List-coloring the Square of a Subcubic Graph

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Abstract

The *square* G^2 of a graph G is the graph with the same vertex set as G and with two vertices adjacent if their distance in G is at most 2. Thomassen showed that for a planar graph G with maximum degree $\Delta(G) = 3$ we have $\chi(G^2) \leq 7$. Kostochka and Woodall conjectured that for every graph, the list-chromatic number of G^2 equals the chromatic number of G^2, that is $\chi_l(G^2) = \chi(G^2)$ for all G. If true, this conjecture (together with Thomassen's result) implies that every planar graph G with $\Delta(G) = 3$ satisfies $\chi_l(G^2) \leq 7$. We prove that every graph (not necessarily planar) with $\Delta(G) = 3$ other than the Petersen graph satisfies $\chi_l(G^2) \leq 8$ (and this is best possible). In addition, we show that if G is a planar graph with $\Delta(G) = 3$ and girth $g(G) \geq 7$, then $\chi_l(G^2) \leq 7$. Dvořák, Škrekovski, and Tancer showed that if G is a planar graph with $\Delta(G) = 3$ and girth $g(G) \geq 10$, then $\chi_l(G^2) \leq 6$. We improve the girth bound to show that if G is a planar graph with $\Delta(G) = 3$ and $g(G) \geq 9$, then $\chi_l(G^2) \leq 6$. All of our proofs can be easily translated into linear-time coloring algorithms.

1 Introduction

We study the problem of coloring the square of a graph. (We consider simple undirected graphs.) Since each component of a graph can be colored independently, we only consider connected graphs. The *square* of a graph G, denoted G^2, has the same vertex set as G and has an edge between two vertices if the distance between them in G is at most 2. We use $\chi(G)$ to denote chromatic number G. We use $\Delta(G)$ to denote the largest degree in G. We say a graph G is subcubic, if $\Delta(G) \leq 3$.

Wegner [17] initiated the study of the chromatic number for squares of planar graphs. This topic has been actively studied lately due to his conjecture.

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Conjecture. *(Wegner [17]) Let G be a planar graph. The chromatic number $\chi(G^2)$ of G^2 is at most 7 if $\Delta(G) = 3$, at most $\Delta(G) + 5$ if $4 \leq \Delta(G) \leq 7$, and at most $\lfloor \frac{3\Delta(G)}{2} \rfloor + 1$ otherwise.*

Thomassen [16] proved Wegner's conjecture for $\Delta(G) = 3$, but it is still open for all values of $\Delta(G) \geq 4$. The best known upper bounds are due to Molloy and Salavatipour [14]. Better results can be obtained for special classes of planar graphs. Borodin *et al.* [1] and Dvořák *et al.* [4] proved that $\chi(G^2) = \Delta(G) + 1$ if G is a planar graph G with sufficiently large maximum degree and girth at least 7. A natural variation of this problem is to study the list chromatic number of the square of a planar graph.

A *list assignment* for a graph is a function L that assigns each vertex a list of available colors. The graph is *L-colorable* if it has a proper coloring f such that $f(v) \in L(v)$ for all v. A graph is called *k-choosable* if G is L-colorable whenever all lists have size k. The list chromatic number $\chi_l(G)$ is the minimum k such that G is k-choosable. Kostochka and Woodall [11] conjectured that $\chi_l(G^2) = \chi(G^2)$ for every graph G.

We consider the problem of list-coloring G^2 when G is subcubic. If G is subcubic then clearly $\Delta(G^2) \leq (\Delta(G))^2 \leq 9$. It is an easy exercise to show that the Petersen graph is the only subcubic graph G such that $G^2 = K_{10}$. Hence, by the list-coloring version of Brook's Theorem [5] we conclude that if G is subcubic and G is not the Petersen graph, then $\chi_l(G^2) \leq \Delta(G^2) \leq 9$. In fact, we show that this upper bound can be strengthened as follows. We say that a subcubic graph is *interesting* if it is not the Petersen graph.

Theorem 1. *If G is an interesting subcubic graph, then $\chi_l(G^2) \leq 8$.*

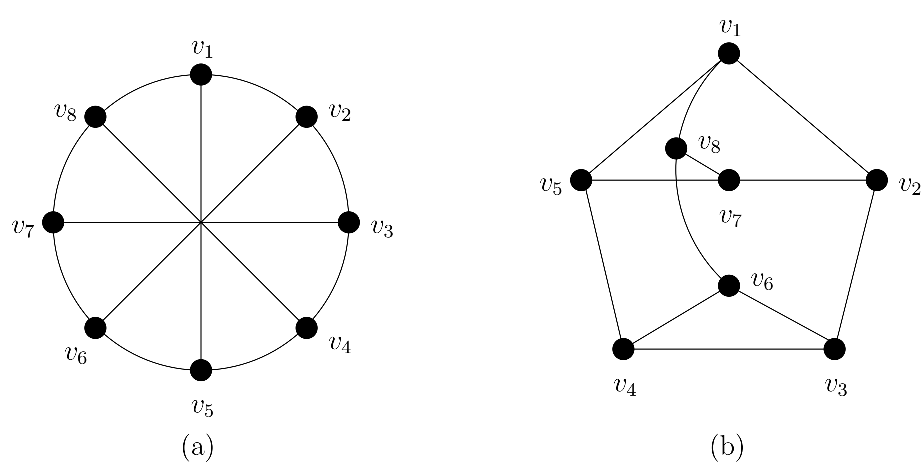

Figure 1. Two graphs, each on 8 vertices; each has K_8 as its square. (a) An 8-cycle $v_1, v_2, \ldots, v_8$ with "diagonals" (i.e. the additional edges are v_iv_{i+4} for each $i \in \{1, 2, 3, 4\}$). This graph has girth 4. (b) This graph has girth 3.

Theorem 1 is best possible, as illustrated by the graphs above. The graph on the left has girth 4. The graph on the right has girth 3. The square of each graph is K_8. Thus, each graph requires lists of size 8. In fact, there are an infinite number of interesting subcubic graphs G such that $\chi_l(G^2) = 8$. Let H be the Petersen graph with an edge removed. Note that $H^2 = K_8$. Hence, any graph G which contains H as a subgraph satisfies $\chi_l(G^2) \geq 8$.

In Section 2 we introduce definitions and themes common to our proofs. In Section 3 we prove Theorem 1. In Section 4 we show that if G is a planar subcubic graph with girth at least 7, then $\chi_l(G^2) \leq 7$. Dvořák, Škrekovski, and Tancer [3] showed that if G is a subcubic planar graph with girth at least 10, then $\chi_l(G^2) \leq 6$. In Section 5 we extend their result by lowering the girth bound from 10 to 9.

2 Preliminaries

We use n, e, and f to denote the number of vertices, edges, and faces in a graph. A *partial (proper) coloring* is the same as a proper coloring except that some vertices may be uncolored. We use $g(G)$ to denote the girth of graph G. When the context is clear, we simply write g. We use k-vertex to denote a vertex of degree k. We use Ad to denote the average degree of a graph. Similarly, we use $Mad(G)$ to denote the maximum average degree of G; that is, the maximum of $2|E(H)|/|V(H)|$ over all induced subgraphs H of G. We use $N[v]$ to denote the closed neighborhood of v in G^2. We use $G[V_1]$ to denote the subgraph of G induced by vertex set V_1.

Throughout the paper, we use the idea of *saving a color* at a vertex v. By this we mean that we assign colors to two neighbors of v in G^2 but we only reduce the list of colors available at vertex v by one. A typical example of this occurs when v is adjacent to vertices v_1 and v_2 in G^2, v_1 is not adjacent to v_2 in G^2, and $|L(v_1)| + |L(v_2)| > |L(v)|$. This inequality implies that either $L(v_1)$ and $L(v_2)$ have a common color or that some color appears in $L(v_1) \cup L(v_2)$ but not in $L(v)$. In the first case, we save by using the same color on vertices v_1 and v_2. In the second case, we use a color in $(L(v_1) \cup L(v_2)) \setminus L(v)$ on the vertex where it appears and we color the other vertex arbitrarily.

We say a graph G is k-minimal if G^2 is not k-choosable, but the square of every proper subgraph of G is k-choosable. A *configuration* is an induced subgraph. We say that a configuration is k-reducible if it cannot appear in a k-minimal graph (we will be interested in $k \in \{6, 7, 8\}$). We say that a configuration is $6'$-reducible if it cannot appear in a 6-minimal graph with girth at least 7. Note that for every $k \geq 4$ a 1-vertex cannot appear in a k-minimal graph (if G contains a 1-vertex x, by hypothesis we can color $(G \setminus \{x\})^2$, then we can extend the coloring to x since in this case $(G \setminus \{x\})^2 = G^2 \setminus \{x\}$). Hence, in the rest of this paper, we assume our graphs have no 1-vertices.

Note that the definition of k-minimal requires that for every subgraph H the square of $G \setminus V(H)$ is k-choosable, but does not require the stronger statement that for every subgraph H the graph $G^2 \setminus V(H)$ is k-choosable. This is a subtle, but significant distinction. To avoid trouble, in Sections 4 and 5 we will only consider reducible configurations H such that $G^2 \setminus V(H) = (G \setminus V(H))^2$; otherwise, we may face difficulties as in the next paragraph.

We give a fallacious proof that $\chi_l(G^2) \leq 7$ for every subcubic planar graph G with girth at least 6. Clearly, a vertex of degree 2 is a 7-reducible configuration (and so is a vertex of degree 1), since it has degree at most 6 in G^2. Let G be a 7-minimal subcubic planar graph of girth at least 6. Since, G is planar and has girth at least 6, G has a vertex v of degree at most 2 (by Lemma 2). By hypothesis, we can color $G^2 \setminus \{v\}$. Since v has at most 6 neighbors in G^2 we can extend the coloring to v.

The flaw in this proof is that by hypothesis, we can color $(G \setminus \{v\})^2$, which may have one fewer edges than $G^2 \setminus \{v\}$; in particular, if v is adjacent to vertices u and w, then $G^2 \setminus \{v\}$ contains the edge uw, but $(G \setminus \{v\})^2$ does not. We may be tempted to add the edge uw to the graph $G^2 \setminus \{v\}$; however, if we do, the new graph may not satisfy the girth restriction.

In both Section 4 and Section 5 we make use of upper bounds on $Mad(G)$. To prove these bounds, we use the following well-known lemma.

Lemma 2. *If G is a planar graph with girth at least g, then $Mad(G) < \frac{2g}{g-2}$.*

Proof: Every subgraph of G is a planar graph with girth at least g; hence, it is enough to show that $Ad = \frac{2e}{n} < \frac{2g}{g-2}$. From Euler's formula we have $f - e + n = 2$. By summing the lengths of all the faces, we get $2e \geq fg$. Combining these gives the following inequality.

$$\begin{aligned} e &< e + 2 \leq \frac{2e}{Ad} + \frac{2e}{g} \\ 1 &< \frac{2}{Ad} + \frac{2}{g} \\ Ad &< \frac{2g}{g-2} \end{aligned}$$

□

In Section 3, we show that given a graph G with lists of size 8, we can greedily color all but a few vertices of G, each near a central location. The "hard work" in Section 3 is showing that we can extend the coloring to these last few uncolored central vertices.

The outlines of Section 4 and Section 5 are very similar. In each section, we exhibit four reducible configurations; recall that a reducible configuration cannot occur in a k-minimal graph. Next, we show that if a subcubic planar graph with girth at least 7 (resp. 9) does not contain any of these reducible configurations, then G has $Mad(G) \geq \frac{14}{5}$ ($\frac{18}{7}$). This implies that each subcubic planar graph of girth 7 (resp. 9) contains a reducible configuration. It follows that there are no k-minimal graphs (for $k \in \{6', 7\}$), and so the theorems are true.

3 General subcubic graphs

We begin this section by proving a number of structural lemmas about 8-minimal subcubic graphs. We conclude by showing that if G is an interesting subcubic graph, then $\chi_l(G^2) \leq 8$.

Lemma 3. *If G is a subcubic graph, then for any edge uv we have $\chi_l(G^2 \setminus \{u, v\}) \leq 8$.*

Proof: For every vertex w other than u and v, we define the *distance class* of w to be the distance in G from w to edge uv. We greedily color the vertices of $G^2 \setminus \{u, v\}$ in order of decreasing distance class. We claim that lists of size 8 suffice. Note that $|N[w]| \leq 10$ for every vertex w. If at least two vertices in $N[w]$ are uncolored when we color w, then we need at most $10 - 2 = 8$ colors at vertex w. Say w is in distance class at least 2. Let x and y be the first two vertices on a shortest path in G from w to edge uv. Since vertices x and y are in lower distance classes than w, they are both uncolored when we color w. Hence, we need at most $10 - 2 = 8$ colors at vertex w. If w is in distance class 1, then u and v are uncolored when we color w. So again we need only $10 - 2 = 8$ colors. □

Lemma 3 shows that if G is a subcubic graph, then lists of size 8 are sufficient to color all but two adjacent vertices of G^2. Hence, if H is any subgraph that contains an edge, then we can color $G^2 \setminus V(H)$ from lists of size 8. The next lemma relies on the same idea as Lemma 3, but generalizes the context in which it applies. Given a graph G and a partial coloring of G^2, we define *excess*$(v) = 1+$ (the number of colors available at vertex v) $-$ (the number of uncolored neighbors of v in G^2). Note that for any subcubic graph G and any such partial coloring, every vertex v has excess$(v) \geq 0$. Intuitively, the excess of a vertex v measures how many colors we have "saved" on v (colors are saved either from using the same color on two neighbors of v or simply because v has fewer than 9 neighbors in G^2). For example, if two neighbors of v in G^2 use the same color, then excess$(v) \geq 1$. Similarly, if v lies on a 4-cycle or a 3-cycle, then excess$(v) \geq 1$ or excess$(v) \geq 2$, respectively. Vertices with positive excess play a special role in finishing a partial coloring.

Lemma 4. *Let G be a subcubic graph and let L be a list assignment with lists of size 8. Suppose that G^2 has a partial coloring from L. Suppose also that vertices u and v are uncolored, are adjacent in G^2, and that excess$(u) \geq 1$ and excess$(v) \geq 2$. If we can order the uncolored vertices so that each vertex except u and v is succeeded in the order by at least 2 adjacent vertices in G^2, then we can finish the partial coloring.*

Proof: We will color the vertices greedily according to the order. Recall that for each vertex w, we have $|N[w]| \leq 10$. Since at least two vertices in $N[w]$ will be uncolored at the time we color w, we will have a color available to use on each vertex w (other than u and v). Since u and v are the only vertices not succeeded by 2 adjacent vertices in G^2, they must be the last two vertices in the order. Because $\text{excess}(u) \geq 1$ and $\text{excess}(v) \geq 2$, we can finish the coloring by greedily coloring u, then v. □

A simple but useful instance where Lemma 4 applies is when the uncolored vertices induce a connected subgraph and vertices u and v are adjacent (we order the vertices by decreasing distance (within the subgraph) from edge uv). Whenever we say that we can greedily finish a coloring, we will be using Lemma 4. Often, we will specify an order for the uncolored vertices; when we do not give an order it is because they induce a connected subgraph. The next two lemmas exhibit small configurations which allow us to apply Lemma 4.

Lemma 5. *If G is an 8-minimal subcubic graph, then G is 3-regular.*

Proof: Say u is a vertex with $d(u) \leq 2$. Let v be a neighbor of u. Note that $\text{excess}(v) \geq 1$ and $\text{excess}(u) \geq 3$. So by Lemma 4, we can list-color G^2 from lists of size 8. □

Lemma 6. *If G is an 8-minimal subcubic graph, then $g(G) > 3$.*

Proof: Say G contains a 3-cycle uvw. Note that $\text{excess}(u), \text{excess}(v), \text{excess}(w) \geq 2$. So by Lemma 4, we can list-color G^2 from lists of size 8. □

Lemma 7. *If G is an 8-minimal subcubic graph, then $g(G) > 4$.*

Proof: Suppose that G is a counterexample. Let each vertex have a list of size 8. Observe that if vertex v lies on a 4-cycle, then $\text{excess}(v) \geq 1$. Note that if v lies on two 4-cycles, then $\text{excess}(v) \geq 2$. Suppose that v_1 lies on two 4-cycles and v_2 is adjacent to v_1 on some 4-cycle. Since $\text{excess}(v_2) \geq 1$ and $\text{excess}(v_1) \geq 2$, we can greedily color G. Hence, we assume that no vertex lies on two 4-cycles. Let $\mathcal{C}$ be a 4-cycle in G. Label the vertices of $\mathcal{C}$ as v_1, v_2, v_3, v_4. Recall that G is 3-regular (by Lemma 5). Let u_i be the neighbor of v_i not on $\mathcal{C}$. We can assume the u_is are distinct, since otherwise either G contains a 3-cycle or some vertex lies on two 4-cycles. By Lemma 3, we color all vertices except the u_is and v_is; call this coloring c. Let $L(v)$ denote the list of remaining colors available at each uncolored vertex v.

Case 1: Suppose that $\text{distance}(u_1, v_3) = 3$. Note that $|L(v_i)| \geq 6$ and $|L(u_i)| \geq 2$. We assume that equality holds for v_1 (otherwise we throw away colors until it does). Since $|L(u_1)| + |L(v_3)| > |L(v_1)|$, we can choose color c_1 for u_1 and color c_2 for v_3 so that $|L(v_1) \setminus \{c_1, c_2\}| \geq 5$. Since $\text{excess}(v_2) \geq 1$ and $\text{excess}(v_1) \geq 2$, we can finish the coloring by Lemma 4 (coloring greedily in the order u_2, u_3, u_4, v_4, v_2, v_1).

Case 2: Suppose instead that $\text{distance}(u_1, v_3) < 3$. Vertices u_1 and u_3 must be adjacent; by symmetry u_2 and u_4 must be adjacent. Now since u_1 and u_3 are adjacent and u_2 and u_4 are adjacent, we have $|L(v_i)| \geq 7$ and $|L(u_i)| \geq 4$ (we assume that equality holds for the v_is). Suppose that $\text{distance}(u_1, u_2) = 3$. Since $|L(u_1)| + |L(u_2)| \geq$

$4 + 4 > 7 = |L(v_1)|$, we can choose color c_1 for u_1 and color c_2 for u_2 such that $|L(v_1) \setminus \{c_1, c_2\}| \geq 6$. Since excess$(v_1) \geq 2$ and excess$(v_2) \geq 1$, we can finish the coloring. Hence, we can assume that distance$(u_1, u_2) < 3$.

Observe that u_1 and u_2 cannot be adjacent, since then v_1 lies on two 4-cycles. Thus, u_1 and u_2 must have a common neighbor. By symmetry, we can assume that u_3 and u_4 have a common neighbor. Since $d(u_1) = 3$ (and we have already accounted for two edges incident to u_1), vertices u_1, u_2, and u_4 must have a common neighbor x. However, then u_2, u_4, and x form a 3-cycle. By Lemma 6, this is a contradiction. □

Lemma 8. *If G is an interesting 8-minimal subcubic graph, then G does not contain two 5-cycles that share three consecutive vertices.*

Proof: Suppose G is a counterexample. Taken together, the two given 5-cycles form a 6-cycle, with one additional vertex adjacent to two vertices of the 6-cycle. Label the vertices of the 6-cycle $v_1, v_2, \ldots, v_6$ and label the final vertex v_7. Let v_7 be adjacent to v_1 and v_4. We consider three cases, depending on how many pairs of vertices on the 6-cycle are distance 3 apart. By Lemma 3, we color all vertices of G^2 except the 7 v_is.

Case 1: Both distance$(v_2, v_5) \geq 3$ and distance$(v_3, v_6) \geq 3$. Let $L(v)$ denote the list of remaining colors available at each uncolored vertex v. In this case, $|L(v_1)| \geq 5$, $|L(v_4)| \geq 5$, $|L(v_7)| \geq 5$ and $|L(v_2)| \geq 4$, $|L(v_3)| \geq 4$, $|L(v_5)| \geq 4$, $|L(v_6)| \geq 4$. We assume equality holds. We consider two subcases.

Subcase 1.1: $L(v_2) \cap L(v_5) \neq \emptyset$ or $L(v_3) \cap L(v_6) \neq \emptyset$. Without loss of generality, we may assume that $L(v_2) \cap L(v_5) \neq \emptyset$. Color v_2 and v_5 with some color $c_1 \in L(v_2) \cap L(v_5)$. Since $|L(v_3) \setminus \{c_1\}| + |L(v_6) \setminus \{c_1\}| > |L(v_7)|$, we can choose color c_2 for v_3 and color c_3 for v_6 such that $|L(v_7) \setminus \{c_1, c_2, c_3\}| \geq 3$. Greedily color the remaining vertices in the order v_1, v_4, v_7.

Subcase 1.2: $L(v_2) \cap L(v_5) = \emptyset$ and $L(v_3) \cap L(v_6) = \emptyset$. Color v_1, v_4, v_7 so that no two vertices among v_2, v_3, v_5, v_6 have only one available color remaining. Call these new lists $L'(v)$. Note that $|L'(v_2)| + |L'(v_5)| \geq 5$ and $|L'(v_3)| + |L'(v_6)| \geq 5$. Hence we can color v_2, v_3, v_5, v_6.

Case 2: Exactly one of distance(v_2, v_5) or distance(v_3, v_6) is 2. Without loss of generality, we may assume that distance$(v_2, v_5) \geq 3$ and distance$(v_3, v_6) = 2$. Recall from Lemma 5 that G is 3-regular. Let u_2, u_5, and u_7 be the vertices not yet named that are adjacent to v_2, v_5, and v_7, respectively. We cannot have $u_2 = u_5$, since we have distance$(v_2, v_5) \geq 3$. Note that distance$(u_2, v_4) \geq 3$ unless $u_2 = u_7$. Similarly, distance$(u_5, v_1) \geq 3$ unless $u_5 = u_7$. Moreover, we cannot have $u_2 = u_7$ or $u_5 = u_7$, since this forms a 4-cycle. Hence, distance$(u_2, v_4) = 3$ and distance$(u_5, v_1) = 3$. Uncolor vertex u_2. Let $L(v)$ denote the list of remaining available colors at each vertex v. We have $|L(v_1)| \geq 6$, $|L(v_2)| \geq 5$, $|L(v_3)| \geq 6$, $|L(v_4)| \geq 5$, $|L(v_5)| \geq 4$, $|L(v_6)| \geq 5$, $|L(v_7)| \geq 5$, and $|L(u_2)| \geq 2$. We assume that equality holds. We consider two subcases.

Subcase 2.1: $L(u_2) \cap L(v_4) \neq \emptyset$. Color u_2 and v_4 with some color $c_1 \in L(u_2) \cap L(v_4)$. Now choose color c_2 for v_2 and color c_3 for v_5 such that $|L(v_3) \setminus \{c_1, c_2, c_3\}| \geq 4$. Let $L'(v) = L(v) \setminus \{c_1, c_2, c_3\}$. The new lists satisfy $|L'(v_1)| \geq 3, |L'(v_3)| \geq 4, |L'(v_6)| \geq 2, |L'(v_7)| \geq 2$. Greedily color the remaining vertices in the order v_7, v_6, v_1, v_3.

Subcase 2.2: $L(u_2) \cap L(v_4) = \emptyset$. We have two subcases here. If $L(v_2) \cap L(v_5) \neq \emptyset$, then color v_2 and v_5 with a common color, and then color u_2 and v_4 to save a color at v_3. Now color the remaining vertices as in Subcase 2.1. If $L(v_2) \cap L(v_5) = \emptyset$, then color u_2 and v_4 to save a color at v_3. Now choose colors for v_6 and for v_7 such that vertices v_2 and v_5 each have at least one remaining color. Let $L'(v)$ denote the list of remaining available colors at each vertex v. Note that $|L'(v_1)| \geq 2$, $|L'(v_3)| \geq 3$, and $|L'(v_2)| + |L'(v_5)| \geq 5$ since $L(v_2) \cap L(v_5) = \emptyset$. In each case, we can color v_1, v_2, v_3, v_5.

Case 3: Both distance(v_2, v_5) and distance(v_3, v_6) are 2. Then v_2 and v_5 have a common neighbor, say v_8, and v_3 and v_6 have a common neighbor, say v_9. Let u_7, u_8, and u_9 be the third vertices adjacent to v_7, v_8, and v_9, respectively. We show that either distance$(v_7, v_8) = 3$ or distance$(v_7, v_9) = 3$ or distance$(v_8, v_9) = 3$. Note that distance$(v_7, v_8) = 3$ unless $u_7 = u_8$. Similarly, distance$(v_7, v_9) = 3$ unless $u_7 = u_9$ and distance$(v_8, v_9) = 3$ unless $u_8 = u_9$. However, we cannot have $u_7 = u_8 = u_9$, since G is not the Petersen graph. Hence, by symmetry, assume that $u_7 \neq u_8$. So distance$(v_7, v_8) = 3$. In this case, consider the two 5-cycles: $v_1v_2v_3v_4v_7v_1$ and $v_2v_3v_4v_5v_8v_2$; they share three consecutive vertices such that when labeled as above distance$(v_2, v_5) = 3$. Hence, the graph can be handled as in case 1 or 2. □

Lemma 9. *If G is an interesting 8-minimal subcubic graph, then G does not contain two 5-cycles that share an edge.*

Proof: Suppose G is a counterexample. By Lemmas 5-7, we know that G is 3-regular and that $g(G) \geq 5$. Taken together, these 5-cycles form an 8-cycle, with a chord. Label the vertices of the 8-cycle $v_1, v_2, \ldots, v_8$ with an edge between v_1 and v_5. By Lemmas 7 and 8, we know that distance$(v_2, v_6) = 3$. Similarly, we know that distance$(v_4, v_8) = 3$. By Lemma 3, we color all vertices of G^2 except the 8 v_is. Let $L(v)$ denote the list of remaining available colors at each vertex v. Note that $|L(v_1)| \geq 6$, $|L(v_2)| \geq 4$, $|L(v_3)| \geq 3$, $|L(v_4)| \geq 4$, $|L(v_5)| \geq 6$, $|L(v_6)| \geq 4$, $|L(v_7)| \geq 3$, and $|L(v_8)| \geq 4$. We assume that equality holds.

Case 1: There exists a color $c_1 \in L(v_4) \cap L(v_8)$. Use color c_1 on v_4 and v_8. Since $|L(v_2) \setminus \{c_1\}| + |L(v_6) \setminus \{c_1\}| > |L(v_5) \setminus \{c_1\}|$, we can choose color c_2 for v_2 and color c_3 for v_6 such that $|L(v_5) \setminus \{c_1, c_2, c_3\}| \geq 4$. Now since excess$(v_1) \geq 1$ and excess$(v_5) \geq 2$, we can finish the coloring by Lemma 4.

Case 2: $L(v_4) \cap L(v_8) = \emptyset$. We can choose color c_1 for v_2 and color c_2 for v_6 such that $|L(v_5) \setminus \{c_1, c_2\}| \geq 5$. Note that now excess$(v_5) \geq 1$. Now color v_3 and v_7 arbitrarily with colors from their lists; call them c_3 and c_4, respectively. Since $L(v_4) \cap L(v_8) = \emptyset$, the remaining lists for v_4 and v_8 have sizes summing to at least 4; call these lists $L'(v_4)$ and $L'(v_8)$. If $|L'(v_4)| \geq 3$, then excess$(v_4) \geq |L'(v_4)| - 1 = 2$, so by Lemma 4 we can finish the coloring. Similarly, if $|L'(v_8)| \geq 3$, then $\text{excess}(v_8) \geq |L'(v_8)| - 1 = 2$, so by Lemma 4 we can finish the coloring. So assume that $|L'(v_4)| = |L'(v_8)| = 2$. Arbitrarily color v_1 from its list; call the color c_3. Since $L(v_4) \cap L(v_8) = \emptyset$, either $|L'(v_4) \setminus \{c_3\}| = 2$ or $|L'(v_8 \setminus \{c_3\}| = 2$. In the first case, $\text{excess}(v_4) \geq 2$; in the second case, $\text{excess}(v_8) \geq 2$. In either case, we can greedily finish the coloring by Lemma 4. □

Lemma 10. *If G is an interesting 8-minimal subcubic graph, then $g(G) > 5$.*

Proof: Suppose G is a counterexample. By Lemmas 5-7, we know that G is 3-regular and that $g(G) = 5$. Let $v_1v_2v_3v_4v_5v_1$ be a 5-cycle and let u_i be the neighbor of vertex v_i not on the 5-cycle.

By Lemma 3, we can greedily color all vertices except the u_is and v_is. Let $L(v)$ denote the list of remaining available colors at each vertex v. Note that $|L(u_i)| \geq 2$ and $|L(v_i)| \geq 6$. We assume that equality holds for the v_is. By Lemma 8, we know that $\text{distance}(u_i, v_{i+2}) = \text{distance}(u_i, v_{i+3}) = 3$ for all i (subscripts are modulo 5). By Lemma 9 we also know that $\text{distance}(u_i, u_{i+1}) = 3$.

Case 1: There exists a color $c_1 \in L(u_1) \cap L(v_3)$. Use c_1 on u_1 and v_3. Greedily color vertices u_2, u_3, u_4; call these colors c_2, c_3, c_4, respectively. Now $|L(v_1) \setminus \{c_2, c_3\}| = 4$, $|L(v_2) \setminus \{c_2, c_3, c_4\}| \geq 3$, and $|L(u_5)| \geq 2$. We can choose color c_5 for u_5 and color c_6 for v_2 such that $|L(v_1) \setminus \{c_1, c_2, c_5, c_6\}| \geq 3$. Now greedily color the remaining vertices in the order v_4, v_5, v_1.

Case 2: There exists a color $c_1 \in L(u_1) \cap L(u_2)$. Use color c_1 on u_1 and u_2. Now $|L(v_5) \setminus \{c_1\}| + |L(u_3)| > |L(v_2) \setminus \{c_1\}|$, so we can choose color c_2 for v_5 and color c_3 for v_3 so that $\text{excess}(v_2) \geq 2$. Note that $\text{excess}(v_1) \geq 1$. Hence, after we greedily color u_5, we can extend the partial coloring to the remaining uncolored vertices by Lemma 4.

Case 3: $L(u_i) \cap L(u_{i+1}) = \emptyset$ and $L(u_i) \cap L(v_{i+2}) = \emptyset$ for all i. By symmetry, we can assume $L(u_i) \cap L(v_{i+3}) = \emptyset$ for all i. We now show that we can color each vertex with a distinct color. Suppose not.

By Hall's Theorem [18], there exists a subset of the uncolored vertices V_1 such that $|\cup_{v \in V_1} L(v)| < |V_1|$. Recall that $|L(u_i)| \geq 2$ and $|L(v_i)| = 6$ for all i. Clearly, $|V_1| > 2$. If $|V_1| \leq 6$, then $V_1 \subseteq \{u_1, u_2, u_3, u_4, u_5\}$. Any three u_is contain a pair u_j, u_{j+1}; their lists are disjoint, so $|\cup_{v \in V_1} L(v)| \geq |L(u_j)| + |L(u_{j+1})| \geq 4$. If $|V_1| = 5$, then $V_1 = \{u_1, u_2, u_3, u_4, u_5\}$. However, each color appears on at most two u_is, hence $|\cup_{v \in V_1} L(v)| \geq 10/2 = 5$. So say $|V_1| \geq 7$. The Pigeonhole principle implies that V_1 must contain a pair u_i, v_{i+2}. Since lists $L(u_i)$ and $L(v_{i+2})$ are disjoint, we have $|\cup_{v \in V_1} L(v)| \geq |L(u_i)| + |L(v_{i+2})| = 2 + 6 = 8$. Hence, $|V_1| \geq 9$. Now V_1 must contain a triple u_i, u_{i+1}, v_{i+3}. Since their lists are pairwise disjoint, we get $|\cup_{v \in V_1} L(v)| \geq |L(u_i)| + |L(u_{i+1})| + |L(v_{i+3})| = 2 + 2 + 6 = 10$. This is a contradiction. Thus, we can finish the coloring. □

Now we prove that if G is 8-minimal, then G does not contain a 6-cycle.

Lemma 11. *If G is an interesting 8-minimal subcubic graph, then $g(G) > 6$.*

Proof: Let G be a counterexample. By Lemma 10, we know that $g(G) > 5$. Hence, a counterexample must have girth 6. We show how to color G from lists of size 8. First, we prove that if $H = C_6$, then $\chi_l(H^2) = 3$. Our plan is to first color all vertices except those on the 6-cycle, then color the vertices of the 6-cycle.

Claim: If $H = C_6$, then $\chi_l(H^2) = 3$.

Label the vertices v_1, v_2, v_3, v_4, v_5, v_6 in succession. Let $L(v)$ denote the list of available colors at each vertex v. We consider separately the cases where $L(v_1) \cap L(v_4) \neq \emptyset$ and where $L(v_1) \cap L(v_4) = \emptyset$.

Case 1: There exists a color $c_1 \in L(v_1) \cap L(v_4)$. Use color c_1 on v_1 and v_4. Note that $|L(v_i) \setminus \{c_1\}| \geq 2$ for each $i \in \{2, 3, 5, 6\}$. If there exists a color $c_2 \in (L(v_2) \cap L(v_5)) \setminus \{c_1\}$, then use color c_2 on v_2 and v_5. Now greedily color v_3 and v_6. So suppose there is no color in $(L(v_2) \cap L(v_5)) \setminus \{c_1\}$. Color v_3 arbitrarily; call it color c_3. Either $|L(v_2) \setminus \{c_1, c_3\}| \geq 2$ or $|L(v_5) \setminus \{c_1, c_3\}| \geq 2$. In the first case, greedily color v_5, v_6, v_2. In the second case, greedily color v_2, v_6, v_5.

Case 2: $L(v_1) \cap L(v_4) = \emptyset$. By symmetry, we assume $L(v_2) \cap L(v_5) = \emptyset$ and $L(v_3) \cap L(v_6) = \emptyset$. Color v_1 arbitrarily; call it color c_1. If there exists i such that $|L(v_i) \setminus \{c_1\}| = 2$, then color v_4 from $c_2 \in L(v_4) \setminus L(v_i)$; otherwise color v_4 arbitrarily. Let $L'(v_j) = L(v_j) \setminus \{c_1, c_2\}$ for all $j \in \{2, 3, 5, 6\}$. Note that $|L'(v_2)| + |L'(v_5)| \geq 4$ and $|L'(v_3)| + |L'(v_6)| \geq 4$. Also, note that there is at most one k in $\{2, 3, 5, 6\}$ such that $|L'(k)| = 1$. So by symmetry we consider two subcases.

Subcase 2.1: $|L'(v_j)| \geq 2$ for every $j \in \{2, 3, 5, 6\}$. We can finish as in case 1 above.

Subcase 2.2: $|L'(v_2)| = 1$, $|L'(v_3)| \geq 2$, $|L'(v_6)| \geq 2$, and $|L'(v_5)| \geq 3$. We color greedily in the order v_2, v_3, v_6, v_5.

This finishes the proof of the claim; now we prove the lemma.

Let u and v be adjacent vertices on a 6-cycle $\mathcal{C}$. By Lemma 3, color all vertices except the vertices of $\mathcal{C}$. Since $g(G) = 6$, $\mathcal{C}$ has no chords. Similarly, no two vertices of $\mathcal{C}$ have a common neighbor not on $\mathcal{C}$. Note that each vertex of $\mathcal{C}$ has at least three available colors. Hence, by the Claim we can finish the coloring. □

The fact that $\chi_l(C_6^2) = 3$ is a special case of a theorem by Juvan, Mohar, and Škrekovski [10]. They showed that for any k, if $G = C_{6k}$, then $\chi_l(G^2) = 3$. Their proof uses algebraic methods and is not constructive. This fact is also a special case of a result by Fleischner and Steibitz [7]; their result also relies on algebraic methods.

Lemma 12. *Let $\mathcal{C}$ be a shortest cycle in an interesting 8-minimal subcubic graph G. If u_1 and u_2 are each distance 1 from $\mathcal{C}$, then u_1 and u_2 are nonadjacent.*

Proof: Let $\mathcal{C}$ be a shortest cycle in G. Lemma 11 implies that $|V(\mathcal{C})| \geq 7$. Let $v_1, v_2, \ldots, v_k$ be the vertices of $\mathcal{C}$. Recall that G is 3-regular. Let u_i be the neighbor of v_i that is not on $\mathcal{C}$. Suppose that there exists u_i adjacent to u_j. Let d be the distance from v_i to v_j *along* $\mathcal{C}$. By combining the path $v_i u_i u_j v_j$ with the shortest path along $\mathcal{C}$ from v_i to v_j, we get a cycle of length $3 + d \leq 3 + \lfloor |V(\mathcal{C}|)/2 \rfloor < |V(\mathcal{C})|$. This contradicts the fact that $\mathcal{C}$ is a shortest cycle in G. □

We are now ready to prove Theorem 1.

Theorem 1. *If G is an interesting subcubic graph, then $\chi_l(G^2) \leq 8$.*

Proof: Let G be a counterexample. By Lemma 5, we know that G is 3-regular. By Lemma 11, we know that G has girth at least 7. Let $\mathcal{C}$ be a shortest cycle in G. Let $v_1, v_2, \ldots, v_k$ be the vertices of $\mathcal{C}$. Let u_i be the neighbor of v_i that is not on $\mathcal{C}$. Let H be the union of the v_is and the u_is. By Lemma 3, we can color $G^2 \setminus V(H)$. Let $L(v)$ denote the list of available colors at each vertex v. Note that $|L(v_i)| \geq 6$ and $|L(u_i)| \geq 2$ for all i. We assume that equality holds.

Claim 1: If we can choose color c_1 for u_i and color c_2 for u_{i+1} such that $|L(v_i) \setminus \{c_1, c_2\}| \geq 5$ and $|L(v_{i+1}) \setminus \{c_1, c_2\}| \geq 5$, then we can extend the coloring to all of G^2.

Use colors c_1 and c_2 on u_i and u_{i+1}. Since $|L(u_{i-1})| = 2$ and $|L(v_{i+2}) \setminus \{c_2\}| \geq 5$ and $|L(v_i) \setminus \{c_1, c_2\}| \geq 5$, we can choose color c_3 for u_{i-1} and color c_4 for v_{i+2} so that $|L(v_i) \setminus \{c_1, c_2, c_3, c_4\}| \geq 4$. Color u_{i+2} arbitrarily. Now since $\text{excess}(v_{i+1}) \geq 1$ and $\text{excess}(v_i) \geq 2$, we can greedily finish the coloring by Lemma 4.

Claim 2: If we can choose color c_1 for u_i such that $|L(v_i) \setminus \{c_1\}| = 6$, then we can extend the coloring to all of G.

Use color c_1 on u_i. Since $|L(u_{i-1})| = 2$ and $|L(v_{i+1}) \setminus \{c_1\}| \geq 5$ and $|L(v_{i-1}) \setminus \{c_1\}| \geq 5$, we can chose color c_2 for u_{i-1} and color c_3 for v_{i+1} such that $|L(v_{i-1}) \setminus \{c_1, c_2, c_3\}| \geq 4$. If $c_2 = c_3$, then we use c_2 on vertices u_{i-1} and v_{i+1}; Now $\text{excess}(v_{i+1}) \geq 1$ and $\text{excess}(v_i) \geq 2$. So after we greedily color u_{i+1}, we can finish by Lemma 4. Hence, we can assume $c_2 \neq c_3$. Note that either $c_2 \notin L(v_{i-1})$ or $c_3 \notin L(v_{i-i})$. If $c_2 \notin L(v_{i-1})$, then use c_2 on u_i; now we can finish by Claim 1. Hence, we can assume $c_3 \notin L(v_{i-1})$. Use c_3 on v_{i+1}. Greedily color u_{i+1} and u_{i+2}; call these colors c_4 and c_5, respectively. We may assume that $|L(v_i) \setminus \{c_1, c_3, c_4\}| = 4$ (otherwise, we can finish greedily as above). We also know that $|L(u_{i-1})| = 2$ and $|L(v_{i+2}) \setminus \{c_3, c_4, c_5\}| \geq 3$. Hence, we can choose color c_6 for u_{i-1} and color c_7 for v_{i+2} such that $|L(v_i) \setminus \{c_1, c_3, c_4, c_6, c_7\}| \geq 3$. Now since $\text{excess}(v_{i-1}) \geq 1$ and $\text{excess}(v_i) \geq 2$, we can finish by Lemma 4.

Claim 3: If we can choose color c_1 for u_{i+1} such that $|L(v_i) \setminus \{c_1\}| = 6$, then we can extend the coloring to all of G.

Use color c_1 on u_{i+1}. Since $|L(u_i)| = 2$ and $|L(v_{i+2}) \setminus \{c_1\}| \geq 5$ and $|L(v_{i+1})| = 6$, we can choose color c_2 for u_i and color c_3 for v_{i+2} such that $|L(v_{i+1}) \setminus \{c_1, c_2, c_3\}| \geq 4$. Now we are in the same situation as in the proof of Claim 2. If $c_2 = c_3$, then we use color c_2 on u_i and v_{i+2} and color greedily as in Claim 2. If $c_2 \notin L(v_{i+1}) \setminus \{c_1\}$, then we use c_2 on u_i and we can finish by Claim 1. Hence we must have $c_3 \notin L(v_{i+1})$. Use c_3 on $L(v_{i+2})$. As in Claim 2, we have $|L(v_i) \setminus \{c_1, c_3\}| \geq 5$ and $|L(v_{i+1}) \setminus \{c_1, c_3\}| \geq 5$. Hence, we can finish as in Claim 2.

Remark: Claim 2 and Claim 3 imply that for every i we have $L(u_{i-1}) \cup L(u_i) \cup L(u_{i+1}) \subseteq L(v_i)$. Furthermore, Claim 1 shows that $L(u_i) \cap L(u_{i+1}) = \emptyset$ for all i. To show that $L(u_{i-1})$, $L(u_i)$, and $L(u_{i+1})$ are pairwise disjoint we prove Claim 4.

Claim 4: If we can choose color c_1 for u_{i-1} and color c_2 for u_{i+1} such that $|L(v_i) \setminus \{c_1, c_2\}| \geq 5$, then we can extend the coloring to G.

Use color c_1 on u_{i-1} and color c_2 and u_{i+1}. Since $|L(u_i)| = 2$ and $|L(v_{i+2}) \setminus \{c_2\}| \geq 5$ and $|L(v_{i+1})| = 6$, we can choose color c_3 for u_i and color c_4 for v_{i+2} such that $|L(v_{i+1}) \setminus \{c_2, c_3, c_4\}| \geq 4$. If $c_3 = c_4$, then we use color c_3 on u_i and v_{i+2}; since excess$(v_{i+1}) \geq 1$ and excess$(v_i) \geq 2$, we can finish by Lemma 4. So either $c_3 \notin L(v_{i+1})$ or $c_4 \notin L(v_{i+1})$.

Suppose $c_3 \notin L(v_{i+1})$. Use c_3 on u_i. Since $|L(v_{i-1}) \setminus \{c_1, c_3\}| \geq 4$ and $|L(u_{i+2})| = 2$ and $|L(v_{i+1}) \setminus \{c_3\}| \geq 5$, we can choose color c_5 for v_{i-1} and color c_6 for u_{i+2} such that $|L(v_{i+1}) \setminus \{c_2, c_3, c_5, c_6\}| \geq 4$. Now since excess$(v_i) \geq 1$ and excess$(v_{i+1}) \geq 2$, we can finish by Lemma 4.

Suppose instead that $c_4 \notin L(v_{i+1})$. Use c_4 on v_{i+2}. Color u_{i+2} and u_{i+3} arbitrarily; call these colors c_5 and c_6, respectively. Since $|L(u_i)| = 2$ and $|L(v_{i+3}) \setminus \{c_4, c_5, c_6\}| \geq 3$ and $|L(v_{i+1}) \setminus \{c_2, c_4, c_5\}| = 4$, we can choose color c_7 for u_i and color c_8 for v_{i+3} such that $|L(v_{i+1}) \setminus \{c_2, c_4, c_5, c_7, c_8\}| \geq 3$. Now since excess$(v_i) \geq 1$ and excess$(v_{i+1}) \geq 2$, we can finish by Lemma 4.

Claim 5: We can extend the coloring to G in the following way. Color each u_j arbitrarily; let $c(u_j)$ denote the color we use on each u_j. Now assign a color to each v_j from $L(u_j) \setminus \{c(u_j)\}$.

For each j, Claim 4 implies that $L(u_{j-1})$, $L(u_j)$, and $L(u_{j+1})$ are pairwise disjoint. Hence, each v_j receives a color not in $\{c(u_{j-1}), c(u_j), c(u_{j+1})\}$. Similarly, since $L(u_j)$ is disjoint from $L(u_{j-2}), L(u_{j-1}), L(u_{j+1})$, and $L(u_{j+2})$, vertex v_j receives a color not in $\{c(v_{j-2}), c(v_{j-1}), c(v_{j+1}), c(v_{j+2})\}$. Hence, the coloring of G^2 is valid. □

4 Planar subcubic graphs with girth at least 7

In this section we prove that if G is a subcubic planar graph with girth at least 7, then $\chi_l(G^2) \leq 7$. Lemma 2 implies that such a graph G has $Mad(G) < \frac{14}{5}$. We exhibit four 7-reducible configurations. We show that every subcubic graph with $Mad(G) \leq \frac{14}{5}$ contains at least one of these 7-reducible configurations. This implies the desired theorem.

Lemma 13. *Let G be a* minimal *graph such that $\chi_l(G^2) > 7$. For each vertex v, let $M_1(v)$ and $M_2(v)$ be the number of 2-vertices at distance 1 and distance 2 from v. If v is a 3-vertex, then $2M_1(v) + M_2(v) \leq 2$. If v is a 2-vertex, then $2M_1(v) + M_2(v) = 0$.*

Proof: We list four 7-reducible configurations. We show that if there exists a vertex v such that the quantity $2M_1(v) + M_2(v)$ is larger than claimed, then G contains one of the four 7-reducible configurations.

Configuration 1: If G contains two adjacent 2-vertices v_1 and v_2, then $G[v_1v_2]$ is 7-reducible. Let $H = G \setminus \{v_1, v_2\}$. By hypothesis, H^2 has a proper coloring from its lists. Now greedily color vertex v_1, then vertex v_2.

Configuration 2: If G contains two 2-vertices v_1 and v_2 adjacent to a 3-vertex u, then $G[v_1v_2u]$ is 7-reducible. Let $H = G \setminus \{v_1, v_2, u\}$. By hypothesis, H^2 has a proper coloring from its lists. Now greedily color u, v_1, v_2 (in that order).

Configuration 3: If G contains two adjacent 3-vertices u_1 and u_2 and each u_i is adjacent to a distinct 2-vertex v_i, then $G[v_1v_2u_1u_2]$ is 7-reducible. Let $H = G \setminus \{v_1, v_2, u_1, u_2\}$. By hypothesis, H^2 has a proper coloring from its lists. Now greedily color u_1, u_2, v_1, v_2.

Configuration 4: Say G contains a 3-vertex w that is adjacent to three 3-vertices u_1, u_2, and u_3. If each u_i is adjacent to a distinct 2-vertex v_i, then $G[v_1v_2v_3u_1u_2u_3w]$ is 7-reducible. Let $H = G \setminus \{v_1, v_2, v_3, u_1, u_2, u_3, w\}$. By hypothesis, H^2 has a proper coloring from its lists. Now greedily color u_1, u_2, u_3, w, v_1, v_2, v_3.

Note that if x is a 2-vertex and $M_1(x) + M_2(x) > 0$, then G contains either the first or second reducible configuration. So $2M_1(x) + M_2(x) = 0$ for every 2-vertex x. If y is a 3-vertex and $2M_1(y) + M_2(y) > 2$, then G contains one of the four reducible configurations. □

Theorem 14. *If G is a subcubic graph with $Mad(G) < \frac{14}{5}$, then $\chi_l(G^2) \leq 7$.*

Proof: Let G be a minimal counterexample to the theorem. By Lemma 13, each 3-vertex v satisfies $2M_1(v)+M_2(v) \leq 2$ and each 2-vertex v satisfies $2M_1(v)+M_2(v) = 0$. We use these bounds to show that $Mad(G) \geq \frac{14}{5}$ (which is a contradiction). We use a discharging argument. Let the initial charge $\mu(v) = d(v)$. We have a single discharging rule:

> **R1**: Each 3-vertex gives $\frac{1}{5}$ to each 2-vertex at distance 1 and gives $\frac{1}{10}$ to each 2-vertex at distance 2.

Observe that each 2-vertex is distance at least 3 from every other 2-vertex. Thus, for every 2-vertex v we get $\mu^*(v) = 2 + 2(\frac{1}{5}) + 4(\frac{1}{10}) = \frac{14}{5}$. Recall that each 3-vertex v satisfies $2M_1(v)+M_2(v) \leq 2$. Hence, for every 3-vertex v we get $\mu^*(v) = 3 - \frac{1}{5}M_1(v) - \frac{1}{10}M_2(v) = 3 - \frac{1}{10}(2M_1(v) + M_2(v)) \geq 3 - \frac{1}{5} = \frac{14}{5}$. Since the new charge at each vertex is at least $\frac{14}{5}$, the average charge is at least $\frac{14}{5}$. Hence $Mad(G) \geq \frac{14}{5}$. This is a contradiction, so no counterexample exists. □

Corollary 15. *If G is a planar subcubic graph with girth at least 7, then $\chi_l(G^2) \leq 7$.*

Proof: From Lemma 2, we have $Mad(G) < \frac{14}{5}$. By Theorem 14, this implies that $\chi_l(G^2) \leq 7$. □

5 Planar subcubic graphs with girth at least 9

In this section we prove that if G is a subcubic planar graph with girth at least 9, then $\chi_l(G^2) \leq 6$. Lemma 2 implies that such a graph G has $Mad(G) < \frac{18}{7}$. Recall that a configuration is $6'$-reducible if it cannot appear in a 6-minimal graph with girth at least 7. We exhibit four $6'$-reducible configurations. We show that every subcubic graph with $Mad(G) < \frac{18}{7}$ contains at least one of these $6'$-reducible configurations. This implies the desired theorem.

We will prove that the following four configurations (shown in Figures 2a, 2b, 3a, and 3b) are $6'$-reducible. We begin with a definition: If v is a 3-vertex, then we say that v is of *class i* if v is adjacent to i vertices of degree 2. Note that if v_1 and v_2 are adjacent 2-vertices, then $G[v_1v_2]$ is $6'$-reducible. Hence, we assume that no pair of 2-vertices is adjacent.

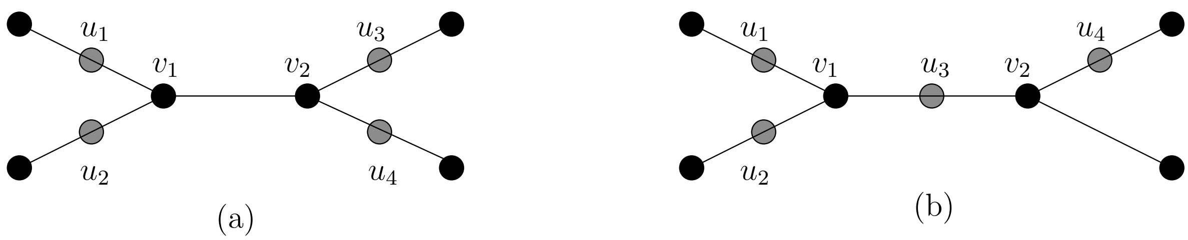

Figure 2. Two $6'$-reducible subgraphs. (a) Two adjacent class 2 vertices v_1 and v_2. (b) A class 3 vertex v_1 and a class 2 vertex v_2 at distance 2.

Lemma 16. *If v_1 and v_2 are adjacent class 2 vertices, then $G[v_1v_2]$ is $6'$-reducible. (This configuration is shown on the left in Figure 2.)*

Proof: Let v_1 and v_2 be adjacent class 2 vertices. Let v_1 be adjacent to vertices u_1 and u_2 and let v_2 be adjacent to vertices u_3 and u_4. Let $H = G \setminus \{v_1, v_2, u_1, u_2, u_3, u_4\}$. By hypothesis, H^2 has a coloring from its lists. Let $L(x)$ denote the list of remaining available colors for each uncolored vertex x in G. Note that $|L(v_i)| \geq 4$ and $|L(u_i)| \geq 3$ for each i. We assume that equality holds (otherwise we throw away colors until it does). Since G has girth at least 7, note that u_1 and u_2 are each distance 3 from each of u_3 and u_4.

Since $|L(v_1)| = 4$ and $|L(u_1)| = 3$, there is a color $c \in L(v_1) \setminus L(u_1)$. Use color c on vertex v_1. The sizes of the new lists (after removing c from each) are $|L(u_1) \setminus \{c\}| = 3$, $|L(v_2) \setminus \{c\}| \geq 3$, and $|L(u_i) \setminus \{c\}| \geq 2$ for $i = 2, 3, 4$. Greedily color the remaining vertices in the order u_3, u_4, v_2, u_2, u_1. This completes the coloring, and proves the lemma. □

Lemma 17. *If v_1 is a class 3 vertex, v_2 is either a class 2 or class 3 vertex, and vertices v_1 and v_2 have a common neighbor u_3, then $G[v_1v_2u_3]$ is $6'$-reducible. (This configuration is shown on the right in Figure 2.) If G contains this configuration and $G^2 \setminus \{u_3\}$ has a proper L-coloring from lists L of size 6, then G^2 has two proper L-colorings ϕ and ψ such that $\phi(u_3) \neq \varphi(u_3)$.*

Proof: Let v_1 be a 3-vertex adjacent to three 2-vertices u_1, u_2, and u_3. Say that v_2 is a 3-vertex adjacent to u_3 and also adjacent to another 2-vertex, u_4. Let $H = G \setminus \{v_1, v_2, u_1, u_2, u_3, u_4\}$. By hypothesis, H^2 has a coloring from its lists. Let $L'(x)$ denote the list of remaining available colors for each uncolored vertex x in G. Note that $|L'(u_1)| \geq 3$, $|L'(u_2)| \geq 3$, $|L'(u_3)| \geq 5$, $|L'(u_4)| \geq 2$, $|L'(v_1)| \geq 4$, and $|L'(v_2)| \geq 2$. We assume that equality holds. (Since G has girth at least 7, note that u_4 is distance at least 3 from each of u_1, u_2, and v_1.)

Since $|L'(v_1)| = 4$ and $|L'(u_1)| = 3$, we can choose a color $c \in L'(v_1) \setminus L'(u_1)$. Use color c on vertex v_1. Greedily color vertex v_2, then vertex u_4. At this point, vertex u_3 has at least two available colors. We can use either available color on u_3 (one choice will give coloring ϕ and the other will give coloring ψ). Now greedily color vertex u_2, then vertex u_1. This completes the coloring, and proves the lemma. □

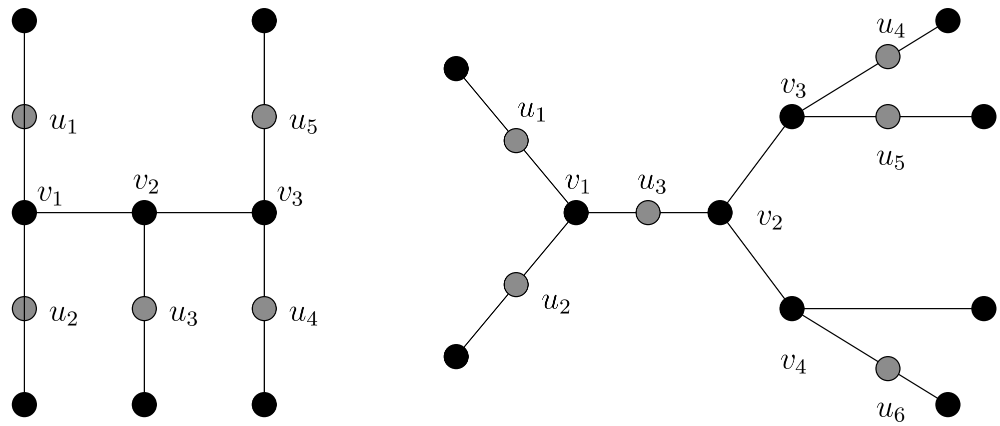

Figure 3. An H-configuration and a Y-configuration; both configurations are $6'$-reducible. (a) An H-configuration: a class 1 vertex v_2 is adjacent to two class 2 vertices v_1 and v_3. (b) A Y-configuration: a class 1 vertex v_2 is adjacent to a class 2 vertex v_3 and a class 1 vertex v_4, and is distance two from a class 3 vertex v_1.

Lemma 18. *We use the term H-configuration to denote a class 1 vertex adjacent to two class 2 vertices. An H-configuration (shown on the left in Figure 3) is $6'$-reducible.*

Proof: Let $H = G \setminus \{v_1, v_2, v_3, u_1, u_2, u_3, u_4, u_5\}$ (see Figure 3). By hypothesis, H^2 has a coloring from its lists. Let $L(x)$ denote the list of remaining available colors for each uncolored vertex x in G. Note that $|L(u_i)| \geq 3$, $|L(v_1)| \geq 4$, $|L(v_3)| \geq 4$, and $|L(v_2)| \geq 5$. We assume that equality holds. Since $|L(v_2)| > |L(u_5)|$, we can choose color $c \in L(v_2) \setminus L(u_5)$. Use color c on vertex v_2. Now greedily color the remaining vertices in the order $u_1, u_2, v_1, u_3, v_3, u_4, u_5$. This completes the coloring, and proves the lemma. □

Lemma 19. *We use the term Y-configuration to denote a class 1 vertex adjacent to a class 2 vertex, adjacent to a class 1 vertex, and distance two from a class 3 vertex. A Y-configuration (shown on the right in Figure 3) is $6'$-reducible.*

Proof: Let $H = G \setminus \{v_1, u_1, u_2, u_3\}$ (see Figure 3). By hypothesis, H^2 has a proper coloring from its lists. Let $L(x)$ denote the list of remaining available colors for each uncolored vertex x. Assume the coloring of H^2 cannot be extended to G^2. Hence $|L(v_1)| = |L(u_1)| = |L(u_2)| = |L(u_3)| = 3$ and $L(v_1) = L(u_1) = L(u_2) = L(u_3)$. (Otherwise the coloring could be extended to G^2.) By Lemma 17, H^2 has a recoloring such that v_2 gets a different color than it currently has. Under this recoloring of H^2, the lists of available colors for v_3, u_1, u_2, and u_3 are no longer identical. Hence, the recoloring of H^2 can be extended to G^2. □

Theorem 20. *If G is a subcubic graph with $Mad(G) < \frac{18}{7}$ and girth at least 7, then $\chi_l(G^2) \leq 6$.*

Proof: Let G be a minimal counterexample to Theorem 20. We show that if G does not contain one of the $6'$-reducible configurations in Lemmas 16, 17, 18, or 19, then $Mad(G) \geq \frac{18}{7}$. We use a discharging argument with initial charge $\mu(v) = d(v)$. We have the following three discharging rules.

R1: Each 3-vertex gives $\frac{2}{7}$ to each adjacent 2-vertex.

R2: Each class 0 vertex gives $\frac{1}{7}$ to each adjacent 3-vertex.

R3: Each class 1 vertex gives $\frac{1}{7}$ to each adjacent class 2 vertex and gives $\frac{1}{7}$ to each class 3 vertex at distance 2.

We must show that for every vertex v, the new charge $\mu^*(v) \geq \frac{18}{7}$.

Recall that each 2-vertex v is adjacent only to 3-vertices. Hence, for a 2-vertex v we have $\mu^*(v) = 2 + 2(\frac{2}{7}) = \frac{18}{7}$. So we consider 3-vertices.

Let v be a 3-vertex. We consider vertices of class 0, class 1, class 2, and class 3 separately.

If v is class 0, then $\mu^*(v) = 3 - 3(\frac{1}{7}) = \frac{18}{7}$.

If v is class 2, then by Lemma 16 vertex v is adjacent to a class 1 vertex or a class 0 vertex. Hence $\mu^*(v) = 3 - 2(\frac{2}{7}) + \frac{1}{7} = \frac{18}{7}$.

If v is class 3, then by Lemma 17 each 3-vertex at distance 2 from v is a class 1 vertex. Hence $\mu^*(v) = 3 - 3(\frac{2}{7}) + 3\frac{1}{7} = \frac{18}{7}$.

Let v be class 1. By Lemma 18, v is adjacent to at most one class 2 vertex. Clearly, v is distance 2 from at most one class 3 vertex. Hence $\mu^*(v) \geq \frac{18}{7}$ unless v is adjacent to a class 2 vertex w and distance 2 from a class 3 vertex x. So we consider this case. Let y be the other 3-vertex adjacent to v. Clearly, y is not class 3 or class 2 (by Lemma 18). If y is class 1, then we have the $6'$-reducible subgraph in Lemma 19. Hence, y must be class 0. In that case y gives $\frac{1}{7}$ to v, so $\mu^*(v) = 3 - \frac{2}{7} - 2(\frac{1}{7}) + \frac{1}{7} = \frac{18}{7}$. Thus, $Mad(G) \geq \frac{18}{7}$. This is a contradiction, so no counterexample exists. □

Corollary 21. *If G is a planar subcubic graph with girth at least 9, then $\chi_l(G^2) \leq 6$.*

Proof: From Lemma 2, we see that $Mad(G) < \frac{18}{7}$. By Theorem 20, this implies that $\chi_l(G^2) \leq 6$. □

6 Efficient Algorithms

Since the proof of Theorem 1 colors all but a constant number of vertices greedily, it is not surprising that the algorithm can be made to run in linear time. For completeness, we give the details.

If G is not 3-regular or G has girth at most 6, then we find a small subgraph H (one listed in Lemmas 5-11) that contains a low degree vertex or a shortest cycle. It is easy to greedily color $G^2 \setminus V(H)$ in linear time (for example, using breadth-first search). Since H has constant size, we can finish the coloring in constant time.

Say instead that G is 3-regular and has girth at least 7. Choose an arbitrary vertex v. Find a shortest cycle through v (for example, using breadth-first search); call it $\mathcal{C}$. Let H be $\mathcal{C}$ and vertices at distance 1 from $\mathcal{C}$. We greedily color $G^2 \setminus V(H)$ in linear time. Using the details given in the proof of Theorem 3, we can finish the coloring in time linear in the size of H.

The proofs of Theorems 14 and 20 are examples of a large class of discharging proofs that can be easily translated into linear time algorithms. The algorithm for each consists of finding a reducible configuration H (7-reducible for Theorem 14 and $6'$-reducible for Theore 20), recursively coloring $G^2 \setminus V(H)$, then extending the coloring to G^2. To achieve a linear running time, we need to find the reducible configuration in amortized constant time. We make no effort to discover the optimal constant k in the kn running time; we only outline the technique to show that the algorithm can be made to run in linear time.

First we decompose G, by removing one reducible configuration after another; when we remove a configuration from G, we add it to a list A (of removed configurations). After decomposing G, we build the graph back up, adding elements of A in the reverse of the order they were removed. When we add back an element of A, we color all of its vertices. In this way, we eventually reach G, with every vertex colored. We call these two stages the decomposing phase and the rebuilding phase. It only remains to specify how we find each configuration that we remove during the decomposing phase.

Our plan is to maintain a list B of instances in the graph of reducible configurations. We begin with a preprocessing phase, in which we store in B every instance of a reducible configuration in the original graph. Using brute force, we can do this in linear time (since we have only a constant number of reducible configurations and each configuration is of bounded size, each vertex can appear in only a constant number of instances of reducible configurations).

When we remove a reducible configuration H from G, we may create new reducible configurations. We can search for these new reducible configurations in constant time (since they must be adjacent to H). We add each of these new reducible configurations to B. In removing H, we may have destroyed one or more reducible configurations in B (for example, if they contained vertices of H). We make no effort to remove the destroyed configurations from B. Thus, at every point in time, B will contain all the reducible configurations in the remaining graph (as well as possibly containing many "destroyed" reducible configurations). To account for this, when we choose a configuration H from B to remove from the remaining graph, we must verify that H is not destroyed. If H is destroyed, we discard it, and proceed to the next configuration in B. We will show that the entire process of decomposing G (and building A) takes linear time. (However, during the process, the time required to find a particular configuration to add to A may not be constant.)

Theorems 14 and 20 guarantee that as we decompose G, list B will never be empty. Our only concern is that perhaps B may contain "too many" destroyed configurations. We show that througout both the preprocessing phase and the decomposing phase, only a linear number of configurations get added to B. In the original graph G, each vertex can appear in only a constant number of reducible configurations; hence, in the preprocessing phase, only a linear number of reducible configurations are added to B.

During the decomposing phase, if we remove a destroyed configuration from B, we discard it without adding any configurations to B. If we remove a valid configuration from B, we add only a constant number of configurations to B. Each time we remove a valid configuration from B, we decrease the number of vertices in the remaining graph; hence we remove only a linear number of valid configurations from B. Thus, during the decomposing phase, we add only a linear number of configurations to B. As a result, the decomposing phase runs in linear time.

During the rebuilding phase, we use constant time to add a configuration back, and constant time to color the configuration's vertices (we do this using the lemma that proved the configuration was reducible). List A contains only a linear number of configurations, hence, the rebuilding phase runs in linear time. Since each of the preprocessing phase, decomposing phase, and rebuilding phase runs in linear time, our complete algorithm runs in linear time.

7 Future Work

As we mentioned in the introduction, Theorem 1 is best possible, since there are infinitely many interesting subcubic graphs G such that $\chi_l(G^2) = 8$ (for example, any graph which contains the Petersen graph with one edge removed). However, it is natural to ask whether the result can be extended to graphs with arbitrary maximum degree. Let G be a graph with maximum degree $\Delta(G) = k$. Since $\Delta(G^2) \leq k^2$, we immediately get that $\chi_l(G^2) \leq k^2 + 1$. If $G^2 \neq K_{k^2+1}$, then by the list-coloring version of Brook's Theorem [5], we have $\chi_l(G^2) \leq k^2$. Hoffman and Singleton [12] made a thorough study of graphs G with maximum degree k such that $G^2 = K_{k^2+1}$. They called these *Moore Graphs*. They showed that a unique Moore Graph exists when $\Delta(G) \in \{2, 3, 7\}$ and possibly when $\Delta(G) = 57$ (which is unknown), but that no Moore Graphs exist for any other value of $\Delta(G)$. (When $\Delta(G) = 3$, the unique Moore Graph is the Petersen Graph). Hence, if $\Delta(G) \notin \{2, 3, 7, 57\}$, we know that $\chi_l(G^2) \leq \Delta(G)^2$. As in Theorem 1, we believe that we can improve this upper bound.

Conjecture 1. *If G is a graph with maximum degree $\Delta(G) = k$ and G is not a Moore Graph, then $\chi_l(G^2) \leq k^2 - 1$.*

Erdős, Fajtlowitcz and Hoffman [6] considered graphs G with maximum degree k such that $G^2 = K_{k^2}$. The proved the following result, which provides evidence in support of our conjecture.

Theorem. *(Erdős, Fajtlowitcz and Hoffman [6]) Apart from the cycle C_4, there is no graph G with maximum degree k such that $G^2 = K_{k^2}$.*

We extend this result to give a bound on the clique number $\omega(G^2)$ of the square of a non-Moore graph G with maximum degree k.

Lemma 22. *If G is not a Moore graph and G has maximum degree $k \geq 3$, then G^2 has clique number $\omega(G^2) \leq k^2 - 1$.*

Proof: If G is a counterexample, then by the Theorem of Erdős, Fajtlowitcz and Hoffman, we know that G^2 properly contains a copy of K_{k^2}. Choose adjacent vertices u and v_1 such that v_1 is in a clique of size k^2 (in G^2) and u is not in that clique; call the clique H. Note that $|N[v_1]| \leq k^2 + 1$, so all vertices in $N[v_1]$ other than u must be in H. Label the neighbors of u as v_is. Note that no v_i is on a 4-cycle. If so, then $|N[v_i]| \leq k^2$; since $u \in N[v_i]$ and $u \notin V(H)$, we get $|V(H)| \leq k^2 - 1$, which is a contradiction.

Note that each neighbor of a vertex v_i (other than u) must be in H. Since no v_i lies on a 4-cycle, each pair v_i, v_j have u as their only common neighbor. So the v_is and their neighbors (other than u) are k^2 vertices in H. But u is within distance 2 of each of these k^2 vertices in H. Hence, adding u to H yields a clique of size $k^2 + 1$. This is a contradiction. □

We believe that Conjecture 1 can probably be proved using an argument similar to our proof of Theorem 1. In fact, arguments from our proof of Theorem 1 easily imply that if G is a counterexample to Conjecture 1, then G is k-regular and has $g(G) \in \{4, 5\}$. However, we do not see a way to handle these remaining cases without resorting to extensive case analysis (which we have not done).

Significant work has also been done proving lower bounds on $\chi_l(G)$. Brown [2] constructed a graph G with maximum degree k and $\chi_l(G^2) \geq k^2 - k + 1$ whenever

$k-1$ is a prime power. By combining results of Brown [2] and Huxley [9], Miller and Širáň [13] showed that for every $\epsilon > 0$ there exists a constant c_ϵ such that for every k there exists a graph G with maximum degree k such that $\chi_l(G^2) \geq k^2 - c_\epsilon k^{19/12+\epsilon}$.

Another area for further work is reducing the girth bounds imposed in Theorems 14 and 20. We know of no subcubic planar graph G with girth at least 4 such that $\chi_l(G^2) = 7$. (If G is the cartesian product $C_3 \Box K_2$, then subdividing an edge of G not in a 3-cycle yields a planar subcubic graph G' such that $\chi_l((G')^2) = 7$). We know of no subcubic planar graph G with girth at least 6 such that $\chi_l(G^2) = 6$.

Finally, we can consider the restriction of Theorem 1 to planar graphs. If G is a planar subcubic graph, then we know that $\chi_l(G^2) \leq 8$. However, we don't know of any planar graphs for which this is tight. This returns us to the question that motivated much of this research and that remains open.

Question 2. *Is it true that every planar subcubic graph G satisfies $\chi_l(G^2) \leq 7$?*

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