Modified Linear Programming and Class 0 Bounds for Graph Pebbling

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August 28, 2015

Abstract

Given a configuration of pebbles on the vertices of a connected graph G, a *pebbling move* removes two pebbles from some vertex and places one pebble on an adjacent vertex. The *pebbling number* of a graph G is the smallest integer k such that for each vertex v and each configuration of k pebbles on G there is a sequence of pebbling moves that places at least one pebble on v.

First, we improve on results of Hurlbert, who introduced a linear optimization technique for graph pebbling. In particular, we use a different set of weight functions, based on graphs more general than trees. We apply this new idea to some graphs from Hurlbert's paper to give improved bounds on their pebbling numbers.

Second, we investigate the structure of Class 0 graphs with few edges. We show that every n-vertex Class 0 graph has at least $\frac{5}{3}n - \frac{11}{3}$ edges. This disproves a conjecture of Blasiak et al. For diameter 2 graphs, we strengthen this lower bound to $2n - 5$, which is best possible. Further, we characterize the graphs where the bound holds with equality and extend the argument to obtain an identical bound for diameter 2 graphs with no cut-vertex.

1 Introduction

Graph pebbling was introduced by Chung in 1989. Following a suggestion of Lagarias and Saks, she computed the pebbling number of Cartesian products of paths to give a combinatorial proof of the following number-theoretic result of Kleitman and Lemke.

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Keywords: pebbling, linear programming, Lemke

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Theorem 1. *[3, 13] Let $\mathbb{Z}_n$ be the cyclic group on n elements and let $|g|$ denote the order of a group element $g \in \mathbb{Z}_n$. For every sequence $g_1, g_2, \ldots, g_n$ of (not necessarily distinct) elements of $\mathbb{Z}_n$, there exists a zero-sum subsequence $(g_k)_{k \in K}$, such that $\sum_{k \in K} \frac{1}{|g_k|} \leq 1$. Here K is the set of indices of the elements in the subsequence.*

Chung developed the pebbling game to give a more natural proof of this theorem. Results of this type are important in this area of number theory, as they generalize zero-sum theorems such as the Erdős-Ginzburg-Ziv theorem [6]. Over the past two decades, pebbling has developed into its own subfield [11, 12], with over 80 papers.

We consider a connected graph G with *pebbles* (indistinguishable markers) on some of its vertices. More precisely, a *configuration* p on a graph G is a function from $V(G)$ to $\mathbb{N} \cup \{0\}$. The *size of p*, denoted $|p|$, is $\sum_{v \in V(G)} p(v)$. A *pebbling move* removes two pebbles from some vertex and places one pebble on an adjacent vertex. A *rooted graph* is a pair (G, r) where G is a graph and $r \in V(G)$ is the *root vertex*. A pebbling configuration p is *solvable* for a rooted graph (G, r) if some configuration p' has at least one pebble on r, and p' can be obtained from p by a sequence of pebbling moves. Otherwise, p is *unsolvable* (or *r-unsolvable*, when the root r is specified.)

The *pebbling number* $\pi(G)$ is the least integer k such that, for any vertex $v \in V(G)$ and any initial configuration p of k pebbles, p is solvable for (G, v). Likewise $\pi(G, r)$ is the pebbling number of G, when the root vertex must be r. A trivial lower bound for $\pi(G)$ is $|V(G)|$: for some root r, we place one pebble on each vertex other than r, for a total of $|V(G)| - 1$ pebbles, but we cannot reach r.

The path, P_n, on n vertices has $\pi(P_n) = 2^{n-1}$. More generally, if graph G has diameter d, then $\pi(G) \geq 2^d$. Let $f(n, d)$ denote the maximum pebbling number of an n-vertex graph with diameter d. Pachter, Snevily, and Voxman [15] proved that $f(n, 2) = n + 1$, and Clarke, Hochberg, and Hurlbert [5] classified all graphs G of diameter 2 with $\pi(G) = n + 1$. Bukh [2] proved that $f(n, 3) = 3n/2 + O(1)$, and Postle, Streib, and Yerger [17] strenthened Bukh's result, proving the exact bound $f(n, 3) = \lfloor 3n/2 \rfloor + 2$. They also gave [17] an asymptotic bound for $f(n, 4)$.

Section 2 gives some necessary preliminaries; in particular it describes a technique of Hurlbert [10] based on linear programming. In Section 3, we improve Hurlbert's method by using a different set of weight functions, based on graphs more general than trees. We apply this new idea to some graphs from his paper to give improved bounds on their pebbling numbers.

In Section 4, we investigate the structure of Class 0 graphs (graphs G with $\pi(G) = |V(G)|$) with few edges. We show that every n-vertex Class 0 graph has at least $\frac{5}{3}n - \frac{11}{3}$ edges. This disproves a conjecture of Blasiak et al [1]. For diameter 2 graphs, we strengthen this bound to $2n - 5$ edges, which is best possible. We characterize the graphs where it holds with equality and extend the argument to obtain an identical bound for diameter 2 graphs with no cut-vertex.

2   Linear Programming Preliminaries

Computing a graph's pebbling number is hard. Watson [18] and Clark and Milans [4] studied the complexity of graph pebbling and some of its variants, including optimal pebbling and cover pebbling. Watson showed that it is NP-complete to determine whether a given configuration is solvable for a given rooted graph (G, r). Clark and Milans refined this result, showing that deciding whether $\pi(G) \leq k$ is Π_2^P-complete; this means that pebbling is in the class of problems computable in polynomial time by a co-NP machine equipped with an oracle for an NP-Complete language.

Hurlbert [10] introduced a new linear programming technique, in hopes of more efficiently computing bounds on pebbling numbers. Before we describe our improvements on it, we briefly explain his method. Let G be a graph and let T be a subtree of G rooted at r. For each $v \in V(T) - r$, let v^+ be the parent of v, the neighbor of v in T that is closer to r. A *tree strategy* is a tree T and an associated nonnegative *weight function* w_T (or w if the context is clear) where $w(r) = 0$ and $w(v^+) = 2w(v)$ for every vertex not adjacent to r. Further, $w(v) = 0$ if $v \notin V(T)$. Let $\mathbf{1}_G$ be the vector on $V(G)$ in which every entry is 1.

Hurlbert [10] proposed a general method for defining such a weight function through tree strategies. He proved the following result (here $\cdot$ denotes dot product).

Lemma 1. *Let T be a tree strategy of G rooted at r, with associated weight function w. If p is an r-unsolvable configuration of pebbles on $V(G)$, then $w \cdot p \leq w \cdot \mathbf{1}_G$.*

The proof idea is easy. Suppose that p is a configuration with $w \cdot p > w \cdot \mathbf{1}_G$. This implies that some vertex v in T has at least two pebbles. Now we make a pebbling move from v toward the root, i.e., from v to v^+, to get a new configuration p'. Since $w(v^+) = 2w(v)$, we have $w \cdot p' = w \cdot p > w \cdot \mathbf{1}_G$. By repeating this process, we can eventually move a pebble to the root, r.

Since every r-unsolvable pebbling configuration p satisfies $w \cdot p \leq w \cdot \mathbf{1}_G$, it follows that $\pi(G, r)$ is bounded above by one plus the number of pebbles in the largest configuration p such that $w \cdot p \leq w \cdot \mathbf{1}_G$. Let $\mathcal{T}_r$ be the set of all tree strategies in G associated with root vertex r. By applying Lemma 1 to all of $\mathcal{T}_r$ simultaneously, we arrive at the following integer linear program:

$$\begin{aligned} \max \quad & \textstyle\sum_{v \neq r} p(v) \\ \text{s.t.} \quad & w \cdot p \leq w \cdot \mathbf{1}_G \\ & \text{for all } T \in \mathcal{T}_r. \end{aligned}$$

Let $z_{G,r}$ be the optimal value of this integer linear program and let $\hat{z}_{G,r}$ be the optimum of the linear relaxation, so that configurations can be rational. Since $z_{G,r} \leq \lfloor \hat{z}_{G,r} \rfloor$, we get the bound $\pi(G, r) \leq z_{G,r} + 1 \leq \lfloor \hat{z}_{G,r} \rfloor + 1$. Let $w_1, \ldots, w_k$ be weight functions of tree strategies for trees (possibly different) rooted at r, and let w' be a convex combination of $w_1, \ldots, w_k$. If p is an r-unsolvable configuration, then $w' \cdot p \leq w' \cdot \mathbf{1}_G$ (otherwise $w_i \cdot p > w_i \cdot \mathbf{1}_G$, for some i, a contradiction). Further, if $w'(v) \geq 1$ for all v, then $|p| \leq \sum_{v \neq r} \lfloor w'(v) \rfloor p(v) \leq \lfloor w' \rfloor \cdot \mathbf{1}_G$. For ease of application, we state this observation in a slightly more general form. We call this the Covering Lemma.

Lemma 2 (Covering Lemma). *For a graph G and a root $r \in V(G)$, let w' be a convex combination of tree strategies for r, and let C and M be positive constants. If $w'(v) \geq C$ for all $v \in V(G) \setminus \{r\}$ and $\sum_{v \in V(G) \setminus \{r\}} w'(v) < M$, then $\pi(G, r) \leq \lfloor \frac{M}{C} \rfloor + 1$. In particular, if $\sum_{v \in V(G) \setminus \{r\}} w'(v) < C|V(G)|$, then $\pi(G, r) = |V(G)|$.*

For any bound on $\pi(G)$ arising from such a w', a *certificate* of the bound consists of the strategies w_i and their coefficents in the convex combination forming w'.

Hurlbert applies this linear programming method more broadly by considering strategies on trees where $w(v^+) \geq 2w(v)$, called *nonbasic strategies*. Since nonbasic strategies are conic combinations of basic strategies [10, Lemma 5], this extension does not strengthen the method. However, it often yields simpler certificates.

3   More General Weight Functions

Here we generalize the notion of weight function from the previous section to allow weight functions for graphs G that are not trees. A *weight function* is a map $w : V(G) \rightarrow \mathbf{R}^+ \cup \{0\}$. A weight function for a graph G and root r is *valid* if $w(r) = 0$ and every r-unsolvable configuration p satisfies $w \cdot p \leq w \cdot \mathbf{1}_G$. Although it is harder to show that one of these more general weight functions is valid, when we can, this often leads to improved pebbling bounds for a variety of graph families. Given a graph G and a root r, it is straightforward to check that the theory developed in the previous section extends to any weight function w such that $w \cdot p \leq w \cdot \mathbf{1}_G$ for every configuration p that is not r-solvable. Our next result establishes a new family of such weight functions. A *k-vertex* is a vertex of degree k.

Lemma 3. *Form G from an even cycle C_{2t} by identifying one vertex with the endpoint of a path of length $s - t$. Let x_t be the resulting 3-vertex and x_0 be the 2-vertex farthest from x_t; now x_0 and x_t split the even cycle into two paths, P_1 and P_2. Label the internal vertices of P_1 as $x'_1, x'_2, \ldots, x'_{t-2}, x'_{t-1}$ and the internal vertices of P_2 as $x''_1, x''_2, \ldots, x''_{t-2}, x''_{t-1}$. Call the 1-vertex r, and let P_3 be the path from x_t to r. Label the internal vertices of P_3 as $x_{t+1}, x_{t+2}, \ldots, x_{s-1}, x_s$. For each $i \neq 0$, give weight 2^i to vertex x_i or vertices x'_i and x''_i. Let $\alpha = \frac{2^s + 2^{t-1} - 2}{2^s - 1}$ and give weight α to x_0. Fix some order on the vertices, and let w be the vector of length $|V(G)|$ where entry i is the weight of vertex i. If p is an r-unsolvable configuration, then $w \cdot p \leq w \cdot \mathbf{1}_G$.*

Proof. Figures 1 and 6 both show examples of this lemma, which we apply later.

Let p be an r-unsolvable configuration. We will show that $w \cdot p \leq w \cdot \mathbf{1}_G$. Let $M = \alpha + 2^{s+1} + 2^t - 4$. Note that $w \cdot \mathbf{1}_G = M$. Let $W_0 = \alpha p(x_0)$, $W_L = \sum_{v \in P_1 \setminus \{x_0, x_t\}} w(v)p(v)$, $W_R = \sum_{v \in P_2 \setminus \{x_0, x_t\}} w(v)p(v)$, and $W_C = \sum_{v \in P_3 \setminus \{r\}} w(v)p(v)$. (Here L, R, and C stand for left, right, and center.) We will show that $W_0 + W_L + W_R + W_C \leq M$.

Claim 1. *If $W_L = 0$ or $W_R = 0$, then the lemma is true.*

By symmetry, assume that $W_L = 0$. Now Lemma 1 implies that $\alpha^{-1} W_0 + W_R + W_C \leq 2^{S+1} - 1$. Multiplying by α gives $W_0 + W_L + W_R + W_C \leq W_0 + W_L + \alpha(W_R + W_C) \leq$

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$\alpha(2^{s+1} - 1) = \alpha + \alpha(2^{s+1} - 2) = \alpha + 2(2^s + 2^{t-1} - 2) = \alpha + 2^{s+1} + 2^t - 4 = M$. This proves the claim.

Claim 2. *If $W_L + \alpha^{-1}W_0 > 2^t - 1$, then the lemma is true.*

Suppose that $W_L + \alpha^{-1}W_0 > 2^t - 1$. We can assume that $W_L < 2^t$; otherwise we can move weight down to x_t, without changing the sum $W_L + W_C$. Now we move some pebbles toward the root and reduce to the case in Claim 1. Specifically, we remove $2^t - W_L$ pebbles from x_0 and place half that many on x_1'. Call the new configuration p' and define W_0', W_L', W_R', and W_C', analogously. Note that $\alpha^{-1}W_0' + W_L' + W_R' + W_C' = \alpha^{-1}W_0 + W_L + W_R + W_C$. Since $W_L' = 2^t$, we can move all weight from internal vertices of P_1 to x_t. This gives a new configuration p'' with $W_L'' = 0$. Again $\alpha^{-1}W_0'' + W_L'' + W_R'' + W_C'' = \alpha^{-1}W_0 + W_L + W_R + W_C$. So the claim holds by Claim 1.

By Claim 2, we now assume that $W_L + \alpha^{-1}W_0 \leq 2^t - 1$. By symmetry, we assume that $W_R + \alpha^{-1}W_0 \leq 2^t - 1$. By Lemma 1, we can also assume that $W_C \leq 2^{s+1} - 2^t$. Adding these inequalities yields

$$W_C + W_L + W_R + 2\alpha^{-1}W_0 \leq 2(2^t - 1) + 2^{s+1} - 2^t$$
$$= 2^{s+1} + s^t - 2. \tag{1}$$

We can assume that $W_0 > 0$, since otherwise the lemma holds by Lemma 1. Thus, we have $W_0 \geq \alpha$, so $(2\alpha^{-1} - 1)W_0 \geq 2 - \alpha$. Subtracting this inequality from (1) gives the desired result. $\qquad \square$

The following observation extends our class of valid weight functions a bit further.

Observation 1. *Let G be a graph and r a root; let w be a weight function on G such that $w \cdot p \leq w \cdot \mathbf{1}_G$ for every r-unsolvable configuration p. Form G' from G by adding a new vertex u adjacent to some vertex u^+ of G (with $u^+ \neq r$), and form w' from w, where $w'(u) = \frac{1}{2}w(u^+)$ and $w'(v) = w(v)$ for every $v \in V(G)$. For every r-unsolvable configuration p' in G', we have $w' \cdot p' \leq w' \cdot \mathbf{1}_{G'}$. Further, we can allow, more generally, that $w'(u) \leq \frac{1}{2}w(u^+)$. We can also attach trees, rather than single vertices.*

Proof. If the new vertex u has more than one pebble, we move as much weight as possible from u to u^+, which does not decrease the total weight on G. This proves the first statement. The second statement follows from taking convex combinations of w and w'. The final statement follows by induction on the size of the tree T that we attach (we just proved the induction step, and the base case, $|T| = 0$, is trivial). $\qquad \square$

3.1 The Cube and the Lemke Graph

To illustrate the usefulness of Lemma 3 and Observation 1, we give two easy applications of this method. We show that $\pi(Q_3) = 8$ and $\pi(L) = 8$, where Q_3 is the 3-dimensional cube and L is the Lemke graph, shown in Figure 3. When using tree strategies alone, Hurlbert's method cannot handle these graphs.
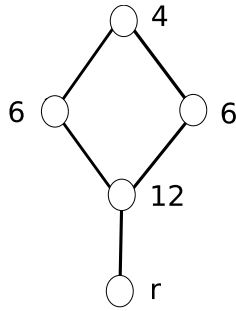
Figure 1: A valid non-tree weight function.

Proposition 1. *If Q_3 is the 3-cube, then $\pi(Q_3) = 8$.*

Proof. Every graph G satisfies $\pi(G) \geq |V(G)|$, so $\pi(Q_3) \geq 8$. Thus, we focus on proving that $\pi(Q_3) \leq 8$.

To show that $\pi(Q_3) \leq 8$, we first note that the weight function in Figure 1 is valid. Since a valid weight function remains valid when multiplied by a positive constant (in this case 3), this statement follows from Lemma 3, with $t = 2$ and $s = 0$.
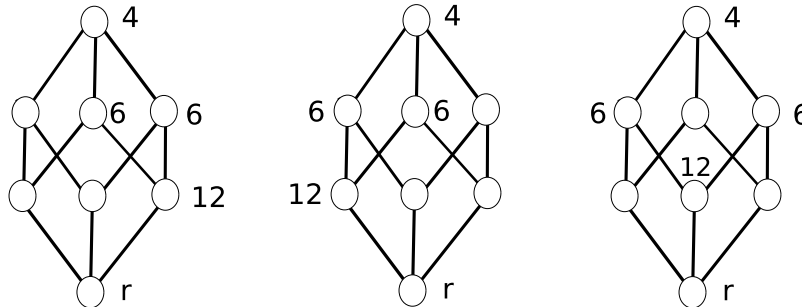

Figure 2: A certificate that $\pi(Q_3) \leq 8$.

The convex combination of the three strategies shown in Figure 2 (each taken with weight 1) yields w' such that $w'(v) = 12$ for all $v \neq r$. Thus, the Covering Lemma shows that $\pi(Q_3) \leq 8$, so Q_3 is Class 0. These three strategies in Figure 2 also serve as a certificate that $\pi(Q_3) \leq 8$, and they yield an efficient algorithm for getting a pebble to r, starting from any configuration p with $|p| \geq 8$. $\qquad\square$

The most famous long-standing pebbling problem is Graham's conjecture: for all graphs G_1 and G_2, $\pi(G_1 \square G_2) \leq \pi(G_1)\pi(G_2)$; here $\square$ denotes the Cartesian product. This conjecture has been verified only for a few classes of graphs. Specifically, it holds when G_1 and G_2 are both cycles [9], both trees [14], both complete bipartite graphs [7], or a fan and a wheel [8].

When considering Graham's conjecture, we are interested in the Lemke Graph, denoted L and shown on the left in Figure 3. This graph is of interest because it is the

smallest graph without the *2-pebbling property*. The exact definition is unimportant for us here; what matters, is that if G has this property, then $\pi(G\square H) \leq \pi(G)\pi(H)$ for every graph H. This makes $L\square L$ a natural candidate for disproving Graham's conjecture.
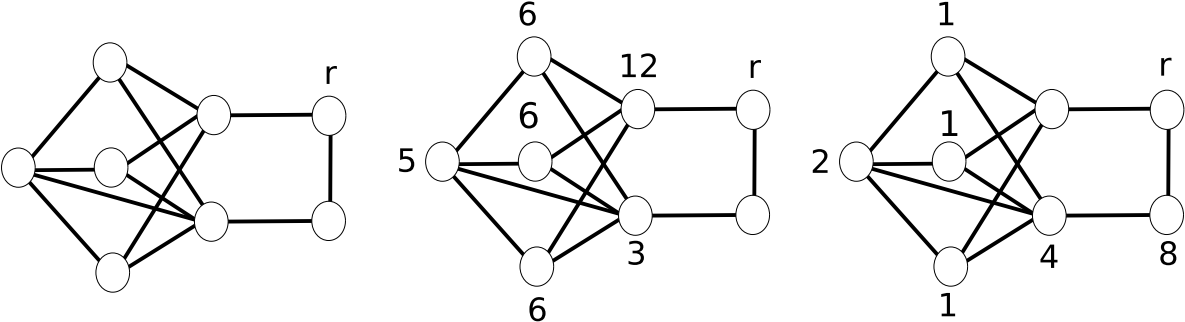

Figure 3: The Lemke graph.

Hurlbert asserted that it is impossible, using tree strategies alone, to obtain the pebbling number of the Lemke graph via the linear programming technique. However, by using this method with more general weight functions, we prove that $\pi(L) = 8$.
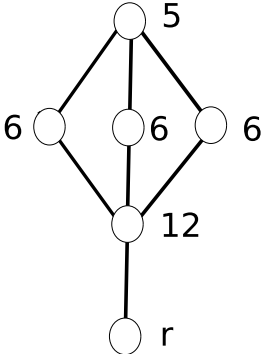

Figure 4: A weight function useful for the Lemke graph.

Theorem 2. *If L is the Lemke graph, then $\pi(L) = 8$.*

Proof. Note that $\pi(L) \geq |V(L)| = 8$, so we focus on proving the upper bound. Hurlbert [10] showed that $\pi(L, v) = 8$ for all vertices $v \in L$ except for r, as shown on the left in Figure 3. So we only need to show that $\pi(L, r) = 8$. Now we need the weight function in Figure 4.

Claim 1. *The weight function in Figure 4 is valid.*

The proof of this claim is very similar to the proof of Observation 1, so we just sketch the ideas. If any vertex weighted 6 has no pebbles, then we invoke the weight

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function in Figure 1, and multiply the resulting inequality by $\frac{5}{4}$ to get one that implies what we want; so we assume that each vertex weighted 6 has a pebble. If the vertex weighted 12 has a pebble, then the vertex weighted 5 has at most one pebble, so we are done. Otherwise, the vertex weighted 5 has at most 3 pebbles; again, we are done. This proves the claim.

The proof that $\pi(L, r) \leq 8$ uses the two strategies in Figure 3. The rightmost is a nonbasic tree strategy. The center strategy is derived from the weight function in Figure 4 by adding a vertex with weight 3 adjacent to some vertex with weight 6. This weight function is valid, by Observation 1. When we sum the weights of the two strategies, each vertex has weight at least 7 and the total weight is 55. Now the Covering Lemma implies that $\pi(L, r) \leq \lfloor \frac{55}{7} \rfloor + 1 = 8$. $\qquad\square$

3.2 Larger Graphs

In this section we determine the pebbling number of the Bruhat graph of order 4. The (weak) *Bruhat graph* of order m has as its vertices the permutations of $\{1, \ldots, m\}$; two vertices are adjacent if the corresponding permutations differ by an adjacent transposition. Since this graph is vertex-transitive, we can choose the root vertex arbitrarily. Using the linear programming method, Hurlbert proved that $\pi(B_4) \leq 72$. By using more general weight functions, we calculate the pebbling number of this graph exactly.
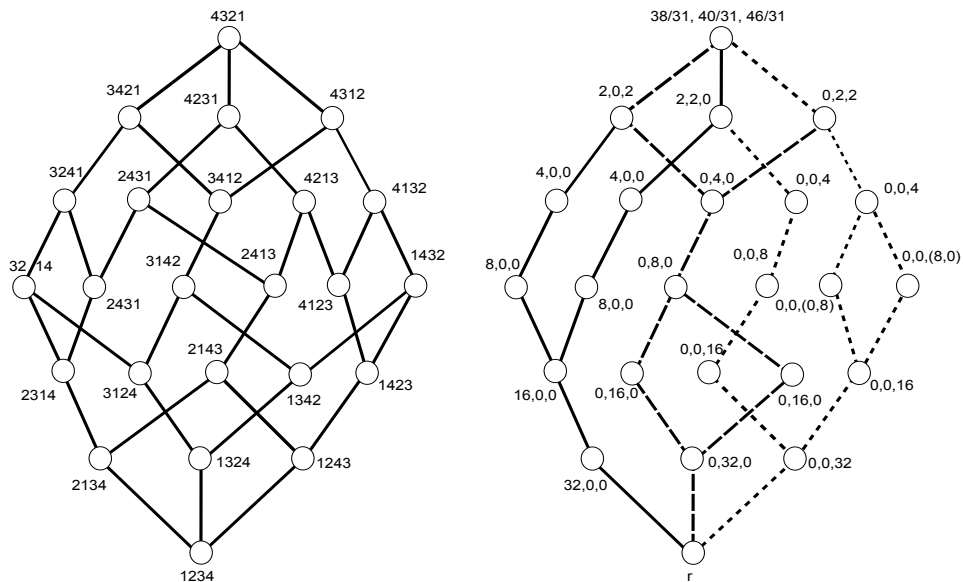

Figure 5: The Bruhat graph of order 4 and a set of strategies proving $\pi(B_4) = 64$.

Theorem 3. *If B_4 is the Bruhat graph of order 4, then $\pi(B_4) = 64$.*

Proof. The diameter of B_4 is 6, so $\pi(B_4) \geq 2^6 = 64$. We need to show that $\pi(B_4) \leq 64$.

Note that the rightmost graph in Figure 6 describes two strategies, as we explain below. We combine these four strategies, as shown in Figure 5, (weighted with multiplicities $\frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}$) to get a weight function w', such that $w'(v) \geq 1$ for all v and $w' \cdot \mathbf{1}_{B_4} = 63$. This proves the desired upper bound $\pi(B_4) \leq 63 + 1$. Thus, we only need to show that the four strategies in Figure 6 are valid.
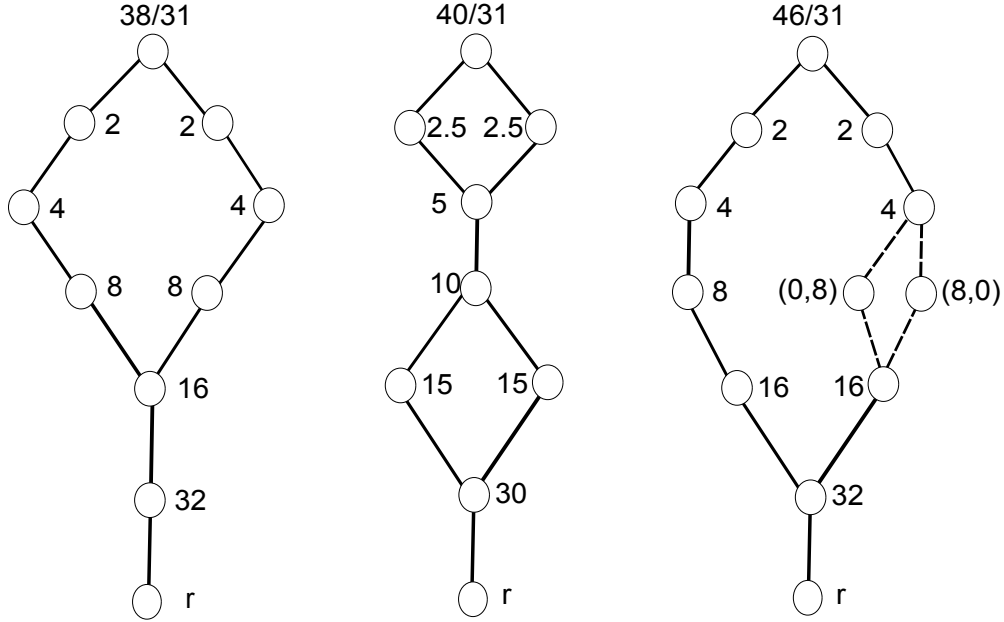

Figure 6: More general weight functions.

The weight function on the right denotes two different weight functions on G; the first includes the vertex labeled $(8,0)$ but not the one labeled $(0,8)$, and the second vice versa. By Lemma 3, the leftmost and rightmost strategies are valid (the former with $t = 4$ and $s = 1$; the latter with $t = 5$ and $s = 0$).

The proof that the middle strategy is valid is similar to the proof of Lemma 3, so we just sketch the ideas. Note that weights 30, 15, 10, and 5 (with the other vertices unweighted) are consistent with Lemma 3 (mulitplied by $\frac{15}{2}$), when $t = 2$ and $s = 0$, and adding a vertex by Observation 1. Thus, we know that if p is r-unsolvable, then these five vertices have weight at most 75. The key observation is that these five vertice can play the role of P_3 in the proof of Lemma 3. Let x_0, x_1', and x_1'' denote the vertex labeled $\frac{40}{31}$ and its two neighbors, respectively. We first consider the case where

x_1' or x_1'', say x_1', has no pebbles. In this case we move as much weight as possible to the vertex labeled 5 from x_0 and x_1''. We also consider the case where both x_1' and x_1'' have pebbles. Either we can reduce to the previous case, or else we get two inequalities. We add these two to the inequality for the bottom 5 vertices, which gives the desired inequality. $\qquad\square$

4   Class 0 Graphs

4.1   Preliminaries

In this section, we study Class 0 graphs. We focus on graphs with diameter at least 2 since those with diameter 0, a single vertex, and diameter 1, a complete graph, are well understood. A graph G is *Class 0* if its pebbling number is equal to its number of vertices, i.e., $\pi(G) = |V(G)|$. Recall that always $\pi(G) \geq |V(G)|$, so Class 0 graphs are those where this trivial lower bound holds with equality. For each vertex v, we write $N(v)$ for the set of vertices adjacent to v, and we write $N[v]$ to denote $N(v) \cup \{v\}$. For a graph G, let $e(G)$ denote the the number of edges in G. In this section, we prove lower bounds on $e(G)$ for all Class 0 graphs.

Blasiak et al. [1] showed that every n-vertex Class 0 graph G has $e(G) \geq \left\lfloor \frac{3n}{2} \right\rfloor$. They also conjectured (see [10, p. 19]) that for some constant C and for all sufficiently large n there exist n-vertex Class 0 graphs with $e(G) \leq \left\lfloor \frac{3n}{2} \right\rfloor + C$. In particular, they defined a family of "generalized Petersen graphs" of arbitrary size and diameter with one vertex of some fixed degree m and all other vertices of degree 3. They conjectured that these graphs are all Class 0. We disprove this conjecture in a very strong sense. Shortly, we prove that for fixed m, all sufficiently large graphs of this form are not Class 0. (Figure 7 shows $P_{8,2}$, one of these generalized Petersen graphs that is not Class 0.) Later in this section, we extend this idea to show that every n-vertex Class 0 graph G has $e(G) \geq \frac{5}{3}n - \frac{11}{3}$. To conclude the section, for all diameter 2 graphs G we strengthen this lower bound to $e(G) \geq 2n - 5$. Further, we characterize the graphs where this bound holds with equality (which include two infinite families).
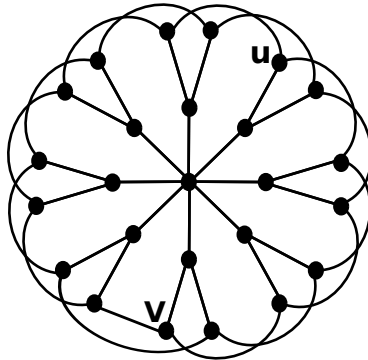

Figure 7: The generalized Petersen graph, $P_{8,2}$, is not Class 0.

Our main tool for proving bounds on $e(G)$ is the following lemma.

Lemma 4 (Small Neighborhood Lemma). *Let G be a Class 0 graph. If $u, v \in V(G)$, $d(u) = 2$, and u and v are distance at least 3 apart, then $d(v) \geq 4$. Similarly, if $u, v \in V(G)$, $d(u) = 3$, u and v are distance at least 4 apart, and each neighbor of v is a 3-vertex, then $d(v) \geq 4$.*

Proof. The proofs for both statements are similar. In each case, we assume the statement is false and construct a configuration with $|V(G)|$ vertices that is u-unsolvable. Consider the first statement first. Suppose, to the contrary, that u and v are as required, but $d(v) \leq 3$. Form configuration p by putting 7 pebbles on v, 0 pebbles on each vertex of $N[u] \cup N(v)$, and 1 pebble on each other vertex. Since $|N[u] \cup N[v]| \leq 7$, this configuration has at least $|V(G)|$ pebbles. Now no pebble can reach u, since at most one pebble can leave $N[r]$. This contradicts that G is Class 0, so $d(v) \geq 4$.

Now consider the second statement. Suppose, to the contrary, that u and v are as required, but $d(v) \leq 3$. Form configuration p by putting 15 pebbles on v, 0 pebbles on each vertex of $N[u] \cup (N[N[v]] \setminus \{v\})$, and 1 pebble on each other vertex. Since $|N[u] \cup N[N[v]]| \leq 15$, the configuration has at least $|V(G)|$ pebbles, but no pebble can reach v, since at most one pebble can leave $N[N[v]]$. This contradicts that G is Class 0. Thus, $d(v) \geq 4$. $\quad\square$

Corollary 1. *For each integer C, there exists an integer n_0, such that if G is any n-vertex graph with $\delta(G) = 3$, $n \geq n_0$, and $e(G) \leq \frac{3}{2}n + C$, then G is not Class 0.*

Proof. We can choose n_0 sufficiently large so that there exists some pair of vertices u, v violating the second statement of the Small Neighborhood Lemma. Specifically, it suffices to find a 3-vertex v such that every vertex within distance four of v is a 3-vertex. To guarantee such a vertex v, we can take, for example, $n_0 = 2C * 3^5$. $\quad\square$

4.2 Diameter at least 3

Now we use the Small Neighborhood Lemma to prove, in Theorem 4, that every n-vertex Class 0 graph G with diameter at least 3 has $e(G) \geq \frac{5}{3}n - \frac{11}{3}$. The case $\delta(G) = 2$ is complicated, so we handle it separately, in Lemma 6. For the case $\delta(G) \leq 1$, we use the following easy lemma from [5].

Lemma 5 ([5]). *Every Class 0 graph G has no cut-vertices. Specifically, $\delta(G) \geq 2$.*

Proof. Let G be a graph with a cut-vertex u and neighbors v_1 and v_2 that are in different components of $G - u$. Consider the distribution p with 3 pebbles on v_1, 0 pebbles on each of u and v_2, and 1 pebble on each other vertex. Distribution p has $|V(G)|$ pebbles, but no pebble can reach v_2, which we now show. If a pebble ever moves to u, then at that point each vertex has at most one pebble, and v_2 has no pebbles. Otherwise, every pebbling move is within the component of $G - u$ containing v_1, so no pebble reaches v_2. Thus no pebble can reach v_2, so G is not Class 0. $\quad\square$

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Lemma 6. *If an n-vertex Class 0 graph G has diameter at least 3 and $\delta(G) = 2$, then $e(G) \geq \frac{5}{3}n - \frac{11}{3}$.*

Proof. Let G be an n-vertex Class 0 graph with diameter at least 3 and $\delta(G) = 2$. We assign each vertex v a charge $\text{ch}(v)$, where $\text{ch}(v) = d(v)$. Now we redistribute these charges, without changing their sum, so that all but a few vertices finish with charge at least $\frac{10}{3}$; the charge of each vertex v after redistributing is $\text{ch}^*(v)$. If at most k vertices finish with charge less than $\frac{10}{3}$ (but all charges are nonnegative), then $e(G) = \frac{1}{2}\sum_{v \in V} \text{ch}(v) = \frac{1}{2}\sum_{v \in V} \text{ch}^*(v) \geq \frac{1}{2}(\frac{10}{3}(n - k)) = \frac{5}{3}n - \frac{5}{3}k$.

Choose $r \in V(G)$ such that $d(r) = 2$. For each positive integer i, let N_i denote the set of vertices at distance i from r. Also, let $N_{3+} = \bigcup_{i \geq 3} N_i$. By the Small Neighborhood Lemma with $u = r$, if $v \in N_{3+}$, then $d(v) \geq 4$.

We redistribute charge according to the following two discharging rules.

1. Each vertex $v \in N_2$ takes charge 1 from some neighbor in N_1. If $d(v) = 2$, then v also takes charge $\frac{1}{3}$ from its other neighbor.

2. Each vertex $v \in N_{3+}$ with $d(v) = 4$ takes charge $\frac{1}{3}$ from each neighbor u with $d(u) \geq 3$.

We show that nearly all vertices finish with charge at least $\frac{10}{3}$. Consider a vertex $v \in V(G) \setminus N[r]$. If $d(v) \geq 5$, then $\text{ch}^*(v) \geq d(v) - \frac{1}{3}d(v) = \frac{2}{3}d(v) \geq \frac{10}{3}$. Now suppose $v \in N_2$ and $d(v) \geq 3$. In this case, $\text{ch}^*(v) \geq d(v) + 1 - \frac{1}{3}(d(v) - 1) = \frac{2}{3}d(v) + \frac{4}{3} \geq \frac{10}{3}$. Suppose instead that $v \in N_2$, $d(v) = 2$, and either v has both neighbors in N_1 or the neighbor of v outside of N_1 has degree at least 3. Now $\text{ch}^*(v) = d(v) + \frac{4}{3} = \frac{10}{3}$.

We show that G has at most two 2-vertices in N_2 with 2-neighbors in N_2. Suppose, to the contrary, that u_1, u_2, and u_3 are 2-vertices in N_2, each with a 2-neighbor in N_2; by symmetry, assume $u_1 u_2 \in E(G)$. By Lemma 5, u_1 and u_2 cannot have a common neighbor $v \in N_1$, since then v would be a cut-vertex. Thus, u_1 and u_2 have distinct neighbors in N_1. However, now u_3 is distance three from either u_1 or u_2; by symmetry, say u_1. Now u_1 and u_3 contradict the Small Neighborhood Lemma. So indeed N_2 has at most two 2-vertices with 2-neighbors in N_2.

Now we consider 4-vertices in N_{3+}. Rather than compute the charges of these 4-vertices individually, we group them together as follows. Let H be the subgraph induced by 4-vertices in N_{3+}, and let H_1 be a component of H with k vertices. If H_1 contains a cycle, then H_1 contains at least k edges, so vertices of H_1 give charge to at most $4k - 2(k) = 2k$ vertices outside H_1. Thus, $\text{ch}^*(H_1) \geq \text{ch}(H_1) - 2k(\frac{1}{3}) = 4k - \frac{2k}{3} = \frac{10}{3}k$. Similarly, if H_1 has some adjacent vertex that is not a 2-vertex, then $\text{ch}^*(H_1) \geq \text{ch}(H_1) - (2k + 1)(\frac{1}{3}) + \frac{1}{3} = \frac{10}{3}k$. Instead, assume that H_1 is a tree and every vertex adjacent to H_1 is a 2-vertex. Recall that each such 2-vertex is in N_2.

If every 2-neighbor of H is adjacent to the same vertex of N_1, call it v, then v is a cut-vertex. Thus, H_1 has 2-neighbors that are adjacent to both vertices of N_1; call these 2-neighbors u_1 and u_2. By the Small Neighborhood Lemma, every pair of 2-vertices in N_2 are adjacent or have a common neighbor. Since u_1 and u_2 are both adjacent to H_1, they can't be adjacent to each other; thus, they must have a common

neighbor, u_3. Further, every 2-vertex in N_2 must be adjacent to u_3. Since $u_3 \in V(H_1)$, u_3 is a 4-vertex, so N_2 has at most four 2-vertices. Thus, H_1 is the only component of H with final charge less than $\frac{10}{3}$ times its size. Furthermore, H_1 has only a single vertex, and $\mathrm{ch}^*(H_1) = 4 - 4(\frac{1}{3}) = \frac{8}{3}$.

Now we compute the total final charge of $V(G)$. For each vertex v not in H, the final *excess* of v is $\mathrm{ch}^*(v) - \frac{10}{3}$. For each component H_i of H with order k, the final excess is $\mathrm{ch}^*(H_i) - \frac{10}{3}k$. We now show that the sum of all final excesses is greater than or equal to $-\frac{22}{3}$, which proves the lemma.

If $v \in N_{3+}$ and $d(v) \geq 5$, then $\mathrm{ch}^*(v) \geq \frac{10}{3}$, so v has nonnegative excess. Each component of H, other than (possibly) H_1, has nonnegative excess. Further, H_1 has excess greater than or equal to $-\frac{2}{3}$. Each $v \in N_2$ with $d(v) \geq 3$ has nonnegative excess. Also, each $v \in N_2$ with $d(v) = 2$ has excess 0, except for at most two adjacent 2-vertices, which each have excess $-\frac{1}{3}$. Finally, the sum of the final charges on $N[r]$ is at least 4 (since r takes no charge from $N(r)$). Thus, the sum of excesses of $N[r]$ is at least $4 - 3(\frac{10}{3}) = -6$. So the sum of excesses over all vertices is at least $2(-\frac{1}{3}) + (-\frac{2}{3}) + (-6) = -\frac{22}{3}$. Thus $\sum_{v \in V(G)} d(v) \geq \frac{10}{3}n - \frac{22}{3}$, so $e(G) \geq \frac{5}{3}n - \frac{11}{3}$. □

Now we prove our main theorem of this section.

Theorem 4. *If G is an n-vertex Class 0 graph with diameter at least 3, then $e(G) \geq \frac{5}{3}n - \frac{11}{3}$.*

Proof. Let G be Class 0 with diameter at least 3. By Lemma 5, $\delta(G) \geq 2$. Lemma 6 proves the bound when $\delta(G) = 2$. If $\delta(G) \geq 4$, then $e(G) \geq \frac{\delta(G)n}{2} \geq 2n$. Thus, we assume that $\delta(G) = 3$.

The proof is similar to that of Lemma 6, but easier. Recall that a *k-vertex* is a vertex of degree k. Similarly, a *k^+-vertex* has degree at least k and a *k-neighbor* of a vertex v is a k-vertex adjacent to v. Choose r to be a 3-vertex with as few vertices at distance 2 as possible. For each integer i, let N_i denote the set of vertices at distance i from r. Also, let $N_{4+} = \bigcup_{i \geq 4} N_i$. We first handle the case $|N_2| \geq 8$, which is short.

Claim 1. *If $|N_2| \geq 8$, then $e(G) \geq \frac{5}{3}n$.*

Since r was chosen among all 3-vertices to minimize N_2, each 3-vertex has either a 5^+-neighbor or at least two 4-neighbors. Thus, we let $\mathrm{ch}(v) = d(v)$ and use the following discharging rule.

1. Each 3-vertex takes $\frac{1}{6}$ from each 4-neighbor and $\frac{1}{3}$ from each 5^+-neighbor.

If $d(v) \geq 5$, then $\mathrm{ch}^*(v) \geq d(v) - \frac{1}{3}d(v) = \frac{2}{3}d(v) \geq \frac{10}{3}$. If $d(v) = 4$, then $\mathrm{ch}^*(v) \geq d(v) - \frac{1}{6}d(v) = 4 - \frac{4}{6} = \frac{10}{3}$. If $d(v) = 3$, then $\mathrm{ch}^*(v) \geq 3 + \frac{1}{3} = \frac{10}{3}$ or $\mathrm{ch}^*(v) \geq 3 + \frac{2}{6} = \frac{10}{3}$. Hence, $e(G) = \frac{1}{2}\sum_{v \in V(G)} \mathrm{ch}(v) = \frac{1}{2}\sum_{v \in V(G)} \mathrm{ch}^*(v) \geq \frac{5}{3}n$. This proves the claim.

Hereafter, we assume that $|N_2| \leq 7$. Now a variation on the Small Neighborhood Lemma implies that $d(v) \geq 4$ for each vertex $v \in N_{4+}$. Suppose instead that $d(v) = 3$ for some vertex $v \in N_{4+}$. Let p be the configuration with 15 pebbles on r, 0

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pebbles on each vertex in $N_1 \cup N_2 \cup N[v]$, and 1 pebble on each other vertex. Since $|\{r\} \cup N_1 \cup N_2 \cup N[v]| \leq 15$, the configuration has at least n pebbles, but no pebble can reach v, since at most one pebble can leave $N[r] \cup N_2$. This contradicts that G is Class 0. Thus, $d(v) \geq 4$ for each $v \in N_{4+}$.

Now we again redistribute charge. We let $\mathrm{ch}(v) = d(v)$ and we use the following two discharging rules.

1. Each vertex in N_2 takes charge 1 from its neighbor in N_1.

2. Each vertex in N_3 takes charge $\frac{1}{3}$ from its neighbor in N_2.

We show that each vertex in $V(G) \setminus N[r]$ finishes with charge at least $\frac{10}{3}$. If $v \in N_{4+}$, then $\mathrm{ch}^*(v) = \mathrm{ch}(v) = d(v) \geq 4$. If $v \in N_3$, then $\mathrm{ch}^*(v) \geq d(v) + \frac{1}{3} \geq \frac{10}{3}$. If $v \in N_2$, then $\mathrm{ch}^*(v) \geq d(v) + 1 - \frac{1}{3}(d(v) - 1) = \frac{2}{3}d(v) + \frac{4}{3} \geq \frac{10}{3}$. The total charge on vertices of $\{r\} \cup N_1$ is $3 + 3(1) = 6$. Thus, the sum of all final charges is at least $\frac{10}{3}(n - 4) + 6 = \frac{10}{3}n - \frac{22}{3}$. Thus, $e(G) \geq \frac{5}{3}n - \frac{11}{3}$. $\qquad\square$

4.3 Diameter 2

We now prove that every n-vertex diameter 2 Class 0 graph G has at least $2n - 5$ edges. This bound is best possible. Before proving this result, we describe some graphs where equality holds. In what follows, we show that these are the *only* graphs where equality holds. To begin, we need the following lemma.

Lemma 7. *Given a graph G and a vertex $v \in V(G)$, form G' from G by adding a new vertex, v', with $N(v') = N(v)$. If G is Class 0, then G' is also Class 0.*

Proof. Let G be Class 0, and form G' from G as in the lemma. We show that G' is Class 0. Let p' be a configuration of size $|V(G')|$ on G' and r be a target vertex in G'.

First suppose that $r \notin \{v, v'\}$. We form configuration p for G as follows. Let $p(w) = p'(w)$ for all $w \in V(G) \setminus \{v\}$, and let $p(v) = \max(p'(v) + p'(v') - 1, 0)$. Now $|p| \geq |V(G)|$, so r is reachable from p in G; let σ be a pebbling sequence that reaches r in G. If σ reaches r from p' in G', then we are done. Otherwise, v must make more moves in σ in G from p than are possible in G' from p'. Now all of these "extra" moves from v can be made instead from v' (precisely because $p(v) = p'(v) + p'(v') - 1$). Thus r is reachable in G' from p'.

Suppose instead that $r \in \{v, v'\}$; by symmetry, assume that $r = v$. We may assume that $p'(v) = 0$ and $p'(v') < 4$. If $p'(v') \leq 1$, then we can proceed as in the previous paragraph. So assume that $p'(v') \in \{2, 3\}$. Since G is Class 0, Lemma 5 implies that $d(v) \geq 2$. Choose $u_1, u_2 \in N(v)$. Since $p'(v') \in \{2, 3\}$, we can assume that $p(u_1) = p(u_2) = 0$. We form p for G as follows. Let $p(w) = p'(w)$ for all $w \in V(G) \setminus \{u_1, u_2\}$ and $p(u_1) = p(u_2) = 1$. Now $|p| \geq |V(G)|$, so v is reachable from p in G; let σ be a pebbling sequence that reaches v in G. If σ makes no moves from u_1 or u_2, then σ also reaches v from p' in G'. So assume that σ makes a move from u_1 or u_2. Form σ' from σ by truncating σ just before the first time that it moves from
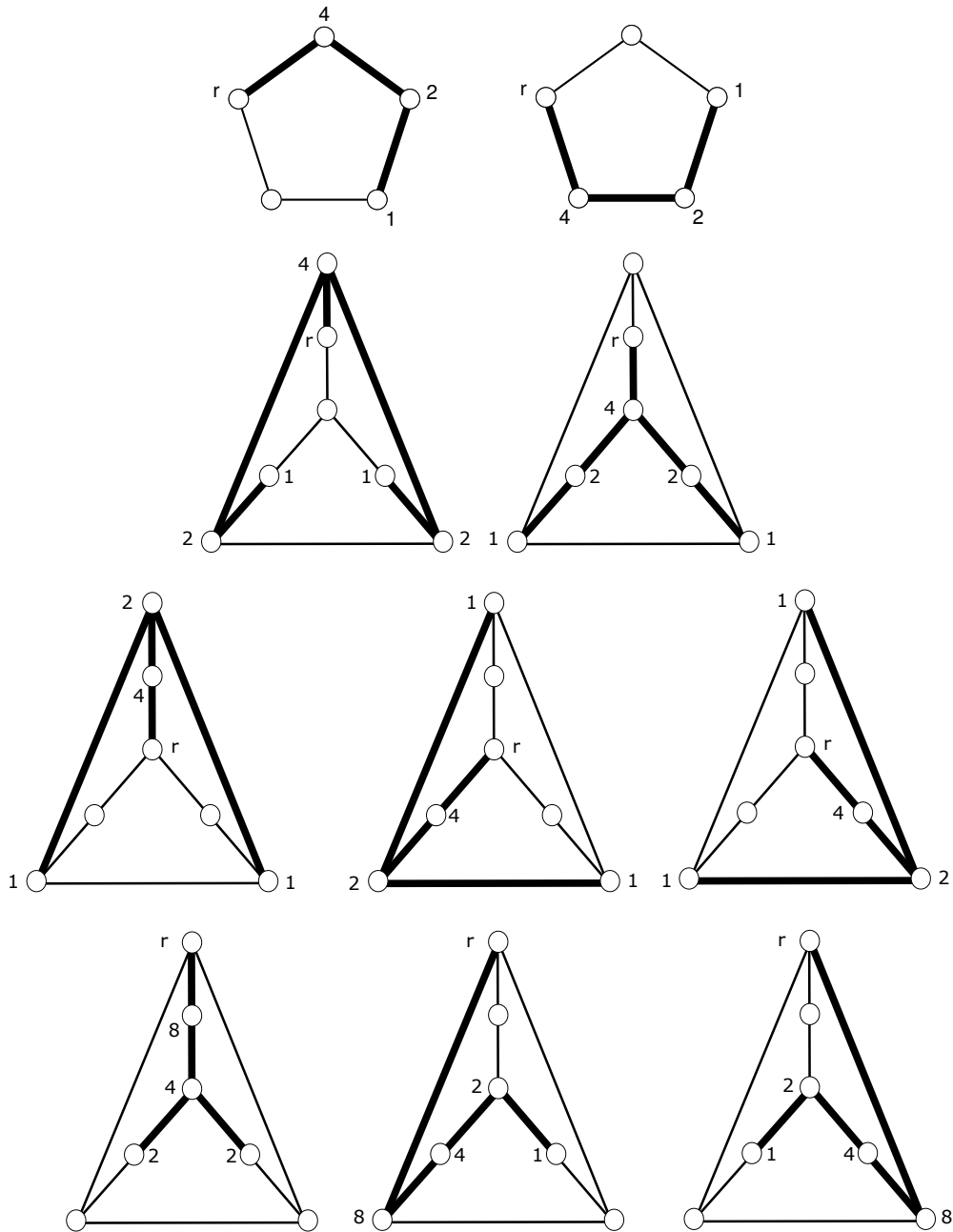
Figure 8: The tree strategies for C_5 and for K_4 with the edges of a $K_{1,3}$ subdivided.

u_1 or u_2, say u_1, and then appending a move from v' to u_1 and a move from u_1 to v. Now σ' reaches v from p' in G. Thus, G' is Class 0. □

Now we use Lemma 7 to show that two infinite families of graphs are all Class 0.

Example 1. The following are two infinite families of Class 0 graphs. Each n-vertex

graph has exactly $2n - 5$ edges. To form an instance of $F_{p,q}$, begin with K_3 and replace the two edges incident to some vertex v with p parallel edges and q parallel edges (where p and q are positive); finally, subdivide each of these $p + q$ new edges. To form an instance of $G_{p,q,r}$, begin with K_4 and replace the three edges incident to some vertex v with p parallel edges, q parallel edges, and r parallel edges (where p, q, and r are positive); finally, subdivide each of these $p + q + r$ new edges.

It is easy to see that each n-vertex graph in $F_{p,q}$ has $2n - 5$ edges, since the 2-vertices induce an independent set (when $p \geq 2$ and $q \geq 2$), and the three high-degree vertices have among them a single edge. Similarly, $G_{p,q,r}$ has $2n - 5$ edges, since the 2-vertices induce an independent set and the four high-degree vertices have among them 3 edges.

We prove that all of $F_{p,q}$ is Class 0, by induction on $p+q$; the induction step follows immediately from Lemma 7. The base case is $F_{1,1}$, which is the 5-cycle. To show that it is Class 0, we use the tree strategies shown in the first row of Figure 8. Since C_5 is vertex-transitive, we can pick the root arbitrarily. Let w be the sum of the weights in the two tree strategies for C_5. Note that $w(v) \geq 3$ for every vertex $v \in V(G) \setminus \{r\}$ and $\sum_{v \in V(G) \setminus \{r\}} w(v) = 14 < 3(4 + 1)$. Thus, by the Covering Lemma, C_5 is Class 0.

We prove that all of $G_{p,q,r}$ is Class 0, by induction on $p + q + r$; the induction step follows immediately from Lemma 7. The base case is $G_{1,1,1}$. To show that $G_{1,1,1}$ is Class 0, we use the tree strategies shown in Figure 8. Up to symmetry, $G_{1,1,1}$ has three types of vertices: a degree 2 vertex, the center degree 3 vertex, and a peripheral degree 3 vertex. The tree strategies for these cases are given in the first, second, and third row below the strategies for C_5.

Let r be a degree 2 vertex, and let $w(v)$ be the sum of the two weight functions in the second row of Figure 8. Note that $w(v) \geq 3$ for all $v \in V(G) \setminus \{r\}$. Further, $\sum_{v \in V(G) \setminus \{r\}} w(v) = 20 < 3(6 + 1)$. Thus, the Covering Lemma implies that $\pi(G_{1,1,1}, r) \leq |V(G_{1,1,1})|$. Now let r be the center vertex, and let $w(v)$ be the sum of the three weight functions in the third row of Figure 8. Note that $w(v) = 4$ for all $v \in V(G) \setminus \{r\}$. Thus, $\sum_{v \in V(G) \setminus \{r\}} w(v) = 24 < 4(6 + 1)$. Thus, the Covering Lemma implies that $\pi(G_{1,1,1}, r) \leq |V(G_{1,1,1})|$. Finally, let r be a peripheral vertex, and let $w(v)$ be the sum of the three weight functions in the fourth row of Figure 8. Note that $w(v) \geq 7$ for all $v \in V(G) \setminus \{r\}$. Further, $\sum_{v \in V(G) \setminus \{r\}} w(v) = 46 < 7(6 + 1)$. Thus, the Covering Lemma implies that $\pi(G_{1,1,1}, r) \leq |V(G_{1,1,1})|$. Since $\pi(G_{1,1,1}, r) \leq |V(G_{1,1,1})|$ for each root r, we conclude that $\pi(G_{1,1,1}) \leq |V(G_{1,1,1})|$. So, $G_{1,1,1}$ is Class 0.

Now we show that every diameter 2 Class 0 graph G has $e(G) \geq 2n - 5$ and characterize when equality holds. Clarke et al. [5, Theorem 2.4] characterized diameter 2 graphs that are not Class 0. It seems likely that we could derive our result from theirs. However, we prefer the proof below, since it seems simpler and more straightforward. Further, the proof below generalizes to diameter 2 graphs with no cut-vertices.

Theorem 5. *Let G be an n-vertex graph with diameter 2. If G has no cut-vertex (in particular, if G is Class 0) then $e(G) \geq 2n - 5$. Further, equality holds if and only if G is the Petersen graph or one of the graphs in Example 1.*

Proof. If $\delta(G) \geq 4$, then $e(G) \geq \frac{4n}{2} = 2n$ and the theorem is true. So we assume $\delta(G) \leq 3$. Lemma 5 implies that $\delta(G) \geq 2$, so $e(G) \geq \frac{n\delta(G)}{2} \geq n$. If $n \leq 5$, then $e(G) \geq n \geq 2n - 5$, so the theorem is true. Thus, we assume $n \geq 6$. We consider two cases: (i) $\delta(G) = 3$ and (ii) $\delta(G) = 2$.

Case 1: $\delta(G) = 3$. Choose $r \in V(G)$ with $d(r) = 3$, and let $S = N(r)$. Each vertex $v \in V(G) \setminus S$ has a neighbor in S and r has 3 neighbors in S, so $\sum_{v \in S} d(v) \geq (n-4) + 3 = n - 1$. Also, $\sum_{v \in V(G) \setminus S} d(v) \geq \sum_{v \in V(G) \setminus S} 3 = 3(n-3)$. So $e(G) = \frac{1}{2} \sum_{v \in V(G)} d(v) \geq \frac{1}{2}((n-1) + 3(n-3)) = \frac{1}{2}(4n - 10) = 2n - 5$.

If equality holds in $e(G) \geq 2n - 5$, then each vertex in $V \setminus S$ has degree 3 and each vertex in $V \setminus (S \cup \{r\})$ has exactly one neighbor in S. Let $\{v_1, v_2, v_3\} = S$, let $S_i = N(v_i) - r$ for each $i \in \{1, 2, 3\}$, and let $H = G[S_1 \cup S_2 \cup S_3]$. Note that H is a disjoint union of cycles, since each vertex has degree 3 and has exactly one neighbor in S. Also $|S_i| \geq 2$ for each i, since $\delta(G) = 3$. Suppose that $|S_1| \geq 3$, and choose $v \in S_3$. Now $|S_1 \cup S_2| \geq 3 + 2 = 5$, so v_3 is distance at least 3 from some vertex of $S_1 \cup S_2$ (precisely because H is a disjoint union of cycles). Hence $|S_1| = 2$ and, by symmetry, $|S_2| = |S_3| = 2$. Similarly, if H consists of two 3-cycles, then some pair of its vertices is distance at least 3 apart. Hence, H is a 6-cycle. Further, each pair of vertices in the same S_i are distance 3 apart in H. Thus, if $e(G) = 2n - 5$, then G is the Petersen graph.

Case 2: $\delta(G) = 2$. Choose $r \in V(G)$ with $d(r) = 2$, and let $\{v_1, v_2\} = N(r)$. We partition $V(G) \setminus N(r)$ into three sets, S_1, S_2, and $S_{1,2}$. (Note that $r \in S_{1,2}$.) Let S_1 consist of all vertices adjacent only to v_1, S_2 of all vertices adjacent only to v_2, and $S_{1,2}$ of all vertices adjacent to both v_1 and v_2. Let H be the subgraph induced by $S_1 \cup S_2$, and let $H_1, \ldots, H_t$ be the components of H. We first show that $e(G) \geq 2n - 4$ if every H_i either contains a cycle or has a vertex adjacent to some vertex of $S_{1,2}$.

We assign each edge to one of its endpoints as follows, so that each vertex other than v_1 and v_2 has at least 2 assigned edges. Each edge with exactly one endpoint in $\{v_1, v_2\}$ is assigned to its other endpoint. Thus, each vertex of $S_{1,2}$ has 2 assigned edges and each vertex of $S_1 \cup S_2$ has 1 assigned edge. Suppose that T is some tree component of H and t is a vertex of T with a neighbor in $S_{1,2}$. We can direct the edges of T so that t has outdegree 0 and each other vertex has outdegree 1. Now we assign to t its edge to $S_{1,2}$ and assign to each other vertex of T its out-edge. When H_i is a component of H with a cycle, the process is similar. We choose some spanning tree T of H_i and choose t to be some vertex incident to an edge of H_i not in T.

So assume that some component H_1 is a tree and has no neighbor in $S_{1,2}$. If $V(H_1) \subseteq S_1$, then v_1 is a cut-vertex, which is forbidden. Hence, $V(H_1) \not\subseteq S_1$; similarly, $V(H_1) \not\subseteq S_2$. Suppose that H has another component, with some vertex w. By symmetry, assume that $w \in S_1$. Since H_1 has vertices in both S_1 and S_2, w is distance at least 3 from some vertex of H_1 in S_2, a contradiction. Thus, H_1 is the only component of H; so from now on, we say H for H_1. Choose t arbitrarily in H, and direct $E(H)$ and assign edges as above. Now t has 1 assigned edge and each other vertex has 2 assigned edges, so $e(G) \geq 2n - 5$. If $v_1 v_2 \in E(G)$, then $e(G) \geq 2n - 4$, so we assume $v_1 v_2 \notin E(G)$. Similarly, if any edge has both endpoints in $S_{1,2}$, then

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$e(G) \geq 2n - 4$, so we assume that $S_{1,2}$ induces an independent set. In what follows, we characterize when equality holds in $e(G) \geq 2n - 5$.

First suppose that H has leaves in both S_1 and S_2; call these u_1 and u_2, respectively. If u_1 and u_2 are adjacent, then H is a single edge, which is possible; this is $F_{1,q}$, where $q = |S_{1,2}|$. Now suppose that u_1 and u_2 are nonadjacent. Since G is diameter 2, u_1 and u_2 have some common neighbor, u_3. By symmetry, assume that $u_3 \in S_1$. Now u_1 and v_2 must have a common neighbor, so $v_1 v_2$ is an edge. However, now $e(G) \geq 2n - 4$.

So assume instead that H has leaves only in one of S_1 and S_2; by symmetry, say S_2. Let S_1' denote the vertices of S_1 adjacent to a leaf. Now S_1' induces a graph with diameter at most 1 (otherwise some leaf is distance at least 3 from some vertex of S_1'). Since H is acyclic, $|S_1'| \leq 2$.

First suppose that $|S_1'| = 1$. Let $\{u_1\} = S_1'$. Now all vertices in S_2 are leaves of H, since H is acyclic. Further, u_1 is the only vertex in S_1. Thus, H is a star centered at u_1. This is possible; $G = F_{p,q}$, where $q = |S_{1,2}|$ and p is the number of leaves of H.

Suppose instead that $|S_1'| = 2$, and let $\{u_1, u_2\} = S_1'$. Now u_1 and u_2 are adjacent, since G has diameter 2; otherwise some leaf in S_2 is distance at least 3 from u_1 or u_2. Again, each vertex $u_3 \in S_2$ must be a leaf, since G is acyclic. Finally, $S_1 = \{u_1, u_2\}$, again since G has diameter 2. Thus, H is a double star, centered at u_1 and u_2, with all leaves in S_2. This is also possible; $G = G_{p,q,r}$, where $p = |S_{1,2}|$ and q and r are (respectively) the numbers of leaves of H adjacent to u_1 and u_2.

Hence, $e(G) = 2n - 5$ implies that H is (i) a single edge, which is $F_{1,q}$, (b) a star with its center in S_1 (by symmetry) and all of its leaves in S_2, which is $F_{p,q}$, or (c) a double star with both of its centers in S_1 and all of its leaves in S_2, which is $G_{p,q,r}$. This finishes the characterization of when $e(G) = 2n - 5$. $\qquad\square$

References

[1] A. Blasiak, A. Czygrinow, A. Fu, D. Herscovici, G. Hurlbert and J.R. Schmitt. Sparse graphs with small pebbling number. Manuscript, (2012).

[2] B. Bukh. Maximum pebbling number of graphs of diameter three. *J. Graph Theory* **52** (2006), 353–357.

[3] F. Chung. Pebbling in hypercubes. *SIAM J. Discrete Math.* **2** (1989), 467–472.

[4] B. Clark and K. Milans. The complexity of graph pebbling, *SIAM J. Discrete Math.* **20** (2006), 769–798.

[5] T.A. Clarke, R.A. Hochberg, and G.H. Hurlbert. Pebbling in diameter two graphs and products of paths. *J. Graph Theory* **25** (1997), 119–128.

[6] P. Erdős, A. Ginzburg, and A. Ziv. A theorem in additive number theory. *Bull. Res. Council Isreal* **10F** (1961) 41–43.

[7] R. Feng and J.Y. Kim. Graham's pebbling conjecture on product of complete bipartite graphs. *Sci. China Ser. A* **44** (2001), 817–822.

[8] R. Feng and J.Y. Kim. Pebbling numbers of some graphs. *Sci. China Ser. A* **45** (2002), 470–478.

[9] D. Herscovici. Graham's pebbling conjecture on products of cycles. *J. Graph Theory* **42** (2003), 141–154.

[10] G. Hurlbert. A linear optimization technique for graph pebbling. Preprint, available at: `http://arxiv.org/abs/1101.5641`, (2011).

[11] G. Hurlbert. A survey of graph pebbling. *Congr. Numer.* **139** (1999), 41–64.

[12] G. Hurlbert. Recent progress in graph pebbling. Graph Theory Notes of New York, **XLIX**, (2005), 25–37.

[13] P. Lemke and D. Kleitman. An addition theorem on the integers modulo n. *J. Number Theory* **31** (1989), 335–345.

[14] D. Moews. Pebbling graphs, *J. Combin. Th. Ser. B* **55** (1992), 244–252.

[15] L. Pachter, H.S. Snevily, and B. Voxman. On pebbling graphs. *Proccedings of the Twenty-Sixth Southeastern International Conference on Combinatorics, Graph Theory and Computing* (Boca Raton, FL, 1995), **107** (1995), 65–80.

[16] L. Postle. Pebbling graphs of fixed diameter. *J. Graph Theory* **75** (2014) 302–310.

[17] L. Postle, N. Streib, and C. Yerger. Pebbling graphs of diameter three and four. *J. Graph Theory* **72** (2013) 398–417.

[18] N. Watson. The complexity of pebbling and cover pebbling. Preprint, available at: `http://arxiv.org/abs/math/0503511`, (2005).