A note on coloring vertex-transitive graphs

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Abstract

We prove bounds on the chromatic number χ of a vertex-transitive graph in terms of its clique number ω and maximum degree Δ. We conjecture that every vertex-transitive graph satisfies $\chi \le \max\left\{\omega, \left\lceil\frac{5\Delta+3}{6}\right\rceil\right\}$ and we prove results supporting this conjecture. Finally, for vertex-transitive graphs with $\Delta \ge 13$ we prove the Borodin-Kostochka conjecture, i.e., $\chi \le \max\{\omega, \Delta - 1\}$.

1 Introduction

Many results and conjectures in the graph coloring literature have the form: *if the chromatic number χ of a graph is close to its maximum degree Δ, then the graph contains a big clique, i.e., ω is large* ([3, 2, 21, 20, 4, 16]). Generically, we call conjectures of this sort *big clique conjectures*. In [19], it was shown that many big clique conjectures hold under the added hypothesis that every vertex is in a medium sized clique. Partial results on big clique conjectures often guarantee a medium sized clique, but not a big clique. But in a vertex-transitive graph, the existence of one medium sized clique implies that every vertex is in a medium sized clique. By applying the idea in [19], we now get a big clique. So, in essence, partial results on big clique conjectures are self-strengthening in the class of vertex-transitive graphs.

In this short note, we give some examples of this phenomenon. There is not much new graph theory here, just combinations of known results that yield facts we did not know. The following conjecture is the best we could hope for. A good deal of evidence supports it, as we will detail below.

Main Conjecture. *If G is vertex-transitive, then* $\chi(G) \le \max\left\{\omega(G), \left\lceil\frac{5\Delta(G)+3}{6}\right\rceil\right\}$.

Our Main Conjecture would be best possible, as shown by Catlin's counterexamples to the Hajós conjecture [5]. Catlin computed the chromatic number of line graphs of odd cycles where each edge has been duplicated k times; in particular, he showed that $\chi(G_{t,k}) = 2k + \left\lceil\frac{k}{t}\right\rceil$

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for $t \geq 2$, where $G_{t,k} := L(kC_{2t+1})$. Since $\Delta(G_{t,k}) = 3k-1$ and $\omega(G_{t,k}) = 2k$, we have $\chi(G_{2,k}) = 2k + \lceil \frac{k}{2} \rceil = \lceil \frac{5k}{2} \rceil = \lceil \frac{15k-2}{6} \rceil = \max\left\{\omega(G_{2,k}), \left\lceil \frac{5\Delta(G_{2,k})+3}{6} \right\rceil\right\}$ for all $k \geq 1$.

Our main result is the following weakening of the Borodin-Kostochka conjecture for vertex-transitive graphs, which we prove in Section 5. This theorem likely holds for all $\Delta \geq 9$ and proving this may be a good deal easier than proving the full Borodin-Kostochka conjecture (note that the Main Conjecture implies the Main Theorem, so our proof of the theorem should weigh as evidence in support of the conjecture).

Main Theorem. *If G is vertex-transitive with $\Delta(G) \geq 13$ and $K_{\Delta(G)} \not\subseteq G$, then $\chi(G) \leq \Delta(G) - 1$.*

As further evidence for the Main Conjecture, we show that the analogous upper bound holds for the fractional chromatic number. Also, we show that the Main Conjecture is true if all vertex-transitive graphs satisfy both Reed's ω, Δ, and χ conjecture and the strong 2Δ-colorability conjecture (see [1]; really we can get by with $\frac{5}{2}\Delta$-colorability). Finally, we show the following.

Theorem 1.1. *There exists $c < 1$, such that for any vertex-transitive graph G, we have $\chi(G) \leq \max\{\omega(G), c(\Delta(G)+1)\}$.*

2 Clustering of maximum cliques

Before coloring anything, we need a better understanding of the structure of maximum cliques in a graph.

2.1 The clique graph

Definition 1. Let G be a graph. For a collection of cliques $\mathcal{Q}$ in G, let $X_{\mathcal{Q}}$ be the intersection graph of $\mathcal{Q}$; that is, the vertex set of $X_{\mathcal{Q}}$ is $\mathcal{Q}$ and there is an edge between $Q_1, Q_2 \in \mathcal{Q}$ iff $Q_1 \neq Q_2$ and Q_1 and Q_2 intersect.

When $\mathcal{Q}$ is a collection of maximum cliques, we get a lot of information about $X_{\mathcal{Q}}$. Kostochka [15] used the following lemma of Hajnal [10] to show that the components of $X_{\mathcal{Q}}$ are complete in a graph with $\omega > \frac{2}{3}(\Delta + 1)$.

Lemma 2.1 (Hajnal [10])**.** *If G is a graph and $\mathcal{Q}$ is a collection of maximum cliques in G, then*

$$\left|\bigcup \mathcal{Q}\right| + \left|\bigcap \mathcal{Q}\right| \geq 2\omega(G).$$

Hajnal's lemma follows by an easy induction. The proof of Kostochka's lemma in [15] is in Russian; for a reproduction of his original proof in English, see [17]. Below we give a shorter proof from [18].

Lemma 2.2 (Kostochka [15])**.** *If $\mathcal{Q}$ is a collection of maximum cliques in a graph G with $\omega(G) > \frac{2}{3}(\Delta(G)+1)$ such that $X_{\mathcal{Q}}$ is connected, then $\cap\mathcal{Q} \neq \emptyset$.*

Proof. Suppose not and choose a counterexample $\mathcal{Q} := \{Q_1, \dots, Q_r\}$ minimizing r. Plainly, $r \geq 3$. Let A be a noncutvertex in $X_\mathcal{Q}$ and B a neighbor of A. Put $\mathcal{Z} := \mathcal{Q} - \{A\}$. Then $X_\mathcal{Z}$ is connected and hence by minimality of r, $\cap\mathcal{Z} \neq \emptyset$. In particular, $|\cup\mathcal{Z}| \leq \Delta(G) + 1$. By assumption, $\cap\mathcal{Q} = \emptyset$, so $|\cap\mathcal{Q}| + |\cup\mathcal{Q}| \leq 0 + (|\cup\mathcal{Z}| + |A - B|) \leq (\Delta(G) + 1) + (\Delta(G) + 1 - \omega(G)) < 2\omega(G)$. This contradicts Lemma 2.1. □

As shown by Christofides, Edwards and King [6], components of $X_\mathcal{Q}$ have nice structure in the $\omega = \frac{2}{3}(\Delta + 1)$ case as well. We'll need this stronger result to get our bounds on coloring vertex-transitive graphs to be tight.

Lemma 2.3 (Christofides, Edwards and King [6])**.** *If $\mathcal{Q}$ is a collection of maximum cliques in a graph G with $\omega(G) \geq \frac{2}{3}(\Delta(G) + 1)$ such that $X_\mathcal{Q}$ is connected, then either*

- *$\cap\mathcal{Q} \neq \emptyset$; or*
- *$\Delta(X_\mathcal{Q}) \leq 2$ and if $B, C \in \mathcal{Q}$ are different neighbors of $A \in \mathcal{Q}$, then $B \cap C = \emptyset$ and $|A \cap B| = |A \cap C| = \frac{1}{2}\omega(G)$.*

2.2 In vertex-transitive graphs

Let G be a vertex-transitive graph and let $\mathcal{Q}$ be the collection of all maximum cliques in G. It is not hard to see that $X_\mathcal{Q}$ is vertex-transitive as well; in fact, we have the following.

Observation 1. *Let G be a vertex-transitive graph and let $\mathcal{Q}$ be the collection of all maximum cliques in G. For each component C of $X_\mathcal{Q}$, put $G_C := G\left[\bigcup V(C)\right]$. Then G_C is vertex-transitive for each component C of $X_\mathcal{Q}$ and $G_{C_1} \cong G_{C_2}$ for components C_1 and C_2 of $X_\mathcal{Q}$.*

A basic consequence of Observation 1 is that if G is vertex-transitive and G_C has a dominating vertex (or universal vertex), then every vertex of G_C is dominating; so G_C is complete. Let G be a vertex-transitive graph with $\omega > \frac{2}{3}(\Delta + 1)$. Suppose that $X_\mathcal{Q}$ has one or more edges. By Kostochka's lemma, $\cap\mathcal{Q}_C$ is nonempty, where $\mathcal{Q}_C$ is the set of maximum cliques in some component G_C. Choose a vertex $v \in \cap\mathcal{Q}_C$, and note that v is adjacent to each vertex in G_C. Since G is vertex-transitive, each vertex of G_C is a dominating vertex in G_C; so, in fact, G_C is a clique, and C is edgeless. Using Lemma 2.3, we get a bit more.

Lemma 2.4. *Let G be a connected vertex-transitive graph and let $\mathcal{Q}$ be the collection of all maximum cliques in G. If $\omega(G) \geq \frac{2}{3}(\Delta(G) + 1)$, then either*

- *$X_\mathcal{Q}$ is edgeless; or*
- *$X_\mathcal{Q}$ is a cycle and G is the graph obtained from $X_\mathcal{Q}$ by blowing up each vertex to a $K_{\frac{1}{2}\omega(G)}$.*

Proof. If $\omega(G) > \frac{2}{3}(\Delta(G) + 1)$, then $X_\mathcal{Q}$ is edgeless as shown above. Hence we may assume $\omega(G) = \frac{2}{3}(\Delta(G) + 1)$. Let Z be a component of $X_\mathcal{Q}$ and put $\mathcal{Z} := V(Z)$. By Lemma 2.3, $\Delta(X_\mathcal{Z}) \leq 2$ and if $B, C \in \mathcal{Z}$ are different neighbors of $A \in \mathcal{Z}$, then $B \cap C = \emptyset$ and $|A \cap B| = |A \cap C| = \frac{1}{2}\omega(G)$. By Observation 1, $X_\mathcal{Z}$ must be a cycle. But then every vertex in G_Z has $\frac{1}{2}\omega(G) + \frac{1}{2}\omega(G) + \frac{1}{2}\omega(G) - 1 = \Delta(G)$ neighbors in G_Z and thus $G = G_Z$. Hence $X_\mathcal{Q} = X_\mathcal{Z}$ is a cycle and G is the graph obtained from $X_\mathcal{Q}$ by blowing up each vertex to a $K_{\frac{1}{2}\omega(G)}$. □

3 The fractional version

The problem of determining chromatic number can be phrased as an integer program: we aim to minimize the total number of colors used, subject to the constraints that (i) each vertex gets colored and (ii) the vertices receiving each color form an independent set. To reach a linear program from this integer program, we relax the constraint that each vertex is colored with a single color, and instead allow a vertex to be colored with a combination of colors, e.g., 1/2 red, 1/3 green, and 1/6 blue. However, we still require that the total weight of any color on any clique is at most 1. The minimum value of this linear program is the fractional chromatic number, denoted χ_f (see [22] for a formal definition and many results on fractional coloring).

It is an easy exercise to show that every vertex-transitive graph G satisfies $\chi_f(G) = \frac{|G|}{\alpha(G)}$, where $|G|$ denotes $|V(G)|$ and $\alpha(G)$ denotes the maximum size of an independent set. We also need Haxell's condition [11] for the existence of an independent transversal.

Lemma 3.1 (Haxell [11])**.** *Let H be a graph and $V_1 \cup \cdots \cup V_r$ a partition of $V(H)$. Suppose that $|V_i| \geq 2\Delta(H)$ for each $i \in [r]$. Then H has an independent set $\{v_1, \ldots, v_r\}$ where $v_i \in V_i$ for each $i \in [r]$.*

Lemma 3.2. *If G is a vertex-transitive graph with $\omega(G) \geq \frac{2}{3}(\Delta(G)+1)$, then $\alpha(G) = \left\lfloor \frac{|G|}{\omega(G)} \right\rfloor$. Moreover, if $\omega(G) > \frac{2}{3}(\Delta(G)+1)$, then $\omega(G)$ divides $|G|$.*

Proof. We may assume that G is connected. Since G is vertex-transitive, every vertex of G is in an $\omega(G)$-clique. First, suppose $\omega(G) > \frac{2}{3}(\Delta(G)+1)$. Then Lemma 2.4 shows that the vertex set of G can be partitioned into cliques $V_1, \ldots, V_r$ with $|V_i| \geq \left\lceil \frac{2}{3}(\Delta(G)+1) \right\rceil$ for each $i \in [r]$. Let H be the graph formed from G by making each V_i independent. Then $\Delta(H) \leq \Delta(G) + 1 - \left\lceil \frac{2}{3}(\Delta(G)+1) \right\rceil$; now by Lemma 3.1, G has an independent set with a vertex in each V_i. Since G is vertex-transitive, all V_i have the same size; so, in fact, $|V_i| = \omega(G)$ for all i. But now $|G| = \alpha(G)\,|V_i| = \alpha(G)\omega(G)$, so we're done.

So instead suppose $\omega(G) = \frac{2}{3}(\Delta(G)+1)$. Now Lemma 2.4 shows that G is obtained from a cycle C by blowing up each vertex of C to a copy of $K_{\frac{1}{2}\omega(G)}$. Hence $\alpha(G) = \left\lfloor \frac{|C|}{2} \right\rfloor = \left\lfloor \frac{|G|}{\omega(G)} \right\rfloor$ as desired. □

Reed's ω, Δ, and χ conjecture states that every graph satisfies

$$\chi \leq \left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil.$$

In [16], Molloy and Reed proved this upper bound without the round-up for the fractional chromatic number χ_f. Since $\chi_f(G) = \frac{|G|}{\alpha(G)}$ for vertex-transitive graphs, an earlier result of Fajtlowicz [8] suffices for our purposes.

Lemma 3.3 (Fajtlowicz [8])**.** *For every graph G, we have $\alpha(G) \geq \frac{2|G|}{\omega(G)+\Delta(G)+1}$.*

Theorem 3.4. *If G is vertex-transitive, then $\alpha(G) \geq \frac{|G|}{\max\left\{\omega(G), \frac{5}{6}(\Delta(G)+1)\right\}}$.*

Proof. Suppose $\omega(G) > \frac{2}{3}(\Delta(G)+1)$. Then Lemma 3.2 shows $\alpha(G) = \frac{|G|}{\omega(G)}$ and we're done. Otherwise, $\omega(G) \leq \frac{2}{3}(\Delta(G)+1)$ and Lemma 3.3 gives $\alpha(G) \geq \frac{2|G|}{\frac{2}{3}(\Delta(G)+1)+\Delta(G)+1} = \frac{|G|}{\frac{5}{6}(\Delta(G)+1)}$ as desired. □

Restating Theorem 3.4 in terms of fractional coloring, we have the following.

Corollary 3.5. *If G is vertex-transitive, then $\chi_f(G) \leq \max\left\{\omega(G), \frac{5}{6}(\Delta(G)+1)\right\}$.*

4 Reed's conjecture plus strong coloring

For a positive integer r, a graph G with $|G| = rk$ is called *strongly r-colorable* if for every partition of $V(G)$ into parts of size r there is a proper coloring of G that uses all r colors on each part. If $|G|$ is not a multiple of r, then G is strongly r-colorable iff the graph formed by adding $r\left\lceil\frac{|G|}{r}\right\rceil - |G|$ isolated vertices to G is strongly r-colorable. The *strong chromatic number $s\chi(G)$* is the smallest r for which G is strongly r-colorable. Not surprisingly, if G is strongly r-colorable, then G is also strongly $(r+1)$-colorable, although the proof of this fact is non-trivial [9].

In [12], Haxell proved that the strong chromatic number of any graph is at most $3\Delta - 1$. In [13], she proved further that for every $c > 11/4$ there exists Δ_c such that if G has maximum degree Δ at least Δ_c, then G has strong chromatic number at most $c\Delta$. The strong 2Δ-colorability conjecture [1] says that the strong chromatic number of any graph is at most 2Δ. If true, this conjecture would be sharp. We need the following intermediate conjecture.

Conjecture 4.1. *The strong chromatic number of any vertex-transitive graph is at most $\frac{5}{2}\Delta$.*

We also need Reed's conjecture [20] restricted to vertex-transitive graphs.

Conjecture 4.2. *Every vertex-transitive graph satisfies $\chi \leq \left\lceil\frac{\omega+\Delta+1}{2}\right\rceil$.*

Theorem 4.3. *If Conjecture 4.1 and Conjecture 4.2 both hold, then the Main Conjecture does as well.*

Proof. We may assume that G is connected. Put $\Delta := \Delta(G)$, $\omega := \omega(G)$ and $\chi := \chi(G)$. Suppose $\omega < \frac{2}{3}(\Delta+1)$. So, we have $\omega \leq \frac{2\Delta+1}{3}$ and moreover, when $\Delta \equiv 3 \pmod 6$, we have $\omega \leq \frac{2}{3}\Delta$. Plugging the first inequality into Conjecture 4.2 gives $\chi \leq \left\lceil\frac{5\Delta+4}{6}\right\rceil = \left\lceil\frac{5\Delta+3}{6}\right\rceil$ when $\Delta \not\equiv 3 \pmod 6$; by using the improved upper bound on ω in the remaining case, we again prove the desired upper bound on χ.

Now suppose $\omega \geq \frac{2}{3}(\Delta+1)$ and let $\mathcal{Q}$ be the set of maximum cliques in G. Applying Lemma 2.4, either $X_{\mathcal{Q}}$ is edgeless or G is obtained from an odd cycle by blowing up each vertex to a $K_{\frac{\omega}{2}}$. In the latter case, G is one of Catlin's examples from [5] and the bound holds as mentioned in the introduction. Hence we may assume that $X_{\mathcal{Q}}$ is edgeless; that is, $V(G)$ can be partitioned into $\omega(G)$-cliques.

Suppose $\chi > \omega$. Now we show that Conjecture 4.1 implies the Main Conjecture. Form G' from G by adding vertices to the maximum cliques of G until they all have $\left\lceil\frac{5\Delta+3}{6}\right\rceil$ vertices; each new vertex has no edges outside its clique, and Δ always denotes the maximum degree in G, not in G'. Now form G'' from G' by removing all edges within each maximum clique. Each vertex now has at most $\Delta+1-\omega \leq \frac{1}{3}(\Delta+1)$ neighbors in G' outside of its clique hence, the maximum degree of G'' is at most $\frac{1}{3}(\Delta+1)$. Since $\left\lceil\frac{5\Delta+3}{6}\right\rceil \geq \frac{5}{2}\left(\frac{1}{3}(\Delta+1)\right)$, Conjecture 4.1 implies that G'' is strongly $\left\lceil\frac{5\Delta+3}{6}\right\rceil$-colorable. By taking the V_i's of G'' to be the vertex sets of the maximum cliques in G', we see that G' is $\left\lceil\frac{5\Delta+3}{6}\right\rceil$-colorable, and hence so is G. □

Reed [20] has shown that there is $0 < \epsilon < 1$ such that every graph satisfies $\chi \leq \epsilon\omega + (1-\epsilon)(\Delta+1)$; for a shorter and simpler proof, see [14]. Combining this upper bound with Haxell's $3\Delta - 1$ strong colorability result, we get the following similarly to Theorem 4.3.

Theorem 4.4. *There exists $c < 1$, such that for any vertex-transitive graph G, we have $\chi(G) \leq \max\{\omega(G), c(\Delta(G)+1)\}$.*

5 Borodin-Kostochka for vertex-transitive graphs

In [7], we proved the following.

Theorem 5.1. *If G is a graph with $\Delta(G) \geq 13$ and $K_{\Delta(G)-3} \not\subseteq G$, then $\chi(G) \leq \Delta(G) - 1$.*

In [19], the second author proved the following.

Theorem 5.2. *If G is a graph with $\Delta(G) \geq 9$ and $K_{\Delta(G)} \not\subseteq G$ such that every vertex is in a clique on $\frac{2}{3}\Delta(G) + 2$ vertices, then $\chi(G) \leq \Delta(G) - 1$.*

By combining these theorems, we immediately get that the Borodin-Kostochka conjecture holds for vertex-transitive graphs with $\Delta \geq 15$. We can improve this result using Lemma 2.4 and Haxell's $3\Delta - 1$ strong colorability result.

Main Theorem. *If G is vertex-transitive with $\Delta(G) \geq 13$ and $K_{\Delta(G)} \not\subseteq G$, then $\chi(G) \leq \Delta(G) - 1$.*

Proof. Suppose that $\chi(G) \geq \Delta(G)$. By Lemma 5.1, we have $\omega(G) \geq \Delta(G)-3 > \frac{2}{3}(\Delta(G)+1)$ since $\Delta(G) \geq 13$. Now Lemma 2.4 shows that $X_{\mathcal{Q}}$ is edgeless, where $\mathcal{Q}$ is the collection of all maximum cliques in G.

Suppose $\chi(G) > \Delta(G)$. Form G' from G by adding vertices to the maximum cliques of G until they all have $\Delta(G) - 1$ vertices, where each new vertices has no edges outside its clique. Each vertex has at most $\Delta(G) + 1 - \omega(G) \leq 4$ neighbors outside its clique. Since $\Delta(G) - 1 = 12 \geq 3 * 4 - 1$, Haxell's $3\Delta - 1$ strong colorability result implies that G' is $(\Delta(G) - 1)$-colorable and hence so is G. □

If Conjecture 4.2 holds, then we get $\omega \geq \Delta - 2 > \frac{2}{3}(\Delta + 1)$ when $\Delta \geq 9$. So, since $\Delta + 1 - \omega \leq 3$, the above argument works for $\Delta \geq 9$. That is, Conjecture 4.2 by itself implies the Borodin-Kostochka conjecture for vertex-transitive graphs.

References

[1] R. Aharoni, E. Berger, and R. Ziv, *Independent systems of representatives in weighted graphs*, Combinatorica **27** (2007), no. 3, 253–267.

[2] O.V. Borodin and A.V. Kostochka, *On an upper bound of a graph's chromatic number, depending on the graph's degree and density*, Journal of Combinatorial Theory, Series B **23** (1977), no. 2-3, 247–250.

[3] R.L. Brooks, *On colouring the nodes of a network*, Mathematical Proceedings of the Cambridge Philosophical Society, vol. 37, Cambridge Univ Press, 1941, pp. 194–197.

[4] P.A. Catlin, *Another bound on the chromatic number of a graph*, Discrete Mathematics **24** (1978), no. 1, 1–6.

[5] ______, *Hajós graph-coloring conjecture: variations and counterexamples*, J. Combin. Theory Ser. B **26** (1979), no. 2, 268–274.

[6] D. Christofides, K. Edwards, and A.D. King, *A note on hitting maximum and maximal cliques with a stable set*, Journal of Graph Theory **73** (2013), no. 3, 354–360.

[7] D.W. Cranston and L. Rabern, *Graphs with $\chi = \Delta$ have big cliques*, arXiv preprint `http://arxiv.org/abs/1305.3526` (2013).

[8] S. Fajtlowicz, *Independence, clique size and maximum degree*, Combinatorica **4** (1984), no. 1, 35–38.

[9] M. R. Fellows, *Transversals of vertex partitions in graphs*, SIAM J. Discrete Math. **3** (1990), no. 2, 206–215.

[10] A. Hajnal, *A theorem on k-saturated graphs*, Canadian Journal of Mathematics **17** (1965), no. 5, 720.

[11] P. Haxell, *A note on vertex list colouring*, Combinatorics, Probability and Computing **10** (2001), no. 04, 345–347.

[12] ______, *On the strong chromatic number*, Combinatorics, Probability and Computing **13** (2004), no. 06, 857–865.

[13] P. E. Haxell, *An improved bound for the strong chromatic number*, J. Graph Theory **58** (2008), no. 2, 148–158.

[14] A. D. King and B. A. Reed, *A short proof that χ can be bounded ϵ away from $\Delta + 1$ towards ω*, arXiv preprint, `http://arxiv.org/abs/1211.1410` (2012).

[15] A.V. Kostochka, *Degree, density, and chromatic number*, Metody Diskretn. Analiz **35** (1980), 45–70 (in Russian).

[16] M.S. Molloy and B.A. Reed, *Graph colouring and the probabilistic method*, Springer Verlag, 2002.

[17] L. Rabern, *On hitting all maximum cliques with an independent set*, Journal of Graph Theory **66** (2011), no. 1, 32–37.

[18] ———, *Coloring graphs from almost maximum degree sized palettes*, Arizona State University, 2013.

[19] ———, *Coloring graphs with dense neighborhoods*, Journal of Graph Theory (2013).

[20] B. Reed, *ω, Δ, and χ*, Journal of Graph Theory **27** (1998), no. 4, 177–212.

[21] ———, *A strengthening of Brooks' theorem*, Journal of Combinatorial Theory, Series B **76** (1999), no. 2, 136–149.

[22] E. R. Scheinerman and D. H. Ullman, *Fractional graph theory*, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., New York, 1997, A rational approach to the theory of graphs, With a foreword by Claude Berge, A Wiley-Interscience Publication.