

# Sufficient sparseness conditions for $G^2$ to be $(\Delta + 1)$ -choosable, when $\Delta \geq 5$ .

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## Abstract

We determine the list chromatic number of the square of a graph  $\chi_\ell(G^2)$  in terms of its maximum degree  $\Delta$  when its maximum average degree, denoted  $\text{mad}(G)$ , is sufficiently small. For  $\Delta \geq 6$ , if  $\text{mad}(G) < 2 + \frac{4\Delta-8}{5\Delta+2}$ , then  $\chi_\ell(G^2) = \Delta + 1$ . In particular, if  $G$  is planar with girth  $g \geq 7 + \frac{12}{\Delta-2}$ , then  $\chi_\ell(G^2) = \Delta + 1$ . Under the same conditions,  $\chi_\ell^i(G) = \Delta$ , where  $\chi_\ell^i$  is the list injective chromatic number.

## 1 Introduction

The square of a graph  $G$ , denoted by  $G^2$ , is the graph with  $V(G^2) = V(G)$  and  $E(G^2) = \{uv \mid d_G(u, v) \leq 2\}$ ; in other words, two vertices are adjacent in  $G^2$  if they are at distance at most two in  $G$ . If  $G$  has maximum degree  $\Delta$ , then coloring  $G^2$  requires at least  $\Delta + 1$  colors; the upper bound  $\Delta^2 + 1$  follows from the greedy algorithm. This upper bound is also achieved for a few graphs, for example for the 5-cycle and the Petersen graph.

Regarding the coloring of squares of planar graphs, Wegner [26] posed the following central problem.

**Conjecture 1** (Wegner). *For a planar graph  $G$  of maximum degree  $\Delta$ :*

$$\chi(G^2) \leq \begin{cases} 7, & \Delta = 3; \\ \Delta + 5, & 4 \leq \Delta \leq 7; \\ \lceil \frac{3}{2}\Delta \rceil + 1, & \Delta \geq 8. \end{cases}$$

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In [16] Havet, van den Heuvel, McDiarmid, and Reed showed that the following holds:  $\chi(G^2) \leq \frac{3}{2}\Delta(1+o(1))$ , which is also true for the choice number (defined below). Dvořák, Král', Nejedlý, and Škrekovski [12] showed that the square of every planar graph of girth at least six with sufficiently large maximum degree  $\Delta$  is  $(\Delta+2)$ -colorable. Borodin and Ivanova [7] strengthened this result to prove that for every planar graph  $G$  of girth at least six with maximum degree  $\Delta \geq 24$ , the choice number of  $G^2$  is at most  $\Delta + 2$ . Most recently, Bonamy, Léveque, and Pinlou [2] showed the same conclusion for every planar graph  $G$  with girth at least six and  $\Delta \geq 17$ . In fact, their proof only requires  $\text{mad}(G) < 3$  (defined below). Lih, Wang, and Zhu [22] showed that the square of a  $K_4$ -minor free graph with maximum degree  $\Delta$  has chromatic number at most  $\lfloor \frac{3}{2}\Delta \rfloor + 1$  if  $\Delta \geq 4$  and  $\Delta + 3$  if  $\Delta \in \{2, 3\}$ . Hetherington and Woodall [18] showed that the bounds in [22] also hold for the choice number.

We write  $\Delta$  for the maximum degree of a fixed graph  $G$ . A  $k$ -*vertex* is a vertex of degree  $k$ . Similarly, a  $k^+$ -*vertex* (resp.  $k^-$ -*vertex*) is a vertex of degree at least (resp. at most)  $k$ . A  $k$ -*thread* is a path with  $k$  internal 2-vertices. The *endpoints* of a thread are its first and last vertices. A *weak neighbor* of a vertex  $v$  is one joined to  $v$  by a  $k$ -thread (for some  $k \geq 1$ ), and a *weak  $k$ -neighbor* is a weak neighbor that is a  $k$ -vertex. We write  $N(v)$  for the neighborhood of  $v$  and  $N[v]$  for  $N(v) \cup \{v\}$ .

A proper *coloring* of the vertices of a graph  $G$  is a mapping  $c : V(G) \rightarrow \mathbb{N}$  such that every two adjacent vertices are mapped to different colors. Elements of  $\mathbb{N}$  are *colors*. List coloring was first studied by Vizing [25] and is defined as follows. Let  $G$  be a simple graph. A *list-assignment*  $L$  is an assignment of lists of colors to vertices. A *list-coloring* is a coloring where each vertex  $v \in V(G)$  receives a color from  $L(v)$ , and the graph  $G$  is  *$L$ -choosable* if there is a proper  $L$ -list-coloring. If  $G$  has a list-coloring for every list-assignment with  $|L(v)| \geq k$  for each vertex  $v$ , then  $G$  is  *$k$ -choosable*. The minimum  $k$  such that  $G$  is  $k$ -choosable is the *choice number* of  $G$ , and is denoted by  $\chi_l$ . An *injective coloring* of a graph  $G$  is a mapping  $c : V(G) \rightarrow \mathbb{N}$  such that vertices with a common neighbor are mapped to different colors (but it need *not* be proper). The *injective chromatic number*  $\chi^i(G)$  and *injective choice number*  $\chi_\ell^i(G)$  are defined analogously. For each  $G$ , we have  $\chi_\ell^i(G) \leq \chi_\ell(G^2)$ .

In the proofs of our theorems, we use the discharging method, which was first used by Wernicke [27], and which is most well-known for its central role in the proof of the Four Colour Theorem. Here we apply the discharging method in the more general context of bounded *maximum average degree*, denoted  $\text{mad}(G)$ , which is defined as  $\text{mad}(G) := \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$ , where  $H$  ranges over all subgraphs of  $G$ . A straightforward consequence of Euler's Formula is that every planar graph  $G$  with girth at least  $g$  satisfies  $\text{mad}(G) < \frac{2g}{g-2} = 2 + \frac{4}{g-2}$ . Using this bound on  $\text{mad}$ , our results for planar graphs follow immediately from corresponding results for maximum average degree. The key tool in many of our proofs is global discharging, which relies on reducible configurations that may be arbitrarily large. Global discharging was introduced by Borodin [5], and has been applied widely; for example, see [7] and [9].

Kostochka and Woodall [21] conjectured that every square of a graph has choice number equal to chromatic number, i.e.,  $\chi_l(G^2) = \chi(G^2)$ . This conjecture inspired much research, although recently it has been disproved [20]. For planar graphs, the best upper bound on  $\chi(G^2)$  in terms of  $\Delta$  was succesively improved by Jonas [19], Wong [28],

Van den Heuvel and McGuinness [17], Agnarsson and Halldorsson [1], Borodin et al. [6] and finally by Molloy and Salavatipour [23] to the best known upper bound so far,  $\chi(G^2) \leq \lceil \frac{5}{3}\Delta \rceil + 78$ . For the best asymptotic upper bound, see [16], mentioned above.

The choosability of squares of subcubic planar graphs has been extensively studied by Dvořák, Škrekovski, and Tancer [13], Montassier and Raspaud [24], Havet [15], and Cranston and Kim [8]. In [10], we gave upper bounds on  $\chi_\ell(G^2)$  when  $\Delta(G) = 4$  and  $\text{mad}(G)$  is bounded. In the present paper, we again consider graphs  $G$  with bounded maximum average degree, but now with higher maximum degree. For  $\Delta(G) \geq 6$ , our results are summarized in the following theorem.

**Main Theorem.** *Let  $G$  be a graph with maximum degree  $\Delta \geq 6$ . If  $\text{mad}(G) < 2 + \frac{4\Delta-8}{5\Delta+2}$ , then  $\chi_\ell(G^2) = \Delta + 1$ . In particular, if  $G$  is planar with girth  $g \geq 7 + \frac{12}{\Delta-2}$ , then  $\chi_\ell(G^2) = \Delta + 1$ .*

Besides our Main Theorem, for  $\Delta = 5$  we prove that  $\text{mad}(G) < 2 + 12/29$  implies  $\chi_\ell(G^2) = 6$ . Note that for  $\Delta = 4$ , in [10] we proved that  $\text{mad}(G) < 2 + 2/7$  implies  $\chi_\ell(G^2) = 5$ . We also construct examples with maximum degree  $k$  and  $\text{mad}$  arbitrarily close to  $2 + 2/7$  (resp.  $2 + 12/29$  and  $2 + (4k - 8)/(5k + 2)$ ) that contain none of the reducible configurations we use in the proofs. So to improve the coloring results, we need additional reducible configurations.

The Main Theorem is proved in three parts:  $k = 6$ ,  $k = 7$ , and  $k \geq 8$ . In each part, we assume a counterexample with the fewest vertices, then reach a contradiction. When we remove one or more vertices from this graph, the square of the result can be properly colored from its lists. We elaborate on this approach in the next section.

We mention in passing that each time that we prove that  $\chi_\ell(G^2) = \Delta + 1$ , the proof can be modified to show that  $\chi_\ell^i(G) = \Delta$ . The coloring algorithms are the same, but now each vertex has at least one fewer constraints on its color.

After submitting this paper, we learned that Bonamy, Lévêque, and Pinlou [3] have submitted a paper proving similar results, using similar methods. For  $\Delta \in \{6, 8\}$ , their results match ours. In general, for  $\epsilon > 0$ , they proved there exists  $\Delta_\epsilon$  such that if  $\text{mad}(G) < \frac{14}{5} - \epsilon$  and  $\Delta \geq \Delta_\epsilon$ , then  $\chi_\ell(G^2) = \Delta + 1$  and  $\chi_\ell^i(G) = \Delta$ . However, for general  $\epsilon$  (other than  $\epsilon \in \{\frac{3}{10}, \frac{8}{35}\}$ , which corresponds to  $\Delta \in \{6, 8\}$ ), their bound on  $\Delta_\epsilon$  is slightly weaker than ours. Perhaps the best comparison is as follows. To prove that  $\chi_\ell(G^2) = \Delta + 1$  and  $\chi_\ell^i(G) = \Delta$  when  $\text{mad}(G) < \frac{14}{5} - \epsilon$ , their required lower bound on  $\Delta$  is a little more than  $\frac{5}{3}$  times as large as ours.

## 2 Reducible configurations

A *configuration* is an induced subgraph of a graph  $G$ . A configuration  $C$  is *reducible* if whenever  $G$  contains  $C$ , we can form a graph  $G'$  with fewer edges than  $G$  such that any good coloring of  $G'$  gives rise to a good coloring of  $G$ . Thus a reducible configuration cannot appear in a minimal counterexample. For convenience, we often write *color*  $G - C$  to mean color  $(G - C)^2$  from its assigned lists. To prove that a configuration is reducible, we infer from the minimality of  $G$  that the subgraph  $G - C$  can be properly

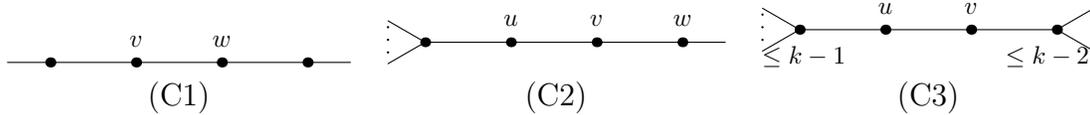


Figure 1: Configurations from Lemma 2

colored, and then prove that this coloring can be extended to a proper coloring of the original graph  $G$ , which gives a contradiction.

A configuration is  $k$ -reducible if it is reducible in the setting of  $k$ -choosability. Clearly a  $k$ -reducible configuration is also  $(k+1)$ -reducible. Perhaps the easiest example is that a 1-vertex  $v$  is  $(\Delta+1)$ -reducible, since  $G-v$  can be colored by minimality, and  $v$  has at most  $\Delta$  neighbors in the square, each forbidding at most one color. Here we show the reducibility of some of the configurations that we use later in the proof. For consistency with our application of these lemmas later on (when  $k$  denotes the maximum degree of a vertex), here we prove that configurations are  $(k+1)$ -reducible.

**Lemma 2.** *For  $k \geq 4$ , the following configurations are  $(k+1)$ -reducible:*

- (C1) a 4-thread;
- (C2) a 3-thread with an endpoint of degree at most  $k-1$ ;
- (C3) a 2-thread with endpoints of degree at most  $k-1$  and  $k-2$ ;

*Proof.* For illustration see Figure 1. For simplicity we assume in (C1)–(C3) that the endpoints of the thread are distinct. This assumption can be justified by showing that any 3-cycle, 4-cycle, or 5-cycle with at most one  $3^+$ -vertex is  $(k+1)$ -reducible. The arguments are similar to those below (but even easier), so we omit them.

The reducibility of (C1) is given in [10] but we repeat it here. Let  $u$  and  $v$  be the middle two vertices of the 4-thread. By the minimality of  $G$  we can color  $G-u-v$ . We say a color is *available* for a vertex  $v$ , if it is in the list of allowable colors for  $v$  and has not already been used on neighbor of  $v$  in  $G^2$ . Now  $u$  and  $v$  each have at least two available colors, so we can easily extend the coloring to  $G^2$ .

Let  $uvw$  be the 3-thread from (C2) with  $u$  adjacent to an endpoint of degree at most  $k-1$ . By minimality of  $G$ , we color  $G-u-v$ . Now  $u$  has at least one available color, and  $v$  has at least two. So color first  $u$  and then  $v$  to get a coloring of  $G^2$ .

Let  $uv$  be the 2-thread from (C3) with  $u$  and  $v$  adjacent to endpoints of degrees at most  $k-1$  and  $k-2$ , respectively. If we color  $G-u-v$ , then  $u$  has at least one available color and  $v$  has at least two. So we can easily extend this coloring to  $G^2$ .  $\square$

**Lemma 3.** *For  $k \geq 5$ , the following configurations are  $(k+1)$ -reducible:*

- (C4) a  $(4\ell)$ -cycle  $v_1v_2 \dots v_{4\ell}$  such that  $d(v_i) \leq k$  when  $4|i$  and  $d(v_i) = 2$  otherwise;
- (C5) a  $(3\ell)$ -cycle  $v_1v_2 \dots v_{3\ell}$  such that  $d(v_i) \leq k-1$  when  $3|i$  and  $d(v_i) = 2$  otherwise;
- (C6) a cycle induced by 1-threads incident to 3-vertices (at both ends) with at least one of these 3-vertices on the cycle incident to a third 1-thread with a 3-vertex at the other end.

*Proof.* See Figure 2 for illustration of configurations (C4)–(C6). Remove all 2-vertices of (C4), and by minimality color the square of the resulting graph. Now the subgraph of  $G^2$  induced by all  $v_i$  with  $i$  odd is a  $2\ell$ -cycle. Each of these vertices has at least two available colors (and even cycles are 2-choosable [14]), so we extend the coloring to them. Finally, we color the  $v_i$  with  $4 \mid (i + 2)$ , each of which has an available color.

Remove all 2-vertices of (C5), and by minimality color the square of the resulting graph. Now the subgraph of  $G^2$  induced by all uncolored  $v_i$  is a  $2\ell$ -cycle. Each of these vertices has at least two available colors, so we extend the coloring to  $G^2$ .

In (C6), let  $v$  be a 3-vertex on the cycle with 3 incident 1-threads leading to 3-vertices. By minimality, we can color  $(G \setminus N[v])^2$ . Now uncolor all 2-vertices on the cycle and color  $v$ . Notice that in  $G^2$  the 2-vertices of (C6) induce two cycles that share an edge, and one of these two cycles has length 3. Moreover the two vertices that belongs to both cycles have at least 3 available colors, and all others have at least 2 colors. By Vizing’s degree-choosability theorem [25], we can extend this coloring to these vertices. We should also mention the case where the 2-neighbor  $u$  of  $v$  not on the cycle has its other 3-neighbor also on the cycle. In this case, in  $G^2$  the 2-vertices induce a cycle with one additional vertex adjacent to four cycle vertices. Again we can complete the coloring by Vizing’s degree-choosability theorem.  $\square$

Now we construct examples to show that the threshold  $2 + \frac{4k-8}{5k+2}$  in the Main Theorem cannot be improved without adding new reducible configurations (or taking a completely different approach for the proof). We construct examples with maximum degree  $k$  and mad arbitrarily close to  $2 + \frac{4k-8}{5k+2}$  that do not contain any of the above reducible configurations. (In fact, our proof of the Main Theorem does use some additional reducible configurations of bounded size, but none of them appear in our examples either.) Example 1 is tight for  $\Delta \in \{4, 5\}$  and Example 2 is tight for  $\Delta \geq 6$ .

**Example 1.** Let  $G$  be a bipartite graph with vertices in part  $A$  of degree  $k - 2$  and vertices in part  $B$  of degree  $k - 3$ . Subdivide each edge of the graph twice. Now add a spanning cycle  $C_1$  through the vertices of  $A$  and a spanning cycle  $C_2$  through the vertices of  $B$ . Subdivide each edge of  $C_1$  three times, and subdivide each edge of  $C_2$

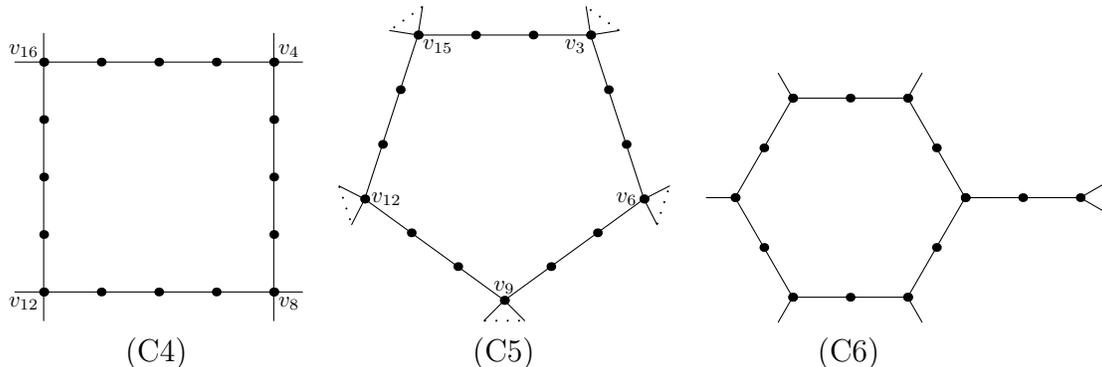


Figure 2: Configurations from Lemma 3

twice. The average degree of this graph is  $3 - (7k - 18)/(2k^2 - 3k - 6)$ , which is  $2 + 2/7$  and  $2 + 12/29$  for  $k = 4$  and  $k = 5$  respectively. However, if we contract just one edge on what was  $C_1$  and one edge on what was  $C_2$ , we get a graph with none of the above reducible configurations.

**Example 2.** Begin with a  $(k - 2)$ -regular graph on a set  $A$  of  $2M$  vertices (for arbitrary fixed  $M$ ) and an independent set  $B$  of size  $M$ . Subdivide each edge incident to  $A$  five times and add one edge from the center vertex of each resulting 5-thread to a vertex of  $B$  so that each vertex of  $B$  now has degree  $k - 2$ . Add a spanning cycle  $C_1$  through the vertices of  $A$  and a spanning cycle  $C_2$  through the vertices of  $B$ . Finally, subdivide each edge of  $C_1$  and  $C_2$  three times. The average degree of this graph is  $2 + (4k - 8)/(5k + 2)$ . If we contract one edge each on what was  $C_1$  and  $C_2$ , the resulting graph has none of the reducible configurations above. Further, it contains only vertices of degrees 2, 3, and  $k$ .

We suspect that in this way we can construct graphs with arbitrarily high girth—probably we can adapt the construction of regular graphs with arbitrary degree and arbitrary girth. If so, then any set of reducible configurations that appears in all graphs formed by this construction must contain new arbitrarily large reducible configurations.

### 3 Maximum degree 5

**Theorem 4.** *If  $\Delta \leq 5$  and  $\text{mad}(G) < 12/29$ , then  $\chi_\ell(G^2) \leq 6$ .*

*Proof.* Assume that the theorem is false and let  $G$  be a minimal counterexample. Note the following properties of  $G$ . By reducible configurations (C1) and (C2),  $G$  contains no 4-thread and each 3-thread has both endpoints of degree 5. By configuration (C4), the subgraph induced by 3-threads is acyclic. Hence, we can assign each 5-vertex to sponsor at most one incident 3-thread so that every 3-thread is sponsored. Let  $F$  denote the subgraph induced by vertices incident to (or on) 3-threads. Let  $v$  be a 5-vertex that is a leaf in  $F$ . Assign  $v$  to sponsor its incident 3-thread; now delete  $v$  and its 3-thread, and recurse.

Now we use reducibility of (C5) and (C6). Similar to 5-vertices sponsoring 3-threads, we assign to each 2-thread with 4-vertices at both ends an incident 4-vertex to sponsor it. Likewise, we assign to each 1-thread with 3-vertices at both ends an incident 3-vertex to sponsor it.

We use discharging with each vertex  $v$  getting initial charge  $d(v)$  and with the following discharging rules. Charge sent to a thread will be equally distributed among the 2-vertices of that thread.

- (R1) Every 5-vertex sends charge  $13/29$  to each incident thread or adjacent 3-vertex, and it sends extra charge  $10/29$  to its sponsored 3-thread (if it exists).
- (R2) Every 4-vertex sends charge  $11/29$  to each incident thread or adjacent 3-vertex, and it sends extra charge  $2/29$  to its sponsored 2-thread (if it exists).

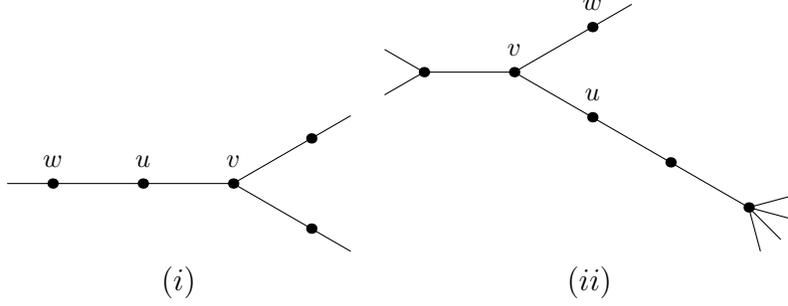


Figure 3: Reducible configurations from Theorem 4

(R3) Every 3-vertex sends charge  $11/29$  to each incident 2-thread, it sends charge  $1/29$  to each incident 1-thread leading to a weak 4-neighbor, and also it sends charge  $12/29$  to its sponsored 1-thread (if it exists).

Now we show that every vertex  $v$  finishes with charge  $\mu^*(v)$  at least  $2 + 12/29$ . If  $d(v) = 5$ , then  $\mu^*(v) \geq 5 - 5(13/29) - (10/29) = 2 + 12/29$ . If  $d(v) = 4$ , then  $\mu^*(v) \geq 4 - 4(11/29) - (2/29) = 2 + 12/29$ . Now we consider the two remaining possibilities  $d(v) = 3$  and  $d(v) = 2$ .

Suppose  $d(v) = 3$ . Consider the possibility that  $v$  is incident with three threads. If all of them are 1-threads, then  $\mu^*(v) \geq 3 - 12/29 - 2(1/29) > 2 + 12/29$ . If at least one is a 2-thread, then let the 2-thread be  $uw$ , with  $u$  adjacent to  $v$ . See Figure 3(i). Remove  $u$  and by minimality color  $G - u$ . Now recolor  $v$  with a distinct color from  $w$  if necessary, and then color  $u$ . So  $v$  has a  $3^+$ -neighbor.

If  $v$  has a  $4^+$ -neighbor, then  $\mu^*(v) \geq 3 + 11/29 - 11/29 - 12/29 > 2 + 12/29$ . Similarly, if  $v$  has at least two 3-neighbors, then  $\mu^*(v) \geq 3 - 12/29 > 2 + 12/29$ . So  $v$  must have a 3-neighbor and two 2-neighbors. If  $v$  has a 3-neighbor and two incident 1-threads, then  $\mu^*(v) \geq 3 - 12/29 - 1/29 > 2 + 12/29$ . So  $v$  must have an incident 2-thread and either another incident 2-thread or else an incident 1-thread leading to a weak 3-neighbor (if it leads to a weak  $4^+$ -neighbor, then  $v$  gives away at most  $12/29$ ). Let  $u$  be a neighbor of  $v$  on a 2-thread and let  $w$  be  $v$ 's other 2-neighbor. See Figure 3(ii). By minimality, we can color  $G \setminus \{u, v, w\}$ . Now we color  $v, w$ , and  $u$  in this order.

Finally, suppose  $d(v) = 2$ . We show that each  $\ell$ -thread  $P$  finishes with charge at least  $\ell(2 + 12/29)$ , so that each 2-vertex finishes with charge at least  $2 + 12/29$ . If  $\ell = 3$ , then  $P$  gets charge  $13/29$  from each endpoint and charge  $10/29$  from its sponsor, so  $\mu^*(P) \geq 6 + 2(13/29) + 10/29 = 6 + 36/29 = 3(2 + 12/29)$ . Suppose  $\ell = 2$ . If  $P$  has a 5-vertex as an endpoint, then  $\mu^*(P) \geq 4 + 13/29 + 11/29 = 2(2 + 12/29)$ . If  $P$  has two 4-vertices as endpoints, then  $\mu^*(P) \geq 4 + 2(11/29) + 2/29 = 4 + 24/29 = 2(2 + 12/29)$ . Since (C3) is reducible, we are in one of these cases. Finally, suppose  $\ell = 1$ . If  $P$  has a 5-vertex endpoint, then  $\mu^*(P) \geq 2 + 13/29$ . If  $P$  has a 4-vertex endpoint, then  $\mu^*(P) \geq 2 + 11/29 + 1/29 = 2 + 12/29$ . If  $P$  has two 3-vertex endpoints, then  $P$  gets  $12/29$  from its sponsor, so  $\mu^*(P) \geq 2 + 12/29$ .

So each vertex finishes with charge at least  $2 + 12/29$ . This contradicts the fact that  $\text{mad}(G) < 2 + 12/29$ , and thus completes the proof.  $\square$

## 4 Maximum degree 6

**Theorem 5.** *If  $\Delta \leq 6$  and  $\text{mad}(G) < 5/2$ , then  $\chi_\ell(G^2) \leq 7$ .*

*Proof.* Assume to the contrary that the theorem is false and let  $G$  be a minimal counterexample. A vertex is *high* if its degree is 5 or 6, it is *medium* if its degree is 3 or 4, and it is *low* otherwise. By reducible configurations (C1) and (C2),  $G$  contains no 4-thread and each 3-thread has both endpoints of degree 6. By configuration (C4), the subgraph induced by 3-threads is acyclic. Hence, we can assign each 6-vertex to sponsor at most one incident 3-thread so that every 3-thread is sponsored.

We use discharging with each vertex  $v$  getting initial charge  $d(v)$  and with the following discharging rules.

- (R1) Every high vertex sends charge  $1/2$  in each direction.
- (R2) Every 6-vertex sends extra charge  $1/2$  to its sponsored 3-thread (if it exists).
- (R3) Every medium vertex sends charge  $1/2$  to each incident 2-thread, and it sends charge  $1/4$  to each incident 1-thread with other endpoint a medium vertex.

We now show that every vertex  $v$  finishes with final charge  $\mu^*(v)$  at least  $5/2$ , which will establish the theorem.

If  $d(v) = 6$ , then  $\mu^*(v) \geq 6 - 6(1/2) - 1/2 = 5/2$ . If  $d(v) = 5$ , then  $\mu^*(v) \geq 5 - 5(1/2) = 5/2$ . If  $d(v) = 4$ , then  $v$  finishes with charge at least  $5/2$  unless  $v$  gives charge in all 4 directions and gives charge  $1/2$  in at least 3 of these directions; see Figure 4(i). By minimality, we can color  $G \setminus N[v]$ . Now we easily color  $v$ 's neighbor on a 1-thread (if one exists), followed by  $v$ , followed by its remaining neighbors. This produces a proper coloring of  $G^2$ . So the cases left to consider are  $d(v) = 3$  and  $d(v) = 2$ .

Suppose  $d(v) = 3$ . Consider first the possibility that  $v$  has three 2-neighbors and is incident to at least one 2-thread. Let  $u$  denote its neighbor on a 2-thread; see Figure 4(ii). By minimality, we can color  $G - u$ . To extend the coloring to  $G$ , first recolor  $v$  to avoid the color on  $u$ 's other neighbor, then color  $u$ . If  $v$  has three 2-neighbors and at least one high weak neighbor, then  $\mu^*(v) \geq 3 - 2(1/4) = 5/2$ . So suppose  $v$  has three 2-neighbors and all its weak neighbors are medium. See Figure 4(iii). By minimality,

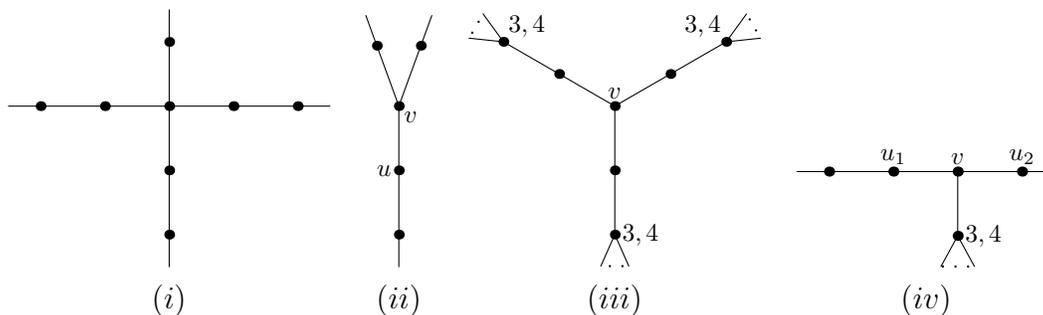


Figure 4: Reducible configurations from Theorem 5

we color  $G \setminus N[v]$ . Now color the vertices in  $N(v)$  in arbitrary order; finally, color  $v$ . Thus, we may assume that  $v$  has a  $3^+$ -neighbor.

If  $v$  has a  $5^+$ -neighbor, then  $\mu^*(v) \geq 3 + 1/2 - 2(1/2) = 5/2$ . So  $v$  has a medium neighbor. If  $v$  has two or more medium neighbors, then  $\mu^*(v) \geq 3 - 1/2 = 5/2$ . So suppose  $v$  has one medium neighbor and two 2-neighbors. If  $v$  has two incident 1-threads, then  $\mu^*(v) \geq 3 - 2(1/4) = 5/2$ . So now assume that  $v$  has an incident 2-thread and the other incident thread is either a 2-thread or a 1-thread leading to a medium weak neighbor. Let  $u_1$  be a neighbor of  $v$  on a 2-thread and let  $u_2$  be the other 2-neighbor of  $v$ . See Figure 4(*iv*). By minimality, we can color  $G \setminus \{u_1, u_2, v\}$ . Now we color  $v, u_2, u_1$  in this order.

Finally, suppose  $d(v) = 2$ . We show that each  $\ell$ -thread  $P$  receives charge at least  $\ell/2$ , so that each 2-vertex finishes with charge  $5/2$ . If  $\ell = 3$ , then  $P$  receives  $1/2$  from each endpoint and an additional  $1/2$  from its sponsor. If  $\ell = 2$ , then  $P$  receives charge  $1/2$  from each endpoint. If  $\ell = 1$ , then either  $P$  receives charge  $1/2$  from a high endpoint or  $P$  receives charge  $1/4$  from both medium endpoints.

Thus, each vertex finishes with charge at least  $5/2$ . This contradicts the fact that  $\text{mad}(G) < 5/2$ , and thus completes the proof.  $\square$

## 5 Maximum degree 7

**Theorem 6.** *If  $\Delta \leq 7$  and  $\text{mad}(G) < 2 + 20/37$ , then  $\chi_\ell(G^2) \leq 8$ .*

*Proof.* Assume to the contrary that the theorem is false and let  $G$  be a minimal counterexample. Again, we use discharging. A vertex is *high* if its degree is 6 or 7, it is *medium* if its degree is 4 or 5, and it is *low* otherwise.

By reducible configurations (C1) and (C2),  $G$  contains no 4-thread and each 3-thread is incident to 7-vertices at both ends. By (C4), the subgraph induced by 3-threads is acyclic. Hence, we can assign each 7-vertex to sponsor at most one incident 3-thread so that every 3-thread is sponsored.

We use discharging with each vertex  $v$  getting initial charge  $d(v)$  and with the following discharging rules.

- (R1) Every high vertex sends charge  $21/37$  in each direction.
- (R2) Every 7-vertex sends extra charge  $18/37$  to its sponsored 3-thread (if it exists).
- (R3) Every 5-vertex sends charge  $20/37$  to each incident 2-thread, and it sends charge  $10/37$  in each direction that does not lead to a 2-thread.
- (R4) Every 4-vertex sends charge  $20/37$  to each incident 2-thread,  $10/37$  to each incident 1-thread, and  $4/37$  to each adjacent 3-vertex.
- (R5) Every 3-vertex sends charge  $19/37$  to each incident 2-thread, and  $10/37$  to each incident 1-thread leading to a weak  $5^-$ -neighbor.

We now show that every vertex finishes with charge  $\mu^*(v)$  at least  $2 + 20/37$ .

If  $d(v) = 7$ , then  $\mu^*(v) \geq 7 - 7(21/37) - (18/37) = 2 + 20/37$ . If  $d(v) = 6$ , then  $\mu^*(v) \geq 6 - 6(21/37) = 2 + 22/37$ . If  $d(v) = 5$ , then  $\mu^*(v) \geq 5 - 4(20/37) - 1(10/37) =$

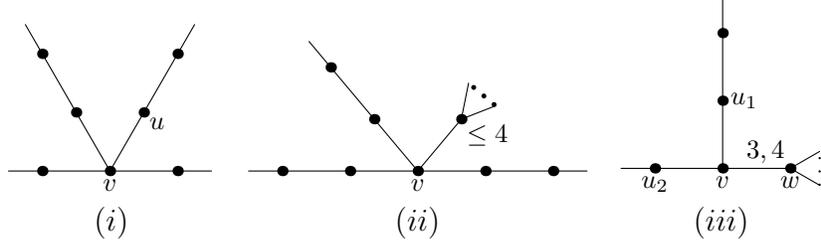


Figure 5: Reducible configurations from Theorem 6

$2 + 21/37$  unless  $v$  has five incident 2-threads. In this case  $N[v]$  is reducible. By minimality, we color  $G \setminus N[v]$ . Then we easily color  $v$ , followed by its neighbors.

Suppose  $d(v) = 4$ . If  $v$  has at most one incident 2-thread, then  $\mu^*(v) \geq 4 - 20/37 - 3(10/37) = 2 + 24/37$ . Similarly, if  $v$  has two incident 2-threads and only one incident 1-thread, then  $\mu^*(v) \geq 4 - 2(20/37) - 10/37 - 4/37 = 2 + 20/37$ . So either  $v$  has two incident 2-threads and two incident 1-threads or  $v$  has at least three incident 2-threads. Consider the first case, shown in Figure 5(i). Let  $u$  be a neighbor of  $v$  on a 2-thread. By minimality, we color  $G - u$ . Now we uncolor  $v$ , then color  $v$  and  $u$ , in that order. Consider now the second case, where  $v$  has at least three incident 2-threads. If  $v$  has a  $5^+$ -neighbor, then  $\mu^*(v) \geq 4 + 10/37 - 3(20/37) = 2 + 24/37$ . If  $v$ 's fourth neighbor is instead a  $4^-$ -vertex, then color  $G - v$  by minimality. See Figure 5(ii). Uncolor three neighbors of  $v$  on 2-threads; now color  $v$ , followed by its uncolored neighbors.

Suppose  $d(v) = 3$ . If  $v$  has three 2-neighbors and at least one of them,  $u$ , is on a 2-thread, then color  $G - u$  by minimality; recolor  $v$  to avoid the color on  $u$ 's other neighbor, then color  $u$ . If  $v$  has three incident 1-threads and each leads to a weak  $6^+$ -neighbor, then  $\mu^*(v) = 3 - 0 > 2 + 20/37$ . If  $v$  has three incident 1-threads and one of them, with internal vertex  $u$ , leads to a weak  $5^-$ -vertex, then color  $G - u$  by minimality. Now color  $u$ , then recolor  $v$ . So  $v$  has at most two 2-neighbors.

If  $v$  has exactly one 2-neighbor and it lies on a 1-thread, then  $\mu^*(v) \geq 3 - 10/37 = 2 + 27/37$ . Suppose  $v$  has exactly one 2-neighbor and it lies on a 2-thread. If either of its other neighbors is a  $4^+$ -vertex, then  $\mu^*(v) \geq 3 - 19/37 + 4/37 = 2 + 22/37$ . Otherwise, let  $u$  denote the 2-neighbor. By minimality, color  $G - u$ . Recolor  $v$  to avoid the color on  $u$ 's other neighbor, then color  $u$ . So  $v$  must have exactly two 2-neighbors.

Suppose that  $v$  has exactly two 2-neighbors,  $u_1$  and  $u_2$ . If  $v$ 's third neighbor  $w$  is a high vertex, then  $\mu^*(v) \geq 3 + 21/37 - 2(19/37) = 2 + 20/37$ , so  $w$  must be a  $5^-$ -vertex. If both  $u_i$  lie on 2-threads, then color  $G \setminus \{v, u_1, u_2\}$ . Now color  $v$ ,  $u_1$ , then  $u_2$ . Suppose instead that  $v$  has exactly one incident 2-thread and that  $u_1$  is its neighbor on the 2-thread. If  $v$ 's  $3^+$ -neighbor  $w$  is a  $4^-$ -vertex, then color  $G - u_1$  by minimality; see Figure 5(iii). Now, recolor  $v$  to avoid the color on the other neighbor of  $u_1$ , then color  $u_1$ ; so  $w$  must be a 5-vertex. If  $v$ 's weak neighbor along the 1-thread is a  $6^+$ -vertex, then  $v$  sends no charge along the 1-thread so  $\mu^*(v) \geq 3 + 10/37 - 19/37 = 2 + 28/37$ . Otherwise this weak neighbor via the 1-thread containing  $u_2$  is a  $5^-$ -vertex. Color  $G \setminus \{v, u_1, u_2\}$  by minimality; now color  $v$ ,  $u_2$ , and  $u_1$ , in that order.

Suppose  $v$  has exactly two incident 1-threads. If  $v$ 's third neighbor is a  $4^+$ -vertex, then  $\mu^*(v) \geq 3 + 4/37 - 2(10/37) = 2 + 21/37$ . Similarly, if either of  $v$ 's weak neighbors

is high, then  $\mu^*(v) \geq 3 - 10/37 = 2 + 27/37$ . Thus, assume that  $v$ 's 2-neighbors  $u_1$  and  $u_2$  each lead to weak  $5^-$ -neighbors and that its third neighbor is a 3-vertex. Color  $G \setminus \{v, u_1, u_2\}$  by minimality. We can now color  $u_1, u_2$ , and  $v$ , in that order.

Finally, suppose  $d(v) = 2$ . We show that each  $\ell$ -thread  $P$  receives charge at least  $20\ell/37$ , so that each 2-vertex finishes with charge  $2 + 20/37$ . If  $\ell = 3$ , then  $\mu^*(P) = 6 + 2(21/37) + 18/37 = 3(2 + 20/37)$ . If  $\ell = 2$  and one endpoint of  $P$  is a 3-vertex, then by reducible configuration (C3) the other endpoint must be a 7-vertex, so  $\mu^*(P) = 4 + 21/37 + 19/37 = 2(2 + 20/37)$ . If  $\ell = 2$  and both endpoints of  $P$  are  $4^+$ -vertices, then  $\mu^*(P) \geq 4 + 2(20/37) = 2(2 + 20/37)$ . If  $\ell = 1$ , then  $\mu^*(P) \geq 2 + 2(10/37) = 2 + 20/37$ .

Thus, each vertex finishes with charge at least  $2 + 20/37$ . This contradicts the fact that  $\text{mad}(G) < 2 + 20/37$ , and so completes the proof.  $\square$

## 6 Maximum degree at least 8

**Theorem 7.** *For  $k \geq 8$ , if  $\Delta \leq k$  and  $\text{mad}(G) < 2 + \frac{4k-8}{5k+2}$ , then  $\chi_\ell(G^2) \leq k + 1$ .*

*Proof.* Suppose to the contrary that the theorem is false and let  $G$  be a minimal counterexample. A  $3^+$ -vertex is *high* if its degree is  $k$  or  $k - 1$ , it is *medium* if its degree is between  $k - 2$  and  $7 - \lfloor \frac{16}{k+2} \rfloor$  (inclusive), and otherwise it is *low*. By reducible configurations (C1) and (C2),  $G$  contains no 4-thread and each 3-thread is incident to  $k$ -vertices at both ends. By configuration (C4), the subgraph induced by 3-threads is acyclic. Hence, we can assign each  $k$ -vertex to sponsor at most one incident 3-thread so that every 3-thread is sponsored.

Let  $\alpha = \frac{4k-8}{5k+2}$ . Let  $\beta = \frac{(k-2)-4\alpha}{k-2} = 1 - \frac{16}{5k+2}$ . For  $k \geq 8$ , note that  $\alpha/2 > 8/(5k+2)$  and  $1 > \beta > \alpha > 2\alpha - \beta > 2/5 > \alpha/2 > 1/5 > \beta - \alpha$ . (Verifying these inequalities is tedious, but straightforward.) The following equality also holds:

$$2 - 16/(5k + 2) = 5\alpha - 2\beta. \quad (1)$$

We use discharging with each vertex  $v$  getting initial charge  $d(v)$  and with the following discharging rules.

- (R1) Every high vertex sends charge  $\beta$  in each direction.
- (R2) Every  $k$ -vertex sends charge  $3\alpha - 2\beta$  to its sponsored 3-thread (if it exists).
- (R3) Every medium vertex sends charge  $2\alpha - \beta$  in each direction.
- (R4) Every low vertex  $v$  sends charge  $2\alpha - \beta$  to each incident 2-thread (leading to a weak  $k$ -neighbor, by (C3), since  $v$  is by definition a  $(k - 2)^-$ -vertex),  $\alpha/2$  to each incident 1-thread leading to a low weak neighbor, and  $\beta - \alpha$  to each incident 1-thread leading to a medium weak neighbor.
- (R5) Every 3-vertex receives charge  $8/(5k + 2)$  from each adjacent 5-vertex and charge  $4/(5k + 2)$  from each adjacent 4-vertex.<sup>1</sup>

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<sup>1</sup>When  $k \leq 10$ , in some cases 3-vertices need more charge; that is the point of this rule. When  $k \geq 11$  this rule is not needed, but since  $8/(5k + 2)$  and  $4/(5k + 2)$  rapidly diminish to 0, this rule causes no problems.

We now show that every vertex finishes with charge at least  $2 + \alpha$ . This will contradict the fact that  $\text{mad}(G) < 2 + \alpha$ , and thus prove the theorem.

**Case:**  $d(v) = k$ . Now  $\mu^*(v) \geq k - k\beta - (3\alpha - 2\beta) = k - (k - 2)\beta - 3\alpha = k - ((k - 2) - 4\alpha) - 3\alpha = 2 + \alpha$ .

**Case:**  $d(v) = k - 1$ . Now  $\mu^*(v) \geq (k - 1) - (k - 1)\beta = (k - 1) - \beta - ((k - 2) - 4\alpha) = 1 + 4\alpha - \beta \geq 2 + \alpha$ .

**Case:**  $v$  is medium. Let  $d = d(v)$ . Now  $\mu^*(v) \geq d - d(2\alpha - \beta)$ . This quantity is at least  $2 + \alpha$  when  $d \geq (2 + \alpha)/(1 + \beta - 2\alpha) = (7k - 2)/(k + 2) = 7 - \frac{16}{k+2}$ .

**Case:**  $d(v) = 6$  and  $v$  is low. If  $v$  has at most five 2-neighbors, then  $\mu^*(v) \geq 6 - 5(2\alpha - \beta) \geq 2 + \alpha$ ; note that the last inequality is equivalent to  $4 \geq 11\alpha - 5\beta$ , i.e. it is equivalent to  $(26 + k)/(2 + 5k) \geq 0$ . If  $v$  has at most four incident 2-threads, then  $\mu^*(v) \geq 6 - 4(2\alpha - \beta) - 2(\alpha/2) \geq 2 + \alpha$  by (1). So now  $v$  must have six 2-neighbors and at least five incident 2-threads. Form  $H$  from  $G$  by deleting  $v$  and each of its neighbors on a 2-thread. Color  $H$  by minimality. Now we can color  $v$  (it has at most 7 restrictions on its color). Finally we can color each uncolored neighbor of  $v$  (the last has at most 8 restrictions on its color).

**Case:**  $d(v) = 5$ . If  $v$  has at most four 2-neighbors and at most three of them lie on 2-threads, then  $\mu^*(v) \geq 5 - 3(2\alpha - \beta) - \alpha/2 - 8/(5k + 2) \geq 2 + \alpha$ . This last inequality simplifies to  $16/(2 + 5k) \geq 0$ , which obviously holds. Similarly, if  $v$  has five 2-neighbors at most one of which lies on a 2-thread, then  $\mu^*(v) \geq 5 - (2\alpha - \beta) - 4(\alpha/2) \geq 2 + \alpha$  and the last inequality simplifies to  $32/(2 + 5k) \geq 0$ . If  $v$  has five 2-neighbors and at least two of them lie on 2-threads, then let  $u_1$  and  $u_2$  be neighbors on 2-threads. By minimality, color  $G \setminus \{u_1, u_2\}$ . Now recolor  $v$  to avoid the colors on the neighbors of  $u_1$  and  $u_2$  ( $v$  has at most 8 constraints on its color), then color  $u_1$  and  $u_2$ . Hence  $v$  must have exactly four 2-neighbors, all of which lie on 2-threads. If the final neighbor of  $v$  is a medium vertex, then it gives  $v$  charge  $2\alpha - \beta$ . So  $\mu^*(v) \geq 5 - 4(2\alpha - \beta) + (2\alpha - \beta) \geq 2 + \alpha$ ; note that the last inequality simplifies to  $3 \geq 7\alpha - 3\beta$  and is equivalent to  $2(10 + k)/(2 + 5k) \geq 0$ , which obviously holds for all  $k$ 's we consider. So we can assume the final neighbor  $u$  of  $v$  is a  $6^-$ -vertex and gives no charge to  $v$ . For  $k \leq 14$ , we get  $\mu^*(v) \geq 5 - 4(2\alpha - \beta) - 8/(5k + 2)$ , which is bigger than  $2 + \alpha$  since  $3 - 8/(5k + 2) \geq 9\alpha - 4\beta$ . And for  $k \geq 15$ , we proceed as follows. Color  $G \setminus (N[v] - u)$  by minimality. Now color  $v$  (it has at most 10 constraints on its color), then color each uncolored neighbor of  $v$  (the last has at most 7 constraints on its color).

**Case:**  $d(v) = 4$ . Suppose  $v$  has four 2-neighbors. If  $v$  has an incident 2-thread, then let  $u$  denote  $v$ 's neighbor on the 2-thread. By minimality, color  $G - u$ . Now recolor  $v$  to avoid the color on  $u$ 's other neighbor ( $v$  has at most 7 restrictions on its color), then color  $u$  (which has at most 6 restrictions on its color). So assume instead that  $v$  has four incident 1-threads. Now  $\mu^*(v) \geq 4 - 4(\alpha/2)$ . This quantity is at least  $2 + \alpha$  when  $k \leq 14$ . So suppose  $k \geq 15$ . If  $v$  has a 1-thread leading to a low weak neighbor, then let  $u$  denote the neighbor on this 1-thread. By minimality, color  $G - u$ . Now recolor  $v$  to avoid the color on  $u$ 's other neighbor ( $v$  has at most 7 restrictions on its color), then color  $u$  (which has at most 10 restrictions). So suppose instead that each of the four 1-threads incident to  $v$  leads to a medium or high weak

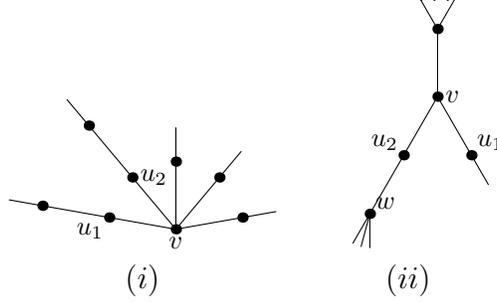


Figure 6: Reducible configurations from Theorem 7

neighbor. Now  $\mu^*(v) \geq 4 - 4(\beta - \alpha) \geq 2 + \alpha$ , where the last inequality is equivalent to  $2(18 + k)/(2 + 5k) \geq 0$ . Thus,  $v$  has at most three 2-neighbors.

If  $v$  has a medium or high neighbor, then  $\mu^*(v) \geq 4 - 3(2\alpha - \beta) + (2\alpha - \beta) \geq 2 + \alpha$  by (1). Similarly, if  $v$  has at most two 2-neighbors, then  $\mu^*(v) \geq 4 - 2(2\alpha - \beta) - 2(4/(5k + 2)) \geq 2 + \alpha$  by (1). Thus,  $v$  has exactly three 2-neighbors and one low neighbor. If  $v$  has three incident 2-threads and its low neighbor  $u$  is a 5<sup>-</sup>-neighbor, then color  $G \setminus (N[v] - u)$  by minimality. Now color  $v$  (it has at most 8 restrictions on its color), then color each of its uncolored neighbors (the last has at most 6 restrictions on its color). Suppose instead that  $v$  has three incident 2-threads and its low neighbor is a 6-vertex; so  $k \geq 15$ . Now  $v$  has at most 9 restrictions on its color, but since  $k \geq 15$ , we can complete the coloring.

So we may assume that  $v$  has at most two incident 2-threads. If  $v$  has no incident 2-threads, then  $\mu^*(v) \geq 4 - 3(\alpha/2) - 4/(5k + 2) \geq 2 + \alpha$ , which is easy to verify. If  $v$  has one incident 2-thread, then  $\mu^*(v) \geq 4 - (2\alpha - \beta) - 2(\alpha/2) - 4/(5k + 2) \geq \alpha + 2$  simplifies to  $1 \geq 7\alpha/2 - 2\beta$ , which holds when  $k \leq 18$ . When  $k \geq 19$ , let  $u$  be  $v$ 's neighbor on its 2-thread. Color  $G - u$  by minimality. Recolor  $v$  to avoid the color on  $u$ 's other neighbor ( $v$  has at most  $6 + 2 + 2 + 1$  constraints on its color), then color  $u$ . So, suppose  $v$  has two incident 2-threads. If  $v$ 's 3<sup>+</sup>-neighbor is a 3-neighbor, then let  $u$  be a neighbor on a 2-thread. By minimality, color  $G - u$ . Recolor  $v$  to avoid the color on  $u$ 's other neighbor ( $v$  has at most 8 restrictions on its color), then color  $u$ . If instead,  $v$ 's final neighbor is a 4<sup>+</sup>-vertex, then  $\mu^*(v) \geq 4 - 2(2\alpha - \beta) - \alpha/2$ . This quantity is at least  $2 + \alpha$  when  $k \leq 10$ . So assume  $k \geq 11$  and let  $u$  be a neighbor on a 2-thread. Color  $G - u$  by minimality. Now recolor  $v$  to avoid the color on  $u$ 's other neighbor ( $v$  has at most 11 restrictions on its color), then color  $u$ .

**Case:**  $d(v) = 3$ . If  $v$  has no 2-neighbors, then  $v$  begins with charge 3 and gives away no charge, so assume that  $v$  has at least one 2-neighbor. First suppose that  $v$  has three 2-neighbors and at least one such neighbor  $u$  is on a 2-thread or on a 1-thread leading to a medium or low weak neighbor. Color  $G - u$  by minimality. Uncolor  $v$  and color  $u$  (it has at most  $(k - 2) + 2$  constraints on its color); now color  $v$  (which has at most 6 constraints on its color). If instead  $v$  has three incident 1-threads and each leads to a high weak neighbor, then  $\mu^*(v) = 3 - 0$ . So  $v$  must have at most two 2-neighbors.

Suppose that  $v$  has exactly two 2-neighbors; call them  $u_1$  and  $u_2$ . If  $u_1$  and  $u_2$  both lie on 2-threads and  $v$ 's third neighbor is a medium or low neighbor, then color

$G \setminus \{v, u_1, u_2\}$  by minimality. Now color  $v$  (which has at most  $(k-2)+2$  constraints on its color), then  $u_1$  and  $u_2$ . If instead  $v$  has a high neighbor, then  $\mu^*(v) = 3 - 2(2\alpha - \beta) + \beta = 3 - 4\alpha + 3\beta \geq 2 + \alpha$  which holds by (1) since  $\beta = 1 - 16/(5k + 2)$ . So  $v$  must not have two incident 2-threads.

Suppose that  $v$  has an incident 1-thread and an incident 2-thread, with  $u_1$  on the 2-thread. If  $v$ 's  $3^+$ -neighbor is low, then color  $G - u_1$  by minimality. Recolor  $v$  to avoid the color on  $u_1$ 's other neighbor. If  $v$ 's  $3^+$ -neighbor has degree at most 5, then  $v$  has at most  $5 + 2 + 1$  constraints on its color; if it has degree 6, then  $v$  has at most  $6 + 2 + 1$  constraints on its color, but now  $k \geq 15$  since a 6-vertex is low, so the recoloring succeeds. Finally, color  $u_1$ . So  $v$ 's  $3^+$ -neighbor is medium or high; if it is high, then  $\mu^*(v) \geq 3 + \beta - (2\alpha - \beta) - \alpha/2 \geq 2 + \alpha$  by (1). So assume  $v$ 's  $3^+$ -neighbor is medium. Let  $u_2$  be the neighbor of  $v$  on a 1-thread and  $w$  the other neighbor of  $u_2$ ; see Figure 6(ii). If  $w$  is medium or high, then  $\mu^*(v) \geq 3 + (2\alpha - \beta) - (2\alpha - \beta) - (\beta - \alpha) \geq 2 + \alpha$ , since the second inequality is equivalent to  $1 \geq \beta$ . So assume  $w$  is low. Now color  $G \setminus \{v, u_1, u_2\}$  by minimality. Color  $v$  (it has at most  $(k - 2) + 1 + 1$  constraints on its color), then  $u_2$ , and finally  $u_1$ .

Suppose that  $v$  has two incident 1-threads. If  $v$ 's  $3^+$ -neighbor is medium or high, then  $\mu^*(v) \geq 3 + (2\alpha - \beta) - 2(\alpha/2) \geq 2 + \alpha$ , which holds since  $1 > \beta$ . So assume this third neighbor is low. If at least one 1-thread leads to a weak neighbor that is low, then let  $u_1$  be the 2-vertex on that 1-thread. Color  $G - u_1$  by minimality. Recolor  $v$  to avoid the color on the neighbor of  $u_1$ ; if  $v$ 's  $3^+$ -neighbor has degree at most 5, then  $v$  has at most  $5 + 2 + 1$  constraints on its color, and if it has degree 6, then  $v$  has at most  $6 + 2 + 1$  constraints on its color, but now  $k \geq 15$  since a 6-vertex is low, so the recoloring succeeds. Now color  $u_1$  (the analysis of constraints on the color of  $u_1$  is analogous to that of  $v$ ). So neither 1-thread leads to a low weak neighbor.

If at least one 1-thread leads to a weak neighbor that is high, then  $\mu^*(v) \geq 3 - (\beta - \alpha) \geq 2 + \alpha$ , which holds since  $1 > \beta$ . So assume that both 1-threads lead to weak neighbors that are medium. Now  $\mu^*(v) \geq 3 - 2(\beta - \alpha)$ . This quantity is at least  $2 + \alpha$  when  $k \leq 22$ ; so assume  $k \geq 23$ . Color  $G \setminus \{v, u_1, u_2\}$  by minimality. Now color  $u_1$  and  $u_2$  (each has at most  $(k - 2) + 1 + 1$  constraints on its color), then color  $v$  (which has at most  $6 + 2 + 2$  constraints on its color). Thus  $v$  must have exactly one 2-neighbor.

Suppose that  $v$  has one 2-neighbor and two  $3^+$ -neighbors. If either  $3^+$ -neighbor is medium or high, then  $\mu^*(v) \geq 3 + (2\alpha - \beta) - (2\alpha - \beta) = 3 > 2 + \alpha$ ; so assume that both  $3^+$ -neighbors are low vertices. Suppose that the 2-neighbor  $u$  lies on a 1-thread. (We assume that  $v$ 's weak neighbor is medium or low, since otherwise  $\mu^*(v) = 3 - 0$ .) Now  $\mu^*(v) \geq 3 - \alpha/2$ . This quantity is at least  $2 + \alpha$  when  $k \leq 14$ . So suppose  $k \geq 15$ . Now color  $G - u$  by minimality, and uncolor  $v$ . First color  $u$  (which has at most  $(k - 2) + 1 + 1$  constraints on its color), then color  $v$  (which has at most  $6 + 6 + 2$  constraints).

So now suppose that  $v$ 's single 2-neighbor  $u$  lies on a 2-thread. As above,  $v$ 's two other neighbors must be low. Suppose  $k \geq 13$ . By minimality, color  $G - u$ . Recolor  $v$  to avoid the color on  $u$ 's other neighbor (it has at most  $6 + 6 + 1$  constraints on its color), then color  $u$  (which has at most 5 constraints). Similarly, if  $11 \leq k \leq 12$ , then 6-vertices are medium. In this case, the same recoloring process works; now however  $v$ 's low neighbors must be  $5^-$ -vertices, so  $v$  has at most  $5 + 5 + 1$  constraints on its color. Now suppose  $k \leq 10$ . If  $v$  has a 5-neighbor, then  $\mu^*(v) \geq 3 - (2\alpha - \beta) + 8/(5k + 2)$ .

This quantity is at least  $2 + \alpha$  when  $k \leq 10$ . If  $v$  has a 4-neighbor, then  $\mu^*(v) \geq 3 - (2\alpha - \beta) + 4/(5k + 2)$ . This quantity is at least  $2 + \alpha$  when  $k = 8$ . If  $v$  has two 3-neighbors, then we again use the recoloring process above. Finally, if  $9 \leq k \leq 10$  and  $v$  has no 5<sup>+</sup>-neighbor, then the recoloring process above works again; this time  $v$  has at most  $4 + 4 + 1$  constraints on its color (which is fine, since  $k \geq 9$ ).

**Case:**  $d(v) = 2$ . We show that each  $\ell$ -thread  $P$  receives charge at least  $\alpha\ell$ , so that each 2-vertex finishes with charge at least  $2 + \alpha$ . If  $\ell = 3$ , then  $\mu^*(P) = 6 + 2\beta + (3\alpha - 2\beta) = 3(2 + \alpha)$ . If  $\ell = 2$ , then by reducible configuration (C3) at least one endpoint of  $P$  must be a high vertex. So  $\mu^*(P) \geq 4 + \beta + (2\alpha - \beta) = 2(2 + \alpha)$ . Finally, suppose  $\ell = 1$ . If at least one endpoint is a medium or high vertex, then  $\mu^*(P) \geq 2 + (2\alpha - \beta) + (\beta - \alpha) = 2 + \alpha$ . If instead both endpoints are low vertices, then  $\mu^*(P) \geq 2 + 2(\alpha/2) = 2 + \alpha$ .

We have shown that every vertex finishes with charge at least  $2 + \alpha$ . This contradicts the fact that  $\text{mad}(G) < 2 + \alpha$ , and so finishes the proof.  $\square$

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