Hamiltonicity in Connected Regular Graphs

Daniel W. Cranston* Suil O†

April 29, 2012

Abstract

In 1980, Jackson proved that every 2-connected k-regular graph with at most $3k$ vertices is Hamiltonian. This result has been extended in several papers. In this note, we determine the minimum number of vertices in a connected k-regular graph that is not Hamiltonian, and we also solve the analogous problem for Hamiltonian paths. Further, we characterize the smallest connected k-regular graphs without a Hamiltonian cycle.

1 Introduction

In 1980, Jackson [2] gave a sufficient condition on the number of vertices in a 2-connected k-regular graph for it to be Hamiltonian. A graph G is *k-connected* if it has more than k vertices and every subgraph obtained by deleting fewer than k vertices is connected. A graph G is *Hamiltonian* if it contains a spanning cycle. For terminology and notation not defined here, we use [6].

Theorem 1.1. (Jackson [2]) *Every 2-connected k-regular graph on at most $3k$ vertices is Hamiltonian.*

Theorem [2] has been extended in several papers. Hilbig [1] extended it to graphs on $3k+3$ vertices with two exceptions. Let P denote the Petersen graph, and let P' denote the graph obtained from P by replacing one vertex v of P by the complete graph K_3 and making each vertex of the K_3 adjacent to a distinct neighbor of v.

Theorem 1.2. (Hilbig [1]) *If G is a 2-connected k-regular graph on at most $3k+3$ vertices and $G \notin \{P, P'\}$, then G is Hamiltonian.*

*Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, Richmond, VA, 23284, dcranston@vcu.edu

†Department of Mathematics, The College of William and Mary, Williamsburg, VA, 23185, so@wm.edu.

In Section 2, we show that every connected k-regular graph on at most $2k+2$ vertices has no cut-vertex, which implies by Theorem 1.1 that it is Hamiltonian. In addition, we characterize connected k-regular graphs on $2k+3$ vertices ($2k+4$ vertices when k is odd) that are non-Hamiltonian.

A *Hamiltonian path* is a spanning path. We also solve the analogous problem for Hamiltonian paths. Recall from Theorem 1.2 that every 2-connected k-regular graph G on at most $3k+3$ vertices is Hamiltonian, except for when $G \in \{P, P'\}$. So to show that every connected k-regular graph on at most $3k+3$ vertices has a Hamiltonian path, it suffices to investigate P, P', and connected k-regular graphs with a cut-vertex.

2 Maximum Number of Vertices for Hamiltonicity

Theorem 2.1. *Every connected k-regular graph on at most $2k+2$ vertices is Hamiltonian. Furthermore, we characterize connected k-regular graphs on $2k+3$ vertices (when k is even) and $2k+4$ vertices (when k is odd) that are non-Hamiltonian.*

Proof. Let G be a connected k-regular graph on at most $2k+2$ vertices. By Theorem 1.1, it suffices to show that G has no cut-vertex. Assume to the contrary that G has a cut-vertex, v. Now $G-v$ has at least two components, say O_1 and O_2. Let $H_i = G[V(O_i) \cup \{v\}]$ for $i \in [2]$. Since all vertices in G have degree k and each vertex in H_i except v has its neighbors in H_i, the number of vertices in H_i is at least $k+1$. If $|H_i| = k+1$, then $H_i = K_{k+1}$, which contradicts the fact that v is a cut-vertex. Thus, each component of $G-v$ has at least $k+1$ vertices, which implies that G has at least $2k+3$ vertices. This is a contradiction. (Note that by the degree sum formula, $|V(G)|$ is even if k is odd. Thus, if G has a cut-vertex and k is odd, then G has at least $2k+4$ vertices.)

Now we characterize the smallest connected k-regular graphs that are not Hamiltonian; these show that the bound $2k+2$ in the theorem is optimal. (Our characterization relies on graphs first defined in [4] and [5].) As shown above, if a connected k-regular graph G is non-Hamiltonian, then G has at least $2k+3$ vertices if k is even, and at least $2k+4$ vertices if k is odd. Specifically, G must have a cut-vertex, v (for otherwise, it is Hamiltonian by Theorem 1.1), and each component of $G-v$ must have at least $k+1$ vertices. When k is even, the degree sum formula shows that v must have an even number of neighbors in each component of $G-v$. A similar argument works when k is odd. Thus, the description below gives a complete characterization of such graphs. We begin with the case when k is even, and the case when k is odd is similar.

Let $k = 2r$ for $r \geq 2$. For $2 \leq t \leq 2r-2$ and t even, let $F_{r,t}$ be a graph on $2r+2$ vertices with one vertex of degree t and the remaining $2r+1$ vertices of degree $2r$. We can form such a graph from a copy of K_{2r+1} by deleting a matching on t vertices, then adding a new vertex adjacent to the t endpoints of the matching. Let $F'_{r,t}$ be a graph on $2r+1$ vertices with $t+1$ vertices of degree $2r$ and $2r-t$ vertices of degree $2r-1$. We form such a graph from a copy of K_{2r+1} by deleting a matching on $2r-t$ vertices.

Let $\mathcal{F}_r$ be the family of $2r$-regular graphs obtained from $F_{r,t}$ and $F'_{r,t}$ by adding edges from the vertex of degree t in $F_{r,t}$ to the $2r-t$ vertices of degree $2r-1$ in $F'_{r,t}$. Since each such graph contains a cut-vertex, the family $\mathcal{F}_r$ consists entirely of $2r$-regular graphs on $2k+3$ vertices that are non-Hamiltonian.

Now let $k = 2r+1$ for $r \geq 1$. For $2 \leq t \leq 2r$ and t even, let $H_{r,t}$ be a graph on $2r+3$ vertices with one vertex of degree t and the remaining $2r+2$ vertices of degree $2r+1$. As above, to form such a graph, begin with a copy of K_{2r+2}, delete a matching on t vertices, then add a new vertex adjacent to the t endpoints of the matching. Let $H'_{r,t}$ be a graph on $2r+3$ vertices with $t+2$ vertices of degree $2r+1$ and $2r+1-t$ vertices of degree $2r$. To form such a graph from a copy of K_{2r+3}, we delete the edges of a spanning subgraph consisting of some nonnegative number of disjoint cycles and exactly $(t+2)/2$ disjoint paths. One (but not the only) way to form such a subgraph, is to take the union of a near perfect matching (on $2r+2$ vertices) and a second disjoint matching on $2r+2-t$ vertices, including the vertex missed by the first matching.

Let $\mathcal{H}_r$ be the family of $(2r+1)$-regular graphs on $2k+4$ vertices obtained from $H_{r,t}$ and $H'_{r,t}$ by adding edges from the vertex of degree t in $H_{r,t}$ to the $2r+1-t$ vertices of degree $2r$ in $H'_{r,t}$. Since each such graph contains a cut-vertex, the family $\mathcal{H}_r$ consists entirely of $(2r+1)$-regular graphs on $2k+4$ vertices that are non-Hamiltonian. □

Theorem 2.1 determines a threshold for the order of a connected k-regular graph that guarantees the graph is Hamiltonian. In fact, for every positive integer $k \geq 3$ and every even integer $n \geq 2k+4$, we can construct connected k-regular graphs on n vertices that are not Hamiltonian. Similarly, for even $k \geq 4$ and odd $n \geq 2k+3$, we can construct connected k-regular graphs on n vertices that are not Hamiltonian.

Our construction is nearly identical to that in the proof of Theorem 2.1. If k is even, we form $F_{r,t}$ starting from any k-regular graph on $n-k-1$ vertices (rather than K_{2r+1}). An easy example of such graphs are circulants. The remainder of the construction is as before. If k is odd, we form $H'_{r,t}$ starting from any $k+1$-regular graph on $n-k-2$ vertices (rather than K_{2r+3}); again circulants are an example. Thus we have determined exactly those orders n for which a connected k-regular graph on n vertices must be Hamiltonian.

We may also wonder which orders n guarantee that a connected k-regular graph on n vertices must have a Hamiltonian path. Now we answer this question; our proof uses Theorem 2.2.

Theorem 2.2. ([3]) *If G is 2-connected with at most $3\Delta(G)-2$ vertices, where $\Delta(G)$ is the maximum degree of G, then G has a cycle containing all vertices of degree $\Delta(G)$.*

Theorem 2.3. *Every connected k-regular graph with at most $3k + 3$ vertices has a Hamiltonian path. Furthermore, we construct connected k-regular graphs on $3k + 4$ vertices (when $k \geq 6$ is even) and on $3k + 5$ vertices (when $k \geq 5$ is odd) that have no Hamiltonian path.*

Proof. Let G be a connected k-regular graph with at most $3k+3$ vertices. If G is 2-connected, then by Theorem 1.2, G has a Hamiltonian cycle, or $G \in \{P, P'\}$. We can easily see that the Petersen graph has a Hamiltonian path, and every such path extends to a Hamiltonian path in P'. So every counterexample G to the theorem must have a cut-vertex.

Assume that G has a cut-vertex, v. If $G - v$ has at least three components, then G cannot have a Hamiltonian path. But by the proof of Theorem 2.1, each component of $G - v$ has at least $k + 1$ vertices, so G has at least $3k + 4$ vertices. Furthermore, if k is odd, then by the degree sum formula, G has at least $3k + 5$ vertices.

So assume that $G - v$ has only two components, say O_1 and O_2. Let $H_1 = G[V(O_1) \cup \{v\}]$ and $H_2 = G[V(O_2) \cup \{v\}]$. Note that all the vertices in H_1 have degree k except for vertex v. If H_1 has a cut-vertex v_1, then the 3 components of $G \setminus \{v, v_1\}$ have orders at least $k + 1$, $k + 1$, and k, so G has order at least $2 + 2(k + 1) + k = 3k + 4$ (and at least $3k + 5$ when k is odd). Thus H_1 (and similarly, H_2) is 2-connected. Now if H_1 has at most $3k - 2$ vertices, then by Theorem 2.2, H_1 has a cycle containing all vertices of H_1 except v; thus H_1 has a Hamiltonian path with endpoint v. The same is true for H_2. Now since $|V(O_i)| \geq k + 1$ (for both $i \in [2]$), if G has at most $4k - 1$ vertices, then G has a Hamiltonian path, since both H_1 and H_2 have Hamiltonian paths with endpoint v.

For $k \geq 4$, we have $4k - 1 \geq 3k + 3$, so any k-regular graph with at most $3k + 3$ vertices has a Hamiltonian path. For $k = 3$, we may assume that $|V(O_1)| \leq |V(O_2)|$. The same argument holds as above unless $|V(G)| = 3k + 3 = 12$ and $|V(O_2)| = 7$, in which case $|V(H_2)| = 8 > 3k - 2$. Now we have $|V(H_1)| = 5$, so $d_{H_1}(v) = 2$, and thus $d_{H_2}(v) = 1$. Now we apply Theorem 2.2 to O_2 to get a cycle through all vertices of O_2 except the neighbor of v. This cycle yields a Hamiltonian path in O_2 that ends at a neighbor of v. Thus, G has a Hamiltonian path.

For the "Furthermore" part, we construct a $(3k + 4)$-vertex connected k-regular graph without a Hamiltonian path when k is even, and a $(3k+5)$-vertex connected k-regular graph without a Hamiltonian path when k is odd.

Let k be even and at least 6. Let F_1 be the graph obtained from K_{k+1} by deleting an edge, and let F_2 be the graph obtained from K_{k+1} by deleting a matching on $k - 4$ vertices. We form a connected k-regular graph F from two copies of F_1 and one copy of F_2 by adding one new vertex v and adding edges from v to all k vertices of degree $k - 1$.

Now let k be odd and at least 5. Let H_1 be F_1 above. Let H_2 be a graph on $k+2$ vertices, with 6 vertices of degree k and $k - 4$ vertices of degree $k - 1$. This is exactly $H'_{r,t}$ from the proof of Theorem 2.1, with $t = 4$ and $r = (k - 1)/2$. Now we form H from two copies of H_1

and one copy of H_2 by adding one new vertex v and adding edges from v to all k vertices of degree $k-1$. □

As in the case of Theorem 2.1, for every $k \geq 3$ and $n \geq 3k+4$, we can modify our constructions to get connected k-regular graphs on n vertices that have no Hamiltonian path (provided that k and n are not both odd).

References

[1] F. Hilbig, Kantenstrukturen in nichthamiltonschen Graphen. Ph.D. Thesis, Technische Universität Berlin (1986).

[2] B. Jackson, Hamilton cycles in regular 2-connected graphs, *J. Combin. Theorey B* **29** (1980), 27–46.

[3] B. Jackson, Cycles through vertices of large maximum degree, *J. Graph Theory* **19** (1995), 157–168.

[4] Suil O, D.B. West, Balloons, cut-edges, matchings, and total domination in regular graphs of odd degree. *Journal of Graph Theory* **64** (2010) 116–131.

[5] Suil O, D.B. West, Matchings, and edge-connectivity in regular graphs. *European J. Combinatorics* **32** (2011) 324–329.

[6] D.B. West, *Introduction to Graph Theory*, Prentice Hall, INC., Upper Sadle River, NJ, 2001.