(z) = w$. The vertex z cannot previously have been burned by the G-Devil, because the imagined H-Devil would have immediately burned w in H to copy that move. Since z is available and $vz \in E(G)$, the G-Angel can move to z. This preserves the property that the H-Angel is on the image of the position of the G-Angel, and the game continues. Since the H-Angel can move forever, the G-Angel also can move forever. $\square$

5 Walker-Blocker on High-dimensional Wedges

In this section we prove that Walker wins the prefix game on wedge posets of dimension at least 14. The *d-dimensional wedge*, written $\mathcal{W}^d$, is the cover digraph of the infinite wedge poset in d dimensions. The vertices are nonnegative integer d-tuples, with $xy \in E(\mathcal{W}^d)$ if y is obtained from x by increasing one coordinate by 1. The root is $(0, \ldots, 0)$.

We compare the wedge with the "power-2" Angel-Devil game on $\mathbb{Z}^2$. An Angel of power k can move to any unburned square that is at most k units away in each horizontal or vertical direction. Thus in the digraph for the power-1 game each vertex has outdegree 8, while in the power-2 game the vertices have outdegree 24. It is known that Devil wins the power-1 game (see Conway [6]), while Angel wins the power-2 game (proved independently by Kloster [8] and Máthé [9]).

To prove our result, we give a robust map from $\mathcal{W}^{24}$ to the digraph for the power-2 Angel on $\mathbb{Z}^2$. Since Angel wins that game, Theorem 4.3 implies that Angel wins on $\mathcal{W}^d$ when $d \geq 24$ (a refinement of the argument lowers the bound to $d \geq 14$). By Theorem 4.1, Walker then wins the prefix game on $\mathcal{W}^d$, hitting all levels.

The construction of our robust map uses the following intuitive idea: if Angel has d different "types of move" in some digraph, and these moves commute, then we can introduce a (highly redundant) coordinate system on the graph by counting how many times Angel has made each type of move. This coordinate system induces a robust map from $\mathcal{W}^d$ into the digraph. The following lemma formalizes the idea.

Lemma 5.1. *If H is a rooted digraph with $V(H) \subset \mathbb{Z}^n$, and $M \subset \mathbb{Z}^n$ is a finite set such that $xy \in E(H)$ implies $y - x \in M$, then there is a robust map from $\mathcal{W}^{|M|}$ to H.*

Proof. Let $d = |M|$, and let $m_1, \ldots, m_d$ be the elements of M. Define $\phi\colon \mathcal{W}^d \to V(H)$ by $\phi(x_1, \ldots, x_d) = h_0 + \sum_{i=1}^{d} x_i m_i$, where h_0 is the root of H. Since $\phi(0, \ldots, 0) = h_0$, the start condition is satisfied. Now consider $(\phi(v), w) \in E(H)$. We must find $z \in V(\mathcal{W}^d)$ such that $\phi(z) = w$ and $vz \in E(G)$. Since $(\phi(x), v) \in E(H)$, the hypothesis guarantees existence of $m_i \in M$ such that $\phi(x) + m_i = v$. With e_i denoting the unit vector with 1 in coordinate i, we have $\phi(x + e_i) = \phi(x) + m_i = v$, and $(x, x + e_i) \in E(\mathcal{W}^d)$. Hence ϕ is robust. $\square$

The underlying digraphs of the classical Angel-Devil game fit the hypothesis of the lemma, yielding the following corollary:

Corollary 5.2. *Angel wins the Angel-Devil game on $\mathcal{W}^d$ for $d \geq 14$ (and hence also Walker wins the prefix game).*

Proof. Since Angel wins the power-2 Angel-Devil game [8, 9], in which Angel always has 24 possible moves expressed as coordinate vectors, Lemma 5.1 and Theorem 4.3 together imply that Angel wins in $\mathcal{W}^{24}$. Furthermore, Angel can win that power-2 Angel-Devil game using only moves changing the horizontal coordinate by at most 2 and the vertical coordinate by at most 1 (proved by Wästlund [10]); hence Angel wins in $\mathcal{W}^{14}$ (and thus in all higher-dimensional wedges). $\square$

In Section 2, we showed that Breaker wins the ordered chain game on the wedge $\mathcal{W}^2$. The question remains: Who wins when $3 \leq d \leq 13$? We conjecture the following.

Conjecture 5.3. *For $d \geq 3$, Walker wins the prefix game on $\mathcal{W}^d$.*

6 Fractional Devils and Biased Games

In this section we consider *b-biased games*, where Blocker makes b moves in response to every move by Walker. Similarly, Devil burns b positions on each move in the b-biased Angel-Devil game. To study b-biased games, we introduce another variation.

For a positive real number p, the *fractional p-Devil game* is played like the Angel-Devil game, but now vertices are not simply burned or unburned. Instead, each vertex has *damage* between 0 and 1, initially 0. Angel may move to a vertex if its damage is less than 1. Devil, on his turn, increases the total damage by at most p on a finite set of vertices (the damage on any one vertex cannot decrease).

In the fractional 1-Devil game, Devil may choose to burn one vertex per turn as in the standard Angel-Devil game, but he may also choose to burn several vertices partially. Thus the fractional Devil is at least as strong as the standard Devil. We do not know whether there are any digraphs on which the fractional 1-Devil wins but the standard Devil loses.

To apply the fractional model, we also generalize the concept of robustness:

Definition 6.1. Let G and H be digraphs with roots g_0 and h_0. For $k \in \mathbb{N}$, a *k-robust map* from G to H is a map $\phi\colon V(G) \to V(H)$ with $\phi(g_0) = h_0$ such that when there is an edge from $\phi(v)$ to w in H, there are at least k vertices z in G with $\phi(z) = w$ and $vz \in E(G)$.

We can now generalize Theorem 4.3:

Theorem 6.2. *Let G and H be two rooted digraphs, let p be a positive real number, and let ϕ be a k-robust map from G to H. If Angel wins the fractional p-Devil game in H, then Angel wins the fractional pk-Devil game in G.*

Proof. We modify the proof of Theorem 4.3 in a straightforward way. When the G-Angel is at v, and the G-devil adds damage $(x_1, \ldots, x_r)$ to vertices $(v_1, \ldots, v_r)$ (totalling at most pk), the simulated H-devil adds damage x_i/k to vertex $\phi(v_i)$, for $1 \leq i \leq r$; this is a legal move. Let moving from $\phi(v)$ to w be the response in the winning strategy for the imaginary H-angel; this requires that damage less than 1 has been done to w. Since damage has been done via ϕ, we conclude that the total damage by the G-Devil on preimages of w has been less than k. Since ϕ is k-robust, there are at least k such preimages that are out-neighbors of v in G, and hence one of them is available as a move for Angel in G. $\square$

Kloster's proof [8] that Angel wins the power-2 Angel-Devil game can be adapted to show that Angel also wins that game against the fractional 1-Devil. Hence Angel wins the fractional 1-Devil game in $\mathcal{W}^{24}$, by Lemma 5.1 and Theorem 6.2. We transform this winning strategy into a winning strategy against the (biased) fractional b-Devil in $\mathcal{W}^{24b}$ by constructing highly robust maps and applying Theorem 6.2 again.

Lemma 6.3. *For $d, k \in \mathbb{N}$, there is a k-robust map from $\mathcal{W}^{kd}$ to $\mathcal{W}^d$.*

Proof. Define ϕ by mapping $(x_1, \ldots, x_{kd})$ to the d-tuple whose ith entry is $\sum_{j=0}^{k-1} x_{jd+i}$. To see that ϕ is k-robust, consider $(\phi(v), w) \in E(\mathcal{W}^d)$. Note that $w = \phi(v) + e_h$ for some h, where e_h is 1 at index h and 0 everywhere else. The k vertices of the form $v + e_{h+jd}$, where the subscript is taken modulo kd, are all outneighbors of v in $\mathcal{W}^{kd}$ that map to w. $\square$

Note that every partition of the kd coordinates into d classes of size k induces a k-robust map; let the ith coordinate of the image be the sum of the k coordinates in the ith class of the partition.

Theorem 6.4. *Angel wins the fractional b-Devil game in $\mathcal{W}^{24b}$ and hence also wins the b-biased Angel-Devil game in $\mathcal{W}^{24b}$.*

Proof. Lemma 6.3 provides a b-robust map from $\mathcal{W}^{24b}$ to $\mathcal{W}^{24}$. Since Kloster's proof [8] implies that Angel wins the fractional 1-Devil game in $\mathcal{W}^{24}$, Theorem 6.2 implies that Angel wins against the fractional b-Devil in $\mathcal{W}^{24b}$. Since the fractional b-Devil is at least as strong as the Devil of bias b, Angel wins the b-biased game in $\mathcal{W}^{24b}$. $\square$

There are easy relationships between some biased games on wedges.

Proposition 6.5. *If Blocker wins the b-biased prefix game on $\mathcal{W}^d$, then Blocker wins the $(b+1)$-biased prefix game on $\mathcal{W}^{d+1}$.*

Proof. On each turn Blocker plays the outneighbor of Walker's previous move with coordinate $d+1$ increased and uses the remaining b moves to play the strategy for d dimensions. Any move by Walker that increases the last coordinate immediately loses a level, so Walker does best by playing the game in d dimensions. □

Define the *gap* of a biased Walker-Blocker game to be $d-b$, where d is the dimension and b is the bias. This suggests a natural question: Given a gap g, what is the smallest dimension d such that gap g suffices for Blocker to hold Walker to a fraction of the levels strictly less than 1? Let $d(g)$ denote this minimum dimension. We know that $d(0)=1$, $d(1)=2$, and $3 \le d(2) \le 5$. The fact that $d(1)=2$ follows from Theorem 3.1, since gap 1 in two dimensions is an unbiased game. The other results are trivial, except for the fact that $d(2) \le 5$, which we prove below.

Proposition 6.6. *With bias 3, Blocker holds Walker to at most $4/5$ of the levels in the ordered chain game on $\mathcal{W}^5$, and hence $d(2) \le 5$.*

Proof. We may assume that Walker first takes 00000. Blocker forces Walker to increase a coordinate that is 0 on each subsequent move to avoid skipping levels, eventually forcing Walker to skip a level. The table below gives the moves by Walker and Blocker, up to symmetry. With the three moves of his first turn and two moves of his second, Blocker occupies all points at the fifth level with largest coordinate 1. Blocker's other moves prevent Walker from moving up one level to reach a point having a coordinate larger than 1. After at most four moves, Walker must skip a level, and the pattern repeats. Thus Blocker can hold Walker to 4/5 of the levels. □

Walker	Blocker		
00000	01111	10111	11011
10000	20000	11101	11110
11000	21000	12000	–
11100	21100	12100	11200

We conclude with Conjecture 6.7, which strengthens Conjecture 5.3.

Conjecture 6.7. *There exists a constant k such that for every dimension d, Maker wins the $(d-k)$-biased ordered chain game on $\mathcal{W}^d$.*

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