

# Sparse graphs are near-bipartite

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## Abstract

A multigraph  $G$  is near-bipartite if  $V(G)$  can be partitioned as  $I, F$  such that  $I$  is an independent set and  $F$  induces a forest. We prove that a multigraph  $G$  is near-bipartite when  $3|W| - 2|E(G[W])| \geq -1$  for every  $W \subseteq V(G)$ , and  $G$  contains no  $K_4$  and no Moser spindle. We prove that a simple graph  $G$  is near-bipartite when  $8|W| - 5|E(G[W])| \geq -4$  for every  $W \subseteq V(G)$ , and  $G$  contains no subgraph from some finite family  $\mathcal{H}$ . We also construct infinite families to show that both results are best possible in a very sharp sense.

## 1 Introduction

A multigraph<sup>1</sup>  $G$  is *near-bipartite* if its vertex set can be partitioned into sets  $I$  and  $F$  such that  $I$  is an independent set and  $F$  induces a forest. This condition is somewhat stronger than being 3-colorable, but the two problems are closely related. We call  $I, F$  a *near-bipartite coloring* of  $G$ , or simply an *nb-coloring*. The goal of this paper is to prove sufficient conditions for multigraphs and simple graphs to be near-bipartite, in terms of their edge-densities; this is akin to the work done for  $k$ -coloring in [17]. Since a near-bipartite coloring of  $G$  restricts to a near-bipartite coloring of each subgraph  $J$  of  $G$ , naturally our edge-density hypothesis for  $G$  should also hold for each subgraph  $J$ . To facilitate a proof by induction, we also allow some vertices to be precolored. That is, we allow vertex subsets  $I_p$  and  $F_p$  such that our near-bipartite coloring  $I, F$  must have  $I_p \subseteq I$  and  $F_p \subseteq F$ . For convenience, let  $U_p = V(G) \setminus (I_p \cup F_p)$ . We prove results for both the class of multigraphs and the class of simple graphs. For simple graphs, to facilitate our proof by induction, we allow some edges to be specified as edge-gadgets. In practice this means that, for each edge-gadget  $vw$ , in every near-bipartite coloring one of  $v$  and  $w$  appears in  $I$  and the other appears in  $F$ ; intuitively, this is the same as if  $vw$  was a multiedge. For a multigraph  $G$  and  $W \subseteq V(G)$ , let  $e(W)$  denote the set of edges with both endpoints in  $W$ . For a simple graph, we let  $e''(W)$  and  $e'(W)$  denote the subsets of  $e(W)$  that are, respectively, edge-gadgets and not edge-gadgets (but still edges). Most of our other terminology and notation is standard, but for reference we collect it in Section 2.4. Now we can define our measures of edge-density, called *potential*, and denoted  $\rho_{m,G}$  and  $\rho_{s,G}$ . (Here  $m$  is for multigraph and  $s$  is for simple graph.)

For a multigraph  $G$  with precoloring  $I_p, F_p$ , for each  $W \subseteq V(G)$  let

$$\rho_{m,G}(W) = 3|W \cap U_p| + |W \cap F_p| - 2|e(W)|$$

and

$$\rho_{s,G}(W) = 8|W \cap U_p| + 3|W \cap F_p| - 5|e'(W)| - 11|e''(W)|.$$

Let  $M_7$  denote the Moser spindle, shown in Figure 1, and let  $\mathcal{H}$  be a finite family of simple graphs that we define in Section 3, none of which is near-bipartite. The following is the main result of this paper.

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<sup>1</sup>Without loss of generality, we assume that each edge has multiplicity at most 2, as we explain at the start of Section 2.

**Main Theorem.** (A) If  $G$  is a multigraph with precoloring  $I_p, F_p$  such that  $\rho_{m,G}(W) \geq -1$  for all  $W \subseteq V(G)$  and  $G$  does not contain  $K_4$  or  $M_7$  as a subgraph, then  $G$  has a near-bipartite coloring  $I, F$  that extends the precoloring  $I_p, F_p$ . Moreover,  $I, F$  can be found in polynomial time.

(B) If  $G$  is a simple graph with precoloring  $I_p, F_p$  such that  $\rho_{s,G}(W) \geq -4$  for all  $W \subseteq V(G)$  and  $G$  does not contain any graph from  $\mathcal{H}$  as a subgraph, then  $G$  has a near-bipartite coloring  $I, F$  that extends the precoloring  $I_p, F_p$ . Moreover,  $I, F$  can be found in polynomial time.

It is NP-complete to decide if a graph is near-bipartite<sup>2</sup>, and this is attributed to Monien [10]. This problem remains NP-complete for several restricted families of graphs. Brandstädt, Brito, Klein, Nogueira, and Protti [9] showed this for perfect graphs, and Bonamy, Dabrowski, Feghali, Johnson, and Paulusma [4] showed it for graphs with diameter 3. Dross, Montassier, and Pinlou [13] showed it for planar graphs, and Yang and Yuan [22] showed it for graphs with maximum degree 4. In contrast, Bonamy, Dabrowski, Feghali, Johnson, and Paulusma [5] showed that for a simple graph  $G$  with  $\Delta(G) \leq 3$  and with no  $K_4$ , an nb-coloring (which exists by the results below) can be found in time  $O(|V(G)|)$ .

Borodin and Glebov [7] proved that if  $G$  is planar with girth at least 5, then  $G$  is near-bipartite. Kawarabayashi and Thomassen [16] extended this result to allow a small set of precolored vertices. Dross, Montassier, and Pinlou [13] conjectured that every planar graph with girth at least 4 is near-bipartite (which would strengthen the result of [7]). Because they each considered different generalizations, multiple groups [3, 6, 8, 11, 22] proved that if  $G$  has no  $K_4$  as a subgraph and  $\Delta(G) \leq 3$ , then  $G$  is near-bipartite. Yang and Yuan [22] characterized near-bipartite graphs with diameter 2. Zaker [24, Theorem 4] proved that  $G$  is near-bipartite if and only if its vertices can be ordered as  $v_1, v_2, \dots, v_n$  such that each triple of edges with a common endpoint  $v_i v_{j_1}, v_i v_{j_2}, v_i v_{j_3}$  does not satisfy  $j_1 < i < j_2 \leq j_3$ .

Finding an nb-coloring  $I, F$  is also called “finding a stable cycle cover” [9]. When we want  $I$  to have bounded size, the problem is called finding an “independent feedback vertex set”, and related work is described in the references of [4].

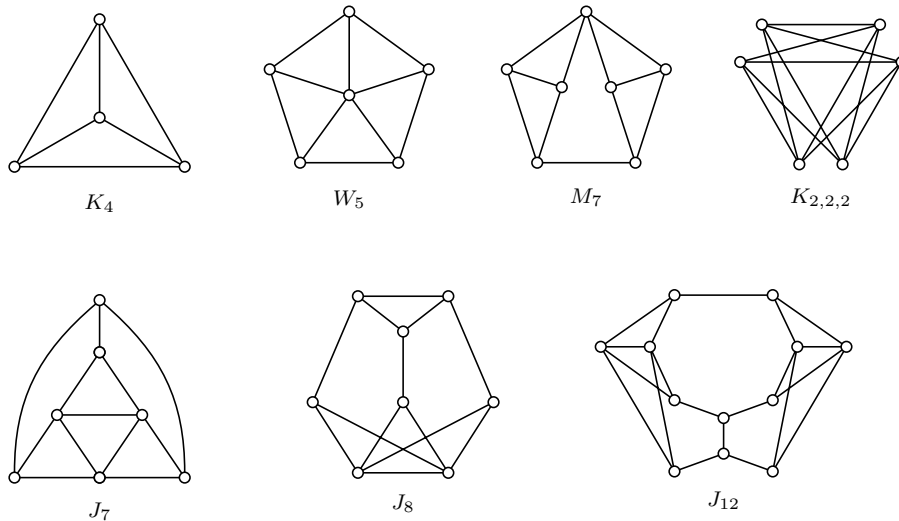


Figure 1: Examples of nb-critical graphs. The graph  $W_5$  is called the 5-wheel and  $M_7$  is called the Moser spindle. All graphs shown are 4-critical, except for  $K_{2,2,2}$ , which is 3-colorable.

The purpose of this paper is to give an algorithm for finding a near-bipartite coloring when  $G$  is sufficiently sparse. This motivates the following definitions. A multigraph is *nb-critical* if it is not near-bipartite, but every proper subgraph is near-bipartite. Figure 1 shows examples of nb-critical graphs. A multigraph  $G$  is  $(a, b)$ -sparse if every nonempty subset of vertices  $W$  satisfies  $|e(W)| \leq a|W| - b$ . A graph is a forest if and only if it is  $(1, 1)$ -sparse. A vertex set  $I$  is independent if and only if  $G[I]$  is  $(0, 0)$ -sparse. Our next two

nb-critical  
(a, b)-  
sparse

<sup>2</sup>This is unsurprising, since nb-coloring is closely connected with 3-coloring, a well-known NP-complete problem.

theorems rephrase parts (A) and (B) of the Main Theorem, state explicit bounds on the running times of algorithms to find the colorings, and also mention constructions to show that both parts are very sharp. We give these constructions in Section 3. In Section 2.3 we describe a key subroutine of our coloring algorithm, but we defer presenting the algorithm in full until Section 5, when we have proved the Main Theorem.

**Theorem 1.1.** *There exists an infinite family of  $(1.5, -1)$ -sparse nb-critical multigraphs. If  $G$  is  $(1.5, -0.5)$ -sparse and has no  $K_4$  and no  $M_7$ , then  $G$  is near-bipartite. We can find an nb-coloring in time  $O(|V(G)|^6)$ .*

A graph  $G$  is 2-degenerate if every nonempty subgraph  $J$  satisfies  $\delta(J) \leq 2$ . Every 2-degenerate graph is near-bipartite, and we can find an nb-coloring in time  $O(|V(G)|)$  using the obvious greedy algorithm. Graphs that are  $(1.5, 0.5)$ -sparse are 2-degenerate, so Theorem 1.1 shows that the greedy algorithm is sufficient in many of the cases where sparsity implies a graph is near-bipartite. Our more impressive result is that we can do better when  $G$  is simple.

**Theorem 1.2.** *There exists an infinite family of  $(1.6, -1)$ -sparse nb-critical simple graphs. There exists a finite graph family  $\mathcal{H}$  such that if  $G$  is a simple graph that is  $(1.6, -0.8)$ -sparse and contains no subgraph isomorphic to a graph in  $\mathcal{H}$ , then  $G$  is near-bipartite. We can find an nb-coloring in time  $O(|V(G)|^{22})$ .*

The most striking aspect of Theorem 1.2 is that we handle the family  $\mathcal{H}$ , which has hundreds of forbidden subgraphs. Each graph in  $\mathcal{H}$  is both nb-critical (and so must be forbidden in such a theorem) and also 4-critical<sup>3</sup>. Although we have not explicitly constructed all graphs in  $\mathcal{H}$ , its recursive definition in Section 3.2 allows us to show that each of these graphs has at most 22 vertices; so  $\mathcal{H}$  is finite. Kostochka and the second author [17] showed that each  $n$ -vertex 4-critical graph  $G$  has  $|E(G)| \geq (5n - 2)/3$ . As we show in Theorem 1.2, each  $n$ -vertex nb-critical graph with  $n \geq 22$  has  $|E(G)| \geq (8n + 4)/5$ . Intuitively, the family  $\mathcal{H}$  is due to the fact that  $(5n - 2)/3 < (8n + 4)/5$  when  $n < 22$ .

Although  $\mathcal{H}$  is finite, it is a natural subset of an infinite family  $\mathcal{H}'$ , and each graph of  $\mathcal{H}'$  is also both nb-critical and 4-critical. Thus, our description of  $\mathcal{H}'$  provides insight into the structure of sparse nb-critical and 4-critical graphs. In view of  $\mathcal{H}'$ , it is natural to ask whether nb-criticality implies 4-criticality, or vice versa. But neither implication is true. In Section 2.1 we construct an infinite family of nb-critical graphs  $H_k$  that are 3-colorable (so not 4-critical). There also exist infinitely many 4-critical graphs such that even after removing multiple (specified) edges from any one of these, it does not become near bipartite<sup>4</sup>.

## 1.1 Proof Outline

To conclude this introduction, we outline the proof of the Main Theorem. The proofs of parts (A) and (B) are similar, but (B) is harder because the family  $\mathcal{H}$  of forbidden subgraphs is much larger. Thus, we just outline the proof of (B).

*(Proof sketch of Main Theorem (B)).* Our proof has three cases. The first two cases use induction on  $|V(G)|$ , and the third case simply constructs an explicit nb-coloring.

**Case 1: There exists  $W \subset V(G)$  with  $2 \leq |W| \leq |V(G)| - 2$  and  $\rho_{s,G}(W) \leq 3$ .** By induction,  $G[W]$  has an nb-coloring  $I_W, F_W$ . We form a new graph  $G'$  from  $G$  by coloring  $G[W]$  with  $I_W, F_W$ , and then identifying each vertex in  $W$  colored  $I$  and identifying each vertex in  $W$  colored  $F$ . We call these new vertices  $w_i$  and  $w_f$ , and they retain their colors. It is easy to check that every nb-coloring of  $G'$  extends to an nb-coloring of  $G$  (by coloring  $G[W]$  with  $I_W, F_W$ ). So the key step is showing that  $G'$  satisfies the hypotheses of the Main Theorem.

Suppose that  $G'$  contains a subset  $W'$  such that  $\rho_{s,G'}(W') \leq -5$ . We can check that also  $\rho_{s,G}(W' \setminus \{w_i, w_f\} \cup W) \leq -5$ , a contradiction. That is, “uncontracting” the set  $W'$  with potential too small in  $G'$  gives a set with potential too small in  $G$ , which contradicts our hypothesis. So suppose instead that  $G'$  contains a subgraph  $H'$  that is forbidden; that is  $H' \in \mathcal{H}$ . If  $H' \notin \{K_4, M_7\}$ , then Corollary 3.8(iii) implies

<sup>3</sup>A graph is 4-critical if it is not 3-colorable, but each of its proper subgraphs is 3-colorable.

<sup>4</sup>For example, we can start with the 4-critical graphs  $G_k$  constructed by Yao and Zhou in [23]. Even if we remove all edges  $x_1 u_i$  and  $y_1 v_j$  with  $4 \leq i, j \leq 2k - 5$  the graph fails to become near-bipartite. The proof of this is a straightforward case analysis (considering the nb-colorings of  $H_{2k}$  and of  $G[\{x_1, x_2, x_3, y_1\}]$ ), but the details are too numerous to include here.

that  $\rho_{s,H'}(V(H')) \leq 0$ , which yields  $\rho_{s,G}((V(H') \setminus \{w_i, w_f\}) \cup W) \leq -5$ , a contradiction. If  $H' \in \{K_4, M_7\}$ , then a short case analysis again reaches a contradiction.

**Case 2:  $G$  contains some “reducible configuration” (and Case 1 does not apply).** Since Case 1 does not apply, we know that  $\rho_{s,G}(W) \geq 4$  for all  $W \subseteq V(G)$  with  $2 \leq |W| \leq |V(G)| - 2$ . We call this inequality our “gap lemma”, since it implies a gap between the lower bound on  $\rho_{s,G}$  required by the hypothesis (–4) and the actual value of  $\rho_{s,G}$  (at least 4). A reducible configuration is one that allows us to proceed by induction. An easy example is an uncolored vertex  $v$  of degree at most 2. By induction,  $G - v$  has an nb-coloring  $I', F'$ . To extend this coloring to  $G$ , we color  $v$  with  $F$  unless all of its neighbors are colored  $F$ ; in that case we color  $v$  with  $I$ . Our gap lemma has the following powerful consequence: For any  $W \subsetneq V(G)$  and any  $w \in W$  that is uncolored, we can color  $G[W]$  with  $w$  colored  $I$  and we can also color  $G[W]$  with  $w$  colored  $F$ . This is because precoloring a vertex decreases its potential (and that of any set containing it) by at most 8. So the gap lemma implies that each vertex subset (containing the precolored vertex  $w$ ) has potential at least  $4 - 8 = -4$ . Thus, the Main Theorem still applies, even after precoloring  $w$ .

Let  $L$  denote the set of degree 3 vertices that are uncolored and not incident to any edge-gadget. We claim that  $G[L]$  is a forest. Suppose, to the contrary, that  $G[L]$  contains a cycle  $C$ . Since  $G$  contains no subgraph in  $\mathcal{H}$ , cycle  $C$  has successive vertices  $v_1$  and  $v_2$  such that their neighbors outside of  $C$ , say  $z_1$  and  $z_2$  are not linked (this is a technical term defined when constructing the family of forbidden subgraphs; it means that adding the edge  $z_1 z_2$  would create a copy of a subgraph in  $\mathcal{H}$ ). Now we form a new graph  $G(C, z_1, z_2)$  from  $G$  by deleting  $V(C)$  and adding edge  $z_1 z_2$ ; if  $z_1 z_2$  already exists, then we replace it with an edge-gadget. Since  $z_1$  and  $z_2$  are not linked,  $G(C, z_1, z_2)$  satisfies the hypotheses of the Main Theorem. It is straightforward to check that every nb-coloring of  $G(C, z_1, z_2)$  extends to an nb-coloring of  $G$ .

**Case 3: Neither Case 1 nor Case 2 applies.** We use discharging to show that  $G$  is very nearly an uncolored graph with no edge-gadgets and consists of an independent set of vertices of degree 4 and a set of vertices of degree 3 that induces a forest. In this case, we can color the independent set with  $I$  and color the forest with  $F$ . If  $G$  exactly matches this description, then  $\rho_{s,G}(V(G)) = -\ell$ , where  $\ell$  is the number of components in the forest. Further, each place in the graph that differs from this description slightly decreases  $\rho_{s,G}(V(G))$ . By hypothesis,  $\rho_{s,G}(V(G)) \geq -4$ , so this number of differences is small (as is  $\ell$ ). In each case, we explicitly construct an nb-coloring of  $G$ .  $\square$

In Section 5 we translate the proof of our Main Theorem into a polynomial-time algorithm. Implementing most of the steps is straightforward. But two parts of this process merit more comment. In Section 2.3, we show how to find a vertex subset  $W$  with minimum potential; we can also further require that  $|W|$  be at least some constant distance away from 0 or from  $|V(G)|$ . This task reduces to a series of max-flow/min-cut problems, each of which runs in time  $O(|V(G)|^3 \log |V(G)|)$ . Finally, to check whether two vertices are linked, we simply use brute force. This relies on the fact that each graph in  $\mathcal{H}$  has at most 22 vertices, so  $\mathcal{H}$  has only finitely many graphs. Thus we can answer this question in time  $O(|V(G)|^{20})$ .

## 2 Preliminaries

In Section 2.1 we construct the sharpness examples promised in Theorems 1.1 and 1.2. In Section 2.2 we motivate our choice of coefficients in the definitions of  $\rho_{m,G}$  and  $\rho_{s,G}$ , and record for reference the potentials of many small graphs. Section 2.3 presents an algorithm for finding a vertex subset with lowest potential; this will be useful in Section 5, where we convert our proofs that certain graphs have nb-colorings into algorithms to construct those nb-colorings. Finally, Section 2.4 collects all of our definitions, most of which are standard. To simplify our notation throughout, we assume that any sets  $I$  and  $F$  are disjoint. This assumption is free, since induced subgraphs of forests are forests. We also assume that each pair of vertices is joined by at most two edges, since allowing further parallel edges puts no further constraints on the coloring.

## 2.1 Sparse nb-critical Graphs

Here we describe the sharpness examples in Theorems 1.1 and 1.2. For each  $k \geq 1$ , we construct a family of graphs  $G_k$  as follows. The top of Figure 2 shows  $G_3$ . Let  $V(G_k) = \{a, b, v_1, \dots, v_{2k}, c, d\}$  and

$G_k$

$$E(G_k) = \{ab, ab, av_1, bv_1, v_{2k}c, v_{2k}d, cd, cd\} \cup \{v_1v_2, v_2v_3, \dots, v_{2k-1}v_{2k}\} \cup \{v_1v_2, v_3v_4, \dots, v_{2k-1}v_{2k}\}.$$

To check that each  $G_k$  is  $(1.5, -1)$ -sparse, we use induction on  $k$ , as follows. Fix  $W \subseteq V(G_k)$ . Suppose that  $W$  contains  $v_i, v_{i+1}$ , for some  $i \leq 2k-2$ . Let  $W' = W/\{v_i, v_{i+1}, v_{i+2}\}$  and  $G'_k = G_k/\{v_i, v_{i+1}, v_{i+2}\}$ ; here “/” denotes contraction. Note that  $G'_k \cong G_{k-1}$ . By hypothesis,  $e(W') \leq 1.5|W'| + 1$ . Thus  $e(W) \leq e(W') + 3 \leq 1.5|W'| + 3 + 1 = 1.5|W| + 1$ . The case when no such  $i$  exists is straightforward, as is the base case. So  $G_k$  is  $(1.5, -1)$ -sparse, as desired.

We claim that each  $G_k$  is nb-critical. To begin we show that  $G_k$  is not near-bipartite. The key observation, which is easy to check, is that when  $I, F$  is an nb-coloring of  $G_k$

$$\text{if } vw \text{ is a multiedge, then } |I \cap \{v, w\}| = |F \cap \{v, w\}| = 1. \quad (1)$$

Assume, contrary to our claim, that  $G$  has an nb-coloring  $I, F$ . Applying (1) to multiedge  $ab$  shows that  $|\{a, b\} \cap I| = 1$ , which implies  $v_1 \in F$ . Similarly,  $|\{c, d\} \cap I| = 1$ , so  $v_{2k} \in F$ . We prove by induction that  $v_{2i-1} \in F$  for all  $i$ , which contradicts (1) for multiedge  $v_{2k-1}v_{2k}$ . Assume, by hypothesis, that  $v_{2i-3} \in F$ . (The base case is when  $i = 2$ .) Applying (1) to  $v_{2i-3}v_{2i-2}$  shows that  $v_{2i-2} \in I$ ; this, in turn, means that  $v_{2i-1} \in F$ , as desired. So  $v_{2k-1}v_{2k} \in F$ , which is a contradiction. Thus,  $G_k$  is not near-bipartite.

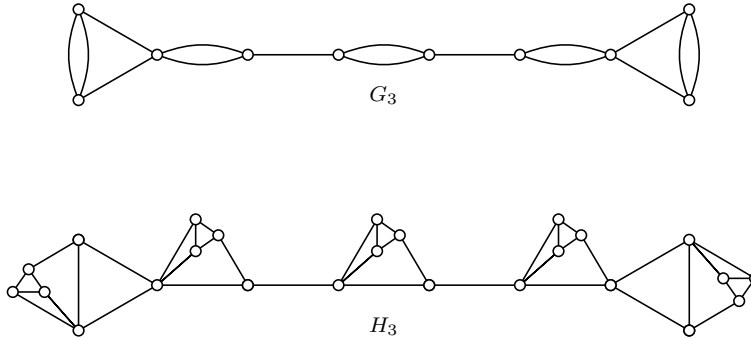


Figure 2: Two examples of nb-critical graphs. Top:  $G_3$  is a multigraph, with vertices  $a, b, v_1, \dots, v_6, c, d$  in order from left to right. Bottom:  $H_3$  is formed from  $G_3$  by replacing each pair of parallel edges by a multiedge-replacement.

To see that each subgraph  $G_k - v_i v_{i+1}$  is near-bipartite, we color greedily in the order  $\{a, b, v_1, \dots, v_i, d, c, v_{2k}, \dots, v_{i+1}\}$ , adding each vertex to any set where it does not contradict the definition of  $I, F$ -coloring. For each other edge  $e$ , we can color  $G - e$  similarly. This completes the proof that each  $G_k$  is nb-critical.

We now construct a family  $H_k$  of simple nb-critical graphs. The bottom of Figure 2 shows  $H_3$ . To do this, we define a *multiedge-replacement* for endpoints  $a, b$  as vertices  $x_{ab}, y_{ab}, z_{ab}$  and edges  $ab, ax_{ab}, ay_{ab}, x_{ab}y_{ab}, x_{ab}z_{ab}, y_{ab}z_{ab}, z_{ab}b$ . We say it is *rooted* at  $a$  and  $b$  and that they are its roots. As an example of an multiedge-replacement, consider the 5 leftmost (or 5 rightmost) vertices in  $H_3$  and the edges they induce, as shown on the bottom in Figure 2. To construct  $H_k$  we replace each multiedge of  $G_k$  with a multiedge-replacement. (These multiedge-replacements allow us to simulate multiedges in simple graphs.) It is straightforward to show by induction on  $k$  that each  $H_k$  is  $(1.6, -1)$ -sparse.

$H_k$   
multiedge-  
replacement

The proof that  $H_k$  is nb-critical follows from the proof that  $G_k$  is nb-critical, together with the fact (proved below) that in any nb-coloring  $I, F$  of a multiedge-replacement,

$$\text{if the multiedge-replacement is rooted and } v \text{ and } w, \text{ then } |I \cap \{v, w\}| = |F \cap \{v, w\}| = 1. \quad (2)$$

We also need the observation that removing any edge from a multiedge-replacement allows an nb-coloring with both roots colored  $F$ ; this is easy to check directly. This observation implies that every proper subgraph of  $H_k$  is near-bipartite.

We now prove (2). If  $z_{vw} \in I$ , then  $\{w, x_{vw}, y_{vw}\} \subseteq F$ . So the circuit  $v, x_{vw}, y_{vw}$  implies that  $v \in I$ , and (2) holds. If instead  $z_{vw} \in F$ , then the circuit  $x_{vw}, y_{vw}, z_{vw}$  forces  $\{x_{vw}, y_{vw}\} \not\subseteq F$ ; by symmetry, assume  $x_{vw} \in F$  and  $y_{vw} \in I$ . Thus  $v \in F$ . But now the circuit  $v, w, z_{vw}, x_{vw}$  forces  $w \in I$ . Again, (2) holds. This completes the proof of (2). So  $H_k$  has no nb-coloring precisely because  $G_k$  has no nb-coloring. Thus,  $H_k$  is nb-critical.

## 2.2 Potential Functions

Recall from the introduction that

$$\rho_{m,G}(W) = 3|W \cap U_p| + |W \cap F_p| - 2|e(W)|$$

and

$$\rho_{s,G}(W) = 8|W \cap U_p| + 3|W \cap F_p| - 5|e'(W)| - 11|e''(W)|.$$

Our choice of coefficients in  $\rho_{m,G}$  and  $\rho_{s,G}$  has a simple explanation based on the constructions in the previous section. We begin with  $\rho_{m,G}$ . The ratio  $3/2$  of the coefficients on  $|W \cap U_p|$  and  $e(W)$  arises because  $\lim_{k \rightarrow \infty} |E(G_k)|/|V(G_k)| = 3/2$ . To understand the coefficient 1 on  $|W \cap F_p|$ , consider an arbitrary vertex  $w \in U_p$ . We create vertices  $y_w, y'_w \in U_p$  and add edges  $wy_w, wy'_w, y_wy'_w, y_wy'_w$ ; see left of Figure 3. Because  $y_wy'_w$  is a multiedge, every nb-coloring  $I, F$  of this graph must have  $|I \cap \{y_w, y'_w\}| = 1$ , so  $w \in F$ . Thus, functionally speaking, this construction is equivalent to moving  $w$  from  $U_p$  to  $F_p$ . The weight of  $w$  in  $F_p$  represents the combined contribution to  $\rho_{m,G}$  of  $w, y_w, y'_w$ , and the associated edges: the 3 vertices and 4 edges give us  $3(3) - 2(4) = 1$ .

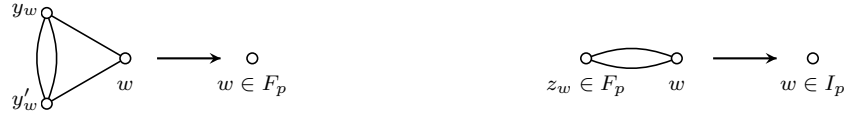


Figure 3: Constructions to require  $w \in F_p$  (left) and  $w \in I_p$  (right).

To understand the coefficient 0 on  $|W \cap I_p|$ , consider an arbitrary vertex  $w \in U_p$ , and create vertex  $z_w \in F_p$  and add edges  $wz_w, wz_w$ ; see right of Figure 3. By construction, every nb-coloring  $I, F$  of this graph must have  $w \in I$ , and so we have mimicked moving  $w$  from  $U_p$  to  $I_p$ . The weight of  $w$  in  $I_p$  represents the combined contribution of  $w, z_w$ , and the two associated edges:  $3 + 1 - 2(2) = 0$ .

To double-check that our coefficients make sense, suppose we want to move a vertex  $v$  from  $U_p$  to  $F_p$ . We can also achieve this by adding a vertex  $w \in I_p$  and adding edge  $vw$ . Functionally, now  $v \in F_p$ , so combining the weights of  $v, w$ , and  $vw$  should give the weight of a single vertex in  $F_p$ , and it does:  $3 + 0 - 2 = 1$ .

Similarly, we can analyze the coefficients of  $\rho_{s,G}$ . Note that  $\lim_{k \rightarrow \infty} |E(H_k)|/|V(H_k)| = 8/5$ . To compute the weight of an edge-gadget, we have  $8(3) - 5(7) = -11$ , since it is simulated by a multiedge-replacement. To effectively move a vertex from  $U_p$  to  $F_p$  or  $I_p$ , we use the same method as above, but with edge-gadgets in place of multiedges. For a vertex in  $F_p$  we count the contributions of 3 vertices, 2 edges, and one additional edge-gadget to get  $8(3) - 2(5) - 11 = 3$ . For a vertex in  $I_p$  we count contributions of one vertex in  $F_p$ , one vertex in  $U_p$ , and one edge-gadget to get  $3 + 8 - 11 = 0$ .

**Example 2.1.** We calculate the potential for several examples (assuming that no vertices are precolored).

- (i)  $\rho_{m,K_k}(V(K_k)) = 3k - 2\binom{k}{2} = 4k - k^2$  and  $\rho_{s,K_k}(V(K_k)) = 8k - 5\binom{k}{2} = \frac{21}{2}k - \frac{5}{2}k^2$ .
- (ii)  $\rho_{m,W_5}(V(W_5)) = 3(6) - 2(10) = -2$  and  $\rho_{s,W_5}(V(W_5)) = 8(6) - 5(10) = -2$ .
- (iii)  $\rho_{m,K_{2,2,2}}(V(K_{2,2,2})) = 3(6) - 2(12) = -6$  and  $\rho_{s,K_{2,2,2}}(V(K_{2,2,2})) = 8(6) - 5(12) = -12$ .

- (iv)  $\rho_{m,M_7}(V(M_7)) = 3(7) - 2(11) = -1$  and  $\rho_{s,M_7}(V(M_7)) = 8(7) - 5(11) = 1$ .
- (v)  $\rho_{m,J_7}(V(J_7)) = 3(7) - 2(12) = -3$  and  $\rho_{s,J_7}(V(J_7)) = 8(7) - 5(12) = -4$ .
- (vi)  $\rho_{m,J_8}(V(J_8)) = 3(8) - 2(13) = -2$  and  $\rho_{s,J_8}(V(J_8)) = 8(8) - 5(13) = -1$ .
- (vii)  $\rho_{m,J_{12}}(V(J_{12})) = 3(12) - 2(20) = -4$  and  $\rho_{s,J_{12}}(V(J_{12})) = 8(12) - 5(20) = -4$ .
- (viii)  $\rho_{m,G_k}(V(G_k)) = 3(2k + 4) - 2(3k + 7) = -2$ ; further  $\rho_{m,G_k}(W) > -2$  for all  $W \subsetneq V(G_k)$ .
- (ix)  $\rho_{s,H_k}(V(H_k)) = 8(5(k + 2)) - 5(8(k + 2) + 1) = -5$ ; further  $\rho_{m,H_k}(W) > -5$  for all  $W \subsetneq V(H_k)$ .

The second statements in (viii) and (ix) are proved by induction on  $k$ .

### 2.3 Computational Aspects of Sparsity

Recall that a graph  $G$  is  $(a, b)$ -sparse if every nonempty  $W \subseteq V(G)$  satisfies  $|e(W)| \leq a|W| - b$ . Similarly,  $G$  is  $(a, b)$ -tight if it is  $(a, b)$ -sparse and  $|E(G)| = a|V(G)| - b$ , and  $G$  is  $(a, b)$ -strictly sparse if it is  $(a, b)$ -sparse and no subgraph is  $(a, b)$ -tight. These sparsity notions have connections to many other concepts. Lee and Streinu [19, §] survey several applications, emphasizing the equivalence between  $(2, 3)$ -tight graphs and Laman graphs for planar bar-and-joint rigidity. Sparsity is also related to minimal bends in vertex contact representations of paths on a grid; see [1, §].

$(a, b)$ -  
sparse  
 $(a, b)$ -tight  
strictly  
sparse

Kostochka and the second author [17] showed how to color  $(\frac{k}{2} - \frac{1}{k-1}, \frac{k(k-3)}{2(k-1)})$ -strictly sparse graphs in polynomial time. Later they proved [18] that certain known critical graphs are in fact  $(\frac{k}{2} - \frac{1}{k-1}, \frac{k(k-3)}{2(k-1)})$ -tight. Their coloring algorithm fits into a larger body of work that uses the so-called “Potential Method” to color sparse graphs. We will use the Potential Method to prove parts (A) and (B) of our Main Theorem. When we color an  $(a, b)$ -sparse graph, a key step is to either find a proper  $(a, b')$ -tight subgraph  $J$ , for specifically chosen  $b' > b$ , or else report that no such  $J$  exists. We may also impose additional constraints, for instance that  $2 \leq |J| \leq |V(G)| - 2$  or that  $|J|$  is maximized or minimized.

The *maximum average degree* of a graph  $G$  is the minimum  $a$  such that  $G$  is  $(a/2, 0)$ -sparse. Researchers have recently discovered new applications for finding a subgraph with maximum average degree, and algorithms achieving this have grown in interest (Google Scholar claims that a paper with a foundational algorithm [14] for this problem has over 250 citations). Finding the subgraph with largest maximum average degree among subgraphs whose order is bounded either from above or below is conjectured to be computationally hard [2], but it can be done in polynomial time [12] if the bounds are  $O(1)$  away from being trivial. We are unaware of any work bounding the subgraph’s order from both above and below simultaneously.

maximum  
average  
degree

Much of the work above generalizes to hypergraphs. Fix a hypergraph  $\mathbb{H}$ , vertex weights  $w_v : V(\mathbb{H}) \rightarrow \mathbf{R}^+$ , and edge weights  $w_e : E(\mathbb{H}) \rightarrow \mathbf{R}^+$ . The potential of a vertex set  $X$ , denoted  $\rho(X)$ , is defined as  $\rho(X) = \sum_{u \in X} w_v(u) - \sum_{f \subseteq X} w_e(f)$ . Hypergraph  $\mathbb{H}$  is  $b$ -sparse if  $\rho(X) \geq b$  for every nonempty vertex subset  $X$ . A graph  $G$  is  $(a, b)$ -sparse if and only if for weights  $w_v \equiv a, w_e \equiv 1$  we have that  $G$  is  $b$ -sparse.

$\rho(X)$   
 $b$ -sparse

Lee and Steinu [19] gave an algorithm to find an  $(a, b)$ -tight subgraph of maximum order when  $0 \leq b < 2a$ , and Streinu and Theran [21] generalized it to hypergraphs. Goldberg [14] gave an algorithm to find a subgraph with largest maximum average degree. The core routine of Goldberg’s algorithm is a max-flow/min-cut method; for a fixed  $a'$  it finds the largest  $b'$  such that the graph is  $(a', b')$ -sparse and returns an  $(a', b')$ -tight subgraph. Goldberg’s algorithm may return the empty subgraph, so it always returns with  $b' \geq 0$ . Kostochka and the second author [17] modified Goldberg’s algorithm to fit the needs of the Potential Method, but they only proved the modifications work for the case needed in that paper. Goldberg [14] also generalized his work to allow for edge weights and “vertex weights,” but his vertex weights are functionally equivalent to the presence of loops and differ from what we do here. To simplify current and future work with the Potential Method, we describe here the most general version of the algorithm in [17].

**Theorem 2.2.** *Fix a hypergraph  $\mathbb{H}$ , vertex weights  $w_v : V(\mathbb{H}) \rightarrow \mathbf{R}^+$ , and edge weights  $w_e : E(\mathbb{H}) \rightarrow \mathbf{R}^+$ . We can find a vertex subset  $W$  such that  $\rho(W) = \min_{U \subseteq V(\mathbb{H})} \rho(U)$  in time  $O((|V(\mathbb{H})| + |E(\mathbb{H})|)^3)$ . If each hyperedge has bounded size, then we can find  $W$  in time  $O((|V(\mathbb{H})| + |E(\mathbb{H})|)^2 \log(|V(\mathbb{H})| + |E(\mathbb{H})|))$ .*

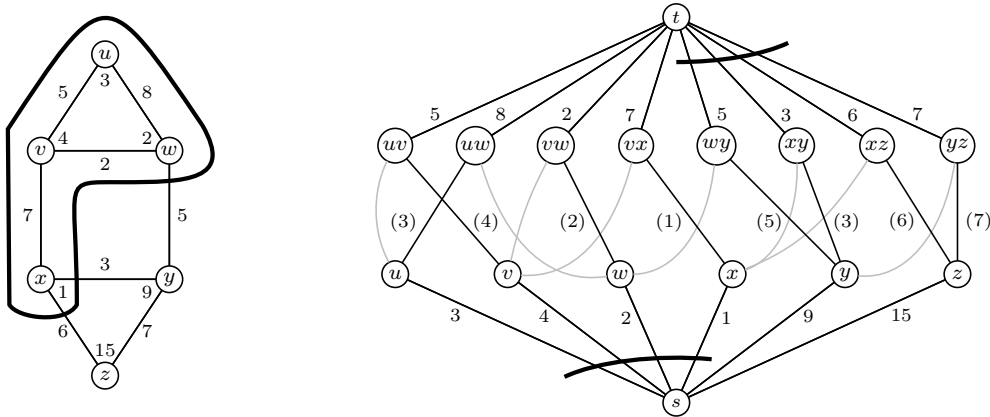


Figure 4: Left: A graph  $G$  with weights on edges and vertices, and its subgraph with minimum potential. The potential of the subgraph of  $G$  indicated is  $3 + 4 + 2 + 1 - (5 + 8 + 2 + 7) = -12$ . Right: A minimum cut and maximum flow—in an auxiliary graph—that correspond to the subgraph of  $G$  with minimum potential. Flow values are shown as  $(a)$  and capacities as  $a$ ; recall that each “center” edge has infinite capacity. Edges without flow value shown have the flow needed to conserve flow at their endpoints. (Curved, light gray edges do not receive any flow, but are drawn for completeness.) The minimum cut has value  $3 + 4 + 2 + 1 + 5 + 3 + 6 + 7 = 31$ . To calculate the minimum potential of a subgraph in  $G$  we subtract the sum of capacities of top edges ( $5 + 8 + 2 + 7 + 5 + 3 + 6 + 7 = 43$ ) from the value of the maximum flow. Thus, the potential is  $31 - 43 = -12$ .

*Proof.* The following is a straightforward adaptation of Goldberg’s argument in [17]; we get to add weights for free. (Figure 4 shows an example.)

Using a Max-flow/Min-cut algorithm, we will find a minimum weight cut  $E'$  in the following auxiliary digraph  $P$ . Let  $V(P) = \{s, t\} \cup V(\mathbb{H}) \cup E(\mathbb{H})$ . For each vertex  $v$  of  $\mathbb{H}$ , add an arc from  $s$  to the corresponding vertex in  $P$  with capacity  $w_v$ . For each edge  $e$  of  $\mathbb{H}$ , add an arc from the corresponding vertex in  $P$  to  $t$  with capacity  $w_e$ . For each vertex  $v$  in an edge  $e$  of  $\mathbb{H}$ , add an arc in  $P$  with infinite capacity from the vertex corresponding to  $v$  to the vertex corresponding to  $e$ .

Let  $w_e^{tot}$  denote the sum of all edge weights in  $\mathbb{H}$ . Observe that if  $v$  is a vertex in an edge  $e$  (in  $\mathbb{H}$ ), then either  $sv$  is in the edge cut  $E'$  of  $P$  or else  $et$  is in  $E'$ . Let  $W = \{v \in V(\mathbb{H}) : sv \in E'\}$ , and note that  $e(W) = \{e \in E(\mathbb{H}) : et \notin E'\}$ . Thus, the weight of  $E'$  is precisely

$$\begin{aligned}
& \sum_{x \in W} w_v(x) + \sum_{f \notin \mathbb{H}[W]} w_e(f) \\
&= \sum_{x \in W} w_v(x) - \sum_{f \in \mathbb{H}[W]} w_e(f) + \sum_{f \in E(\mathbb{H})} w_e(f) \\
&= \rho(W) + w_e^{tot}.
\end{aligned}$$

The algorithm’s running time is dominated by the cost of finding a minimum  $s - t$  edge-cut in  $P$ . Since  $|V(P)| = |V(\mathbb{H})| + |E(\mathbb{H})| + 2$ , the algorithm of Karzanov [15] runs in time  $O((|V(\mathbb{H})| + |E(\mathbb{H})|)^3)$ . If each hyperedge has bounded size, then  $|E(P)| = O(|V(\mathbb{H})| + |E(\mathbb{H})|)$ , so the algorithm of Sleator and Tarjan [20] runs in time  $O((|V(\mathbb{H})| + |E(\mathbb{H})|)^2 \log(|V(\mathbb{H})| + |E(\mathbb{H})|))$ .  $\square$

We have two immediate uses for the vertex weights. First, we can adapt the algorithm to the problem of extending a precoloring, as discussed in Section 2.2. Second, we can specify vertices as mandatory to include in our subgraph, as we show in the proof of our next result.

**Theorem 2.3.** *Theorem 2.2 can be adapted to allow the condition that we find the largest or smallest subset among optimal sets. Further, for constants  $m_1$  and  $m_2$ , we can also require the subset to have order at least  $m_1$  and at most  $|\mathbb{H}| - m_2$ , where the algorithm now runs in time  $O(|V(\mathbb{H})|^{m_1+m_2}(|V(\mathbb{H})| + |E(\mathbb{H})|)^3)$ .*



(*Proof sketch*). To find an optimal subgraph of maximal order, we increase the weight of each vertex by  $\epsilon$ . To find one of minimal order, we decrease each weight.

Let  $W$  denote the vertex subset returned by the algorithm in Theorem 2.2. To ensure that  $|W| \leq |V(G)| - m_2$ , we remove a set  $X$  of  $m_2$  vertices before running the algorithm. By considering all  $\binom{|V(\mathbb{H})|}{m_2}$  choices for  $X$ , we find our desired  $W$ .

To ensure that  $|W| \geq m_1$ , we choose a set  $Y$  of  $m_1$  vertices and add a new hyperedge over those vertices, with extremely high capacity. Any optimal cut must contain those vertices, so we can account for the weight of this new hyperedge at the end. Again we consider all possible choices for  $Y$ . The theorem follows from the inequality  $\binom{|V(\mathbb{H})|}{m_1} \binom{|V(\mathbb{H})|}{m_2} \leq |V(\mathbb{H})|^{m_1+m_2}$ .  $\square$

**Corollary 2.4.** *Let  $m_1, m_2$  be fixed nonnegative integers. If  $G$  is a connected graph with  $O(|V(G)|)$  edges, then a largest (or smallest) vertex subset  $W$  with smallest potential satisfying  $m_1 \leq |W| \leq |V(G)| - m_2$  can be found in time  $O(|V(G)|^{2+m_1+m_2} \log(|V(G)|))$ .*

## 2.4 Definitions and Notation

For completeness, below we collect our definitions, many of which are standard. A graph  $G$  consists of a vertex set  $V(G)$  and a multiset  $E(G)$  of unordered pairs of vertices, called the edge multiset. An edge  $e$  that is the pair of vertices  $v$  and  $w$  is written as  $e = vw$ . This paper deals with loopless graphs, so if  $vw$  is an edge, then  $v \neq w$ . Two edges  $e_1, e_2$  are *parallel* if they are the same pair of vertices. A *multiedge* is an equivalence class of edges that contains exactly two edges. (Recall that we allow at most two parallel edges joining any pair of vertices, since more parallel edges put no further constraints on the coloring.) A graph is *simple* if it has no multiedges.

A *circuit* of length  $k$  in a graph is a sequence of vertices  $v_1, v_2, \dots, v_{k+1}$  and edges  $e_1, e_2, \dots, e_k$  such that (a)  $v_1 = v_{k+1}$ , (b)  $e_i = v_i v_{i+1}$ , and (c)  $e_i \neq e_j$  when  $i \neq j$ . In particular, a multiedge forms a circuit of length 2. A *forest* is a graph with no circuits.

For a vertex subset  $W \subseteq V(G)$ , let  $e(W) = \{e \in E(G) : e \text{ has both endpoints in } W\}$ . We write  $G[W]$  for the subgraph induced by  $W$ ; that is  $V(G[W]) = W$  and  $E(G[W]) = e(W)$ . A vertex subset  $W$  is *independent* if  $|e(W)| = 0$ . For each vertex  $v$ , let  $d(v)$  denote the number of edges (including edge-gadgets) incident to  $v$ . Specifically, multiedges contribute 2 to the degree of each endpoint, but edge-gadgets only contribute 1. We write  $\Delta(G)$  and  $\delta(G)$  to denote the maximum and minimum degrees, respectively. Let  $N(v)$  denote the set of vertices that share an edge or edge-gadget with  $v$ . If  $G$  is simple, then  $d(v) = |N(v)|$ .

## 3 Constructing $\mathcal{H}$

### 3.1 Linked Vertices

In this subsection and the next, we construct the family  $\mathcal{H}$  of subgraphs forbidden in part (B) of the Main Theorem. On a first pass, the reader may prefer to focus on the proof of part (A), since it uses many of the same ideas, but is much easier than that of part (B). In that case, we recommend skipping to Section 4.

While trying to color  $G$ , we often want to color by minimality a graph  $J$  formed by adding an edge to some proper subgraph of  $G$ . A major hurdle we face is showing that  $J$  satisfies the hypotheses of the Main Theorem. To understand when adding an edge creates a copy of some forbidden subgraph, we study the following notion of linked vertices.

**Definition 3.1.** Let  $H$  be an nb-critical graph. Form  $H'$  from  $H$  by removing a single edge  $vw$ . Vertices  $s, t$  are *linked in  $G$*  if  $G$  contains a subgraph  $H''$  that is isomorphic to  $H'$ , where vertices  $s, t \in V(H'')$  correspond to  $v, w$  in the isomorphism. We call  $H$  the *linking graph*.

As an example, if  $G$  contains a copy of  $K_4 - e$ , then its non-adjacent vertices are linked. The following lemma generalizes a key concept from the proof of (2) in Section 2.1.

**Lemma 3.2.** *If vertices  $s, t$  are linked in a graph  $J$ , then for any nb-coloring  $I, F$  of  $J$ , either*

(i)  $\{s, t\} \subseteq I$ , or

(ii)  $\{s, t\} \subseteq F$  and there exists a path from  $s$  to  $t$  in  $J[F]$ .

*Proof.* We use notation as in Definition 3.1, and let  $e = st$ . Suppose, to the contrary, that  $J$  has an nb-coloring  $I, F$  with  $|\{s, t\} \cap F| \geq 1$  and that if  $s, t \in F$ , then  $G[F]$  has no path from  $s$  to  $t$ . Now  $I, F$  is also an nb-coloring for  $J + e$ . Since  $I, F$  restricts to an nb-coloring for  $H'' + e$ , and  $H'' + e \cong H$ , this contradicts our assumption that  $H$  is not near-bipartite.  $\square$

**Lemma 3.3.** *Using the notation of Definition 3.1, we know that  $\delta(H) \geq 3$ . So for each  $w \in V(H'') \setminus \{s, t\}$  we have  $d_G(w) \geq 3$ .*

*Proof.* The second statement clearly follows from the first, so we prove the first. Suppose, to the contrary, that  $w \in V(H)$  and  $d_H(w) \leq 2$ . By nb-criticality,  $H - w$  has an nb-coloring  $I', F'$ . If  $I'$  contains a neighbor of  $w$  in  $H$ , then let  $I = I'$  and  $F = F' \cup \{w\}$ . Otherwise let  $I = I' \cup \{w\}$  and  $F = F'$ . But now  $I, F$  is an nb-coloring of  $H$ , which contradicts that  $H$  is nb-critical.  $\square$

## 3.2 The Forbidden Subgraphs

To define  $\mathcal{H}$  we first define an infinite family of graphs  $\mathcal{H}'$ . The graphs  $K_4, W_5, J_7$ , and  $J_{12}$  are called *base graphs*. We define  $\mathcal{H}'$  recursively: each graph in  $\mathcal{H}'$  is either a base graph or else is formed by merging smaller graphs in  $\mathcal{H}'$  in a certain way. To explain this construction, we define *specially-linked* vertices (in Definition 3.4); this idea builds on Definition 3.1, but also assumes that the nb-critical graph  $H$  is in  $\mathcal{H}'$ . base graphs

All graphs in  $\mathcal{H}'$  contain no edge-gadgets and only contain uncolored vertices. This assumption will persist throughout this subsection. (However, when we forbid a subgraph in the Main Theorem, we also forbid it with precolored vertices and/or with some edges replace by edge-gadgets, since such variations are no easier to color.)

**Definition 3.4.** If two vertices  $s, t$  are linked in a graph  $J$ , then they are *specially-linked* if the linking nb-critical graph  $H$  (in Definition 3.1) is in  $\mathcal{H}'$ , where  $\mathcal{H}'$  is defined next.<sup>5</sup> specially-linked

A graph  $H$  is in  $\mathcal{H}'$  if (i)  $H$  is one of the four base graphs, or (ii)  $H$  is nb-critical and contains an induced cycle  $C = (x_1, x_2, \dots, x_k)$  such that each of the following three conditions holds:  $\mathcal{H}'$

(ii.a) the length of the cycle,  $k$ , satisfies  $k \in \{3, 5\}$ ,

(ii.b) each vertex in  $C$  has degree 3, and

(ii.c) if  $\{x_{j-1}, x_{j+1}, z_j\}$  denotes  $N(x_j)$ , with indices modulo  $k$ , then  $z_j$  and  $z_{j+1}$  are specially-linked in  $H - C$  (whenever  $z_j \neq z_{j+1}$ ).

The family of graphs  $\mathcal{H}$  is defined as

$$\mathcal{H} = \{H \in \mathcal{H}' : \rho_{s,H}(W) \geq -4 \text{ for all } W \subseteq V(H)\}. \quad \mathcal{H}$$

**Remark 3.5.** *If  $H \in \mathcal{H}'$  and  $H$  is not a base graph, then there exists  $j$  such that  $z_j \neq z_{j+1}$ .*

*Proof.* If not, then  $\{x_1, \dots, x_k, z_1\}$  induces either  $K_4$  or  $W_5$ , which contradicts that  $H$  is nb-critical.  $\square$

Examples of graphs in  $\mathcal{H}'$  include  $M_7$  and  $J_8$ ; the graph  $K_{2,2,2}$  is nb-critical, but is not in  $\mathcal{H}'$  since it is 4-regular, so fails condition (ii.b) in Definition 3.4. In Lemma 3.7 we will show that, among graphs in  $\mathcal{H}'$  that are not base graphs,  $M_7$  is the smallest and  $J_8$  is the second smallest (although we do not prove that  $J_8$  is uniquely the second smallest).

In the introduction, we claimed that each graph in  $\mathcal{H}$  is 4-critical and that  $\mathcal{H}$  is a finite family. We now prove these claims, as well as a few properties of  $\mathcal{H}'$  that we will need later. The most important result from this subsection is Corollary 3.8.

<sup>5</sup>Formally, perhaps we should define specially-linked only after defining  $\mathcal{H}'$ . Explicitly making that substitution in (ii.c) below gives a correct recursive definition of  $\mathcal{H}'$ , but also renders (ii.c) harder to parse.

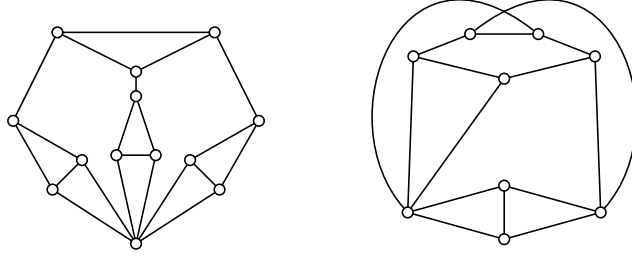


Figure 5: Two graphs in  $\mathcal{H}$ .

**Lemma 3.6.** *Each graph  $H \in \mathcal{H}'$  is 4-critical.*

*Proof.* We use induction on  $|V(H)|$ . It is easy to check that each base graph is 4-critical (due to symmetry, case analysis is quite short).

Now we assume that  $H \in \mathcal{H}'$  and  $H$  is larger than the base graphs. By definition,  $H$  has a cycle  $C$  that satisfies *ii.a*, *ii.b*, and *ii.c* from Definition 3.4. To prove that  $H$  is 4-critical, we show that  $\chi(G) \geq 4$  and that  $\chi(G - e) \leq 3$  for every  $e \in E(G)$ . The latter is easy: since  $H$  is nb-critical,  $H - e$  is near-bipartite, so  $\chi(H - e) \leq 3$ .

Assume, to the contrary, that  $H$  admits a proper 3-coloring  $\varphi$ . By definition, if  $z_i \neq z_{i+1}$ , then  $z_i$  and  $z_{i+1}$  are specially-linked in  $H - C$ . By induction, this implies that the linking graph  $J$  is a 4-critical graph. A basic fact of 4-critical graphs is that for any edge  $vw \in E(J)$ , any proper 3-coloring of  $J - vw$  uses the same color on  $v$  and  $w$ . It follows that  $\varphi(z_i) = \varphi(z_j)$  for all  $i, j \in \{1, \dots, k\}$ . But now  $\varphi(z_1)$  is forbidden from use on each vertex of the odd cycle  $C$ ; since  $C$  has no 2-coloring, this contradicts the existence of  $\varphi$ .  $\square$

**Lemma 3.7.** *If  $H \in \mathcal{H}'$  and  $H \notin \{K_4, W_5, M_7, J_7\}$ , then  $|V(H)| \geq 8$ .*

*Proof.* Fix  $H \in \mathcal{H}' \setminus \{K_4, W_5, M_7, J_7\}$ . Clearly the unique smallest graph in  $\mathcal{H}'$  is  $K_4$ . If  $H \neq J_{12}$ , then Remark 3.5 implies that  $H - C$  contains two linked vertices, so  $H - C$  has at least 4 vertices. Thus,  $H$  has at least 7 vertices. Further,  $H$  has at least 8 vertices unless  $H - C$  is  $K_4 - e$  and  $C$  is a 3-cycle. In this case,  $H$  is  $M_7$ , which contradicts our hypothesis.  $\square$

**Corollary 3.8.** *The families  $\mathcal{H}'$  and  $\mathcal{H}$  satisfy the following four properties.*

- (i) *If  $H \in \mathcal{H}'$ , then  $\rho_{s,H}(V(H)) \leq (10 - |V(H)|)/3 \leq 2$ .*
- (ii) *If  $H \in \mathcal{H}$ , then  $|V(H)| \leq 22$ .*
- (iii) *If  $H \in \mathcal{H}'$  and  $H \notin \{K_4, M_7\}$ , then  $\rho_{s,H}(V(H)) \leq 0$ .*
- (iv) *If  $H \in \mathcal{H}'$  and  $\rho_{s,H}(V(H)) \geq 0$ , then for every  $\emptyset \neq W \subsetneq V(H)$ , we have  $\rho_{s,H}(W) \geq 6$ .*

*Proof.* We start with (i). Lemma 3.6 implies  $H$  is 4-critical. Kostochka and Yancey [17] proved that if  $H$  is 4-critical, then  $|E(H)| \geq (5|V(H)| - 2)/3$ . So  $\rho_{s,H}(V(H)) = 8|V(H)| - 5|E(H)| \leq (24|V(H)| - 25|V(H)| + 10)/3 \leq 2$ ; the final inequality uses that  $|V(H)| \geq 4$ . Now (ii) follows from (i), since  $\rho_{s,H}(V(H)) \geq -4$ .

Next we consider (iii). Kostochka and Yancey [18] constructed a family of 4-critical graphs that they called 4-Ore graphs, and proved that if  $H$  is 4-critical and not 4-Ore, then  $|E(H)| \geq (5|V(H)| - 1)/3$ . They also showed that if  $H$  is 4-Ore, then  $|V(H)| \equiv 1 \pmod{3}$ . Moreover, if  $H$  is 4-Ore and  $|V(H)| \leq 7$ , then  $H$  is  $K_4$  or  $M_7$ .

Recall from Example 2.1 that  $\rho_{s,W_5}(V_{W_5}) = -2$  and  $\rho_{s,S}(V_S) = -4$ . So we assume that  $H \in \mathcal{H}' \setminus \{K_4, W_5, M_7, S_7\}$ . If  $H$  is 4-Ore, then the previous paragraph and Lemma 3.7 imply  $|V(H)| \geq 10$ . So (i) implies (iii). If  $H$  is not 4-Ore, then  $|E(H)| \geq (5|V(H)| - 1)/3$  implies  $\rho_{s,H}(V(H)) \leq (5 - |V(H)|)/3$ . Now  $|V(H)| \geq 8$  again implies (iii).

Finally, consider (iv). We omit the tedious calculations when  $G \in \{K_4, M_7\}$ . The proof of Part (iii) shows that if  $H \in \mathcal{H}'$  and  $\rho_{s,H}(V(H)) = 0$ , then  $H$  is a 4-Ore graph with 10 vertices. It was shown in [18] (see Claim 16) that if  $H$  is 4-Ore and  $\emptyset \neq W \subsetneq V(H)$ , then  $|E(W)| \leq (5|W| - 5)/3$ . Since  $\rho_{s,H}$  is integer valued and  $|W| < |V(H)| = 10$ , part (iv) holds because

$$\rho_{s,H}(W) \geq 8|W| - 5 \left( \frac{5|W| - 5}{3} \right) = \frac{25 - |W|}{3} > 5. \quad \square$$

We omit the work, but case analysis revealed that there are exactly 7 4-Ore graphs with 10 vertices, and all 7 are in  $\mathcal{H}$ . Corollary 3.8(ii) immediately implies the following.

**Remark 3.9.** *There exists finitely many graphs in  $\mathcal{H}$ .*

## 4 Proof of the Main Theorem

In Section 4, we start proving the Main Theorem. The proofs of parts (A) and (B) rely on many common lemmas, which we prove in Section 4.1. To unify our presentation, we write  $\rho_{*,G}$  to denote a statement that holds for both  $\rho_{m,G}$  and  $\rho_{s,G}$ . In Section 4.2 we finish proving part (A). In Sections 4.3 and 4.4 we finish proving part (B). To prove each part of our Main Theorem, we assume it is false, and let  $G$  be a counterexample minimizing  $|V(G)| + |E(G)|$ . Ultimately, we will reach a contradiction, by constructing an nb-coloring  $I, F$  of  $G$ .  $\rho_{*,G}$

### 4.1 Basic Lemmas

In Section 4.1, we have two goals: (i) to show that  $G$  is fairly “well-behaved”, and (ii) to prove our first gap lemma. We say a bit more about each. To facilitate our proofs, we have allowed precolored vertices, as well as edge-gadgets. But we hope that our minimal counterexample  $G$  has few, if any, of these. It is also easy to check that  $\delta(G) \geq 2$ . To get more control on  $G$ , we want to show that  $G$  has few 2-vertices. By “well-behaved” we mean all of these hoped-for properties.

We will often nb-color some proper subgraph  $J$  of  $G$ , by minimality. To get more power in our proof, we would like the option of slightly modifying  $J$  before coloring it. A small modification can only decrease potential by a small amount. For example, adding an edge decreases  $\rho_{m,G}$  by 2 and decreases  $\rho_{s,G}$  by 5. So to allow adding an edge, we must show (for each  $W \subsetneq V(G)$ ), that  $\rho_{m,G}(W) \geq -1 + 2 = 1$  and  $\rho_{s,G}(W) \geq -4 + 5 = 1$ . This is the content of Lemma 4.6. We call this a *gap lemma*, since it establishes a gap between the actual value of  $\rho_{*,G}(W)$  and the lower bound required by the hypothesis of the Main Theorem. In later sections, we prove stronger gap lemmas for both multigraphs and simple graphs, but those proofs all rely on Lemma 4.6. gap lemma

**Lemma 4.1.** *The potential function is submodular, i.e., for any graph  $J$  all  $W_1, W_2 \subseteq V(J)$  satisfy*

$$\rho_{*,J}(W_1 \cap W_2) + \rho_{*,J}(W_1 \cup W_2) \leq \rho_{*,J}(W_1) + \rho_{*,J}(W_2).$$

*Proof.* Each vertex is counted equally many times on both sides of the inequality. Each edge is counted at least as often on the left as on the right. □

**Lemma 4.2.**  *$|V(G)| \geq 3$ ,  $G$  is connected, and  $\delta(G) \geq 2$ .*

*Proof.* The only graphs with at most two vertices with precolorings that do not extend to nb-colorings are (i) when  $I_p = V(G)$  and  $G$  contains an edge and (ii) when  $F_p = V(G)$  and  $G$  contains a multiedge or edge-gadget. In each case,  $\rho_{*,G}(V(G))$  is too small to satisfy the hypothesis of the Main Theorem.

If  $G$  is disconnected, then each component has an nb-coloring by minimality. Together these give an nb-coloring of  $G$ . If  $G$  has a 1-vertex  $v$ , then  $G - v$  has an nb-coloring, and we extend it to  $G$  by adding  $v$  to  $F$ . □

Recall that, for each vertex  $v$ ,  $d(v)$  denotes the number of edges (including edge-gadgets) incident to  $v$ . Specifically, multiedges contribute 2 to the degree of each endpoint, but edge-gadgets only contribute 1. By a *forbidden subgraph*, we mean  $K_4$  or  $M$  in the case of multigraphs, and we mean some graph in the family  $\mathcal{H}$  in the case of simple graphs.

**Lemma 4.3.**  $I_p = \emptyset$ .

*Proof.* Suppose there exists some vertex  $w \in I_p$ . By the lower bound on  $\rho_{*,G}$ , for each edge  $vw$  we know  $v \notin I_p$ . Let  $N = \{v : vw \in E(G)\}$ . Let  $G' = G - w$ , and define a precoloring  $I'_p, F'_p$  as  $I'_p = I_p - \{w\}$  and  $F'_p = F_p \cup N$ . We claim that  $G'$  with the precoloring  $I'_p, F'_p$  satisfies the hypotheses of the Main Theorem. We did not add any edges, so any subgraph contained in  $G'$  is also contained in  $G$ . Let  $W \subseteq V(G')$ , and observe that  $\rho_{*,G'}(W) \geq \rho_{*,G}(W \cup \{w\})$ . This proves the claim. Now by minimality, we can find in polynomial time an nb-coloring  $I', F'$  that extends the precoloring  $I'_p, F'_p$ . Let  $I = I' \cup \{w\}$  and  $F = F'$ .  $\square$

Although we know that  $I_p = \emptyset$  in  $G$ , the notion of  $I_p$  will still be useful. In particular, we will often use minimality to color a graph  $G'$  with a precoloring  $I'_p, F'_p$  such that  $I'_p \neq \emptyset$ .

**Lemma 4.4.**  $|N(v)| \geq 2$  for each  $v \in V(G)$ .

*Proof.* Suppose there exist vertices  $v, w$  such that  $N(v) = \{w\}$ . By minimality,  $G - v$  has an nb-coloring  $I', F'$ . If  $v \notin F_p$ , then we extend  $I', F'$  to  $G$  by coloring  $v$  with the color unused on its neighbor. So assume  $v \in F_p$ . If  $v$  is not incident to a multiedge or an edge-gadget, then  $I', F' \cup \{v\}$  is an nb-coloring of  $G$ .

Now assume that both  $v \in F_p$  and also  $vw$  is either a multiedge or an edge-gadget. If  $w \in F_p$ , then  $\rho_{*,G}(\{v, w\})$  contradicts the hypotheses of the theorem; so assume  $w \notin F_p$ . Let  $G' = G - v$ ,  $F'_p = F_p$ , and  $I'_p = I_p \cup \{w\}$ . We claim that  $G'$  with precoloring  $I'_p, F'_p$  satisfies the hypotheses of the Main Theorem. For if  $w \notin W$ , then  $\rho_{*,G'}(W) = \rho_{*,G}(W)$ ; and if  $w \in W$ , then  $\rho_{*,G'}(W) = \rho_{*,G}(W \cup \{v\})$ . So, by minimality,  $G'$  has an nb-coloring  $I', F'$  that extends the precoloring  $I'_p, F'_p$ . Now  $I', F' \cup \{v\}$  is an nb-coloring of  $G$ , a contradiction.  $\square$

**Lemma 4.5.** If  $v$  is uncolored and not incident to an edge-gadget, then  $d(v) \geq 3$ .

*Proof.* Suppose, to the contrary, that  $v$  is uncolored,  $v$  is not incident to an edge-gadget, and  $d(v) = 2$ . Since  $|N(v)| \geq 2$  by the previous lemma, we denote  $\{x, y\}$  by  $N(v)$ . By minimality,  $G - v$  has an nb-coloring  $I', F'$ . If  $\{x, y\} \subseteq F'$ , then  $I' \cup \{v\}, F'$  is an nb-coloring of  $G$ . Otherwise  $I', F' \cup \{v\}$  is an nb-coloring of  $G$ .  $\square$

Now we can prove our gap lemma.

**Lemma 4.6.** If  $\emptyset \neq W \subsetneq V(G)$ , then  $\rho_{*,G}(W) \geq 1$ .

*Proof.* Suppose, to the contrary, there exists  $W \subsetneq V(G)$  such that  $|W| \geq 1$  and  $\rho_{*,G}(W) \leq 0$ . Among such subsets, choose  $W$  to minimize  $\rho_{*,G}(W)$ . Since  $I_p = \emptyset$ , we must have  $|W| \geq 2$ . Further, if  $|W| = 2$ , then  $E(G[W]) \neq \emptyset$ .

By minimality,  $G[W]$  has an nb-coloring  $I_W, F_W$  with  $F_p \cap W \subseteq F_W$ . Let  $\overline{W} = V(G) \setminus W$ .

**Claim 4.7.** Each  $v \in \overline{W}$  has at most one incident edge (and no edge-gadget) with endpoint in  $W$ .

*Proof.* Suppose, to the contrary, that there exists  $v \in \overline{W}$  with two incident edges, or an incident edge-gadget, with endpoints in  $W$ . Now  $\rho_{*,G}(W \cup \{v\}) < \rho_{*,G}(W)$ . So, by the minimality of  $W$ , we must have  $W \cup \{v\} = V(G)$ . If  $v$  has at least three incident edges into  $W$ , or an edge and another edge-gadget, then  $\rho_{*,G}(W \cup \{v\})$  violates the hypothesis of the Main Theorem:  $\rho_{m,G}(W \cup \{v\}) \leq \rho_{m,G}(W) + 3 - 2(3) = -3$  or  $\rho_{s,G}(W \cup \{v\}) \leq \rho_{s,G}(W) + 8 - 5(3) \leq -7$ . So assume  $v$  has exactly two edges into  $W$  or exactly one edge-gadget and no other edges. Further,  $v \in U_p$ , since otherwise  $\rho_{*,G}(W \cup \{v\})$  is too small. By minimality,  $G - v$  has an nb-coloring. Since  $W = V(G) \setminus \{v\}$ , we can easily extend this coloring to  $G$ , which contradicts that  $G$  is a counterexample. Thus, each  $v \in \overline{W}$  has at most one neighbor in  $W$ , and no incident edge-gadget into  $W$ , as desired.  $\diamond$

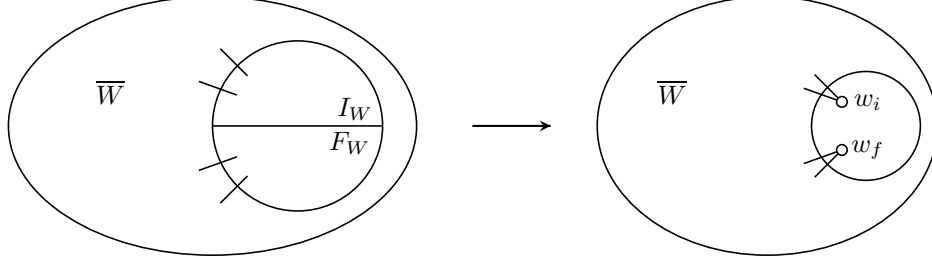


Figure 6: The construction of  $G'$  from  $G$  in the proof of Lemma 4.6.

We construct a graph  $G'$  with vertex set  $\overline{W} \cup \{w_f, w_i\}$ . We give  $G'$  the precoloring  $I'_p, F'_p$ , where  $I'_p = \{w_i\}$  and  $F'_p = (F_p \cap \overline{W}) \cup \{w_f\}$ . The edge set of  $G'$  is

$$E(G') = E(G[\overline{W}]) \cup \{uw_i : ux \in E(G), u \in \overline{W}, x \in I_W\} \cup \{vw_f : vz \in E(G), v \in \overline{W}, z \in F_W\}.$$

If  $w_f$  or  $w_i$  has degree 0, then we delete it. Note that  $G'$  is smaller than  $G$ , since either  $|W| \geq 3 > |\{w_i, w_f\}|$  or else  $|W| = 2$  and  $|E(G'[\{w_i, w_f\}])| = 0 < |E(G[W])|$ . Because  $|N(v) \cap W| \leq 1$  for each  $v \in \overline{W}$ , if  $G$  is a simple graph, then so is  $G'$ .

Suppose  $G'$  has an nb-coloring  $I', F'$  that extends the precoloring  $I'_p, F'_p$ . It is easy to check that  $I' \setminus \{w_i\} \cup I_W, (F' \setminus \{w_f\}) \cup F_W$  is an nb-coloring of  $G$ . This contradicts that  $G$  is a counterexample. So  $G'$  is not near-bipartite. Recall that  $G'$  is smaller than  $G$ . So, to reach a contradiction, we will show that  $G'$ , with precoloring  $I'_p, F'_p$  satisfies the hypotheses of the Main Theorem.

To begin, we show that  $\rho_{m,G'}(W') \geq -1$  and  $\rho_{m,G'}(W') \geq -4$  for all  $W' \subseteq V(G')$ . First assume, to the contrary, that there exists  $W'$  with  $\rho_{m,G'}(W') \leq -2$ . By Lemma 4.1

$$\rho_{m,G}(W' \setminus \{w_i, w_f\} \cup W) \leq \rho_{m,G'}(W') + \rho_{m,G}(W) - \rho_{m,G'}(W' \cap \{w_i, w_f\}). \quad (3)$$

This implies  $\rho_{m,G}(W' \setminus \{w_i, w_f\} \cup W) \leq -2 + 0 - 0 = -2$ , which is a contradiction. Inequality (3) is the key to proving all of our gap lemmas. We use it repeatedly below, often with less detail. Now assume, to the contrary, that there exists  $W'$  with  $\rho_{s,G'}(W') \leq -5$ . Similar to the previous case, now  $\rho_{s,G}(W' \setminus \{w_i, w_f\} \cup W) \leq \rho_{s,G'}(W') + \rho_{s,G}(W) - \rho_{s,G'}(W' \cap \{w_i, w_f\}) \leq -5 + 0 - 0 = -5$ , which is a contradiction.

Now we must show that  $G'$  does not contain forbidden subgraphs. In the case of multigraphs, we must show that  $G'$  contains neither  $K_4$  nor  $M_7$ . Suppose instead that  $G'$  contains  $K_4$  or  $M_7$ , and let  $W'$  denote its vertex set. Recall from Example 2.1 that (with no precolored vertices)  $\rho_{m,K_4}(V(K_4)) = 0$  and  $\rho_{m,M_7}(V(M_7)) = -1$ . So  $\rho_{m,G'}(W') - \rho_{m,G'}(W' \cap \{w_i, w_f\}) \leq 0 - 3$ . Now  $\rho_{m,G}((W' \setminus \{w_f, w_i\}) \cup W) \leq -3 + 0 = -3$ , which is a contradiction.

Finally, we consider the case of simple graphs. We must show that  $G'$  does not contain any graph in  $\mathcal{H}$ . Suppose that it does; call this graph  $H'$ , and let  $W'$  denote its vertex set. By Corollary 3.8(i), we know that (with no precolored vertices)  $\rho_{s,H'}(W') \leq 2$ . Thus,  $\rho_{s,G'}(W') - \rho_{s,G'}(W' \cap \{w_i, w_f\}) \leq 2 - 8 = -6$ . As a result,  $\rho_{m,G}((W' \setminus \{w_f, w_i\}) \cup W) \leq -6 + 0 = -6$ , which is a contradiction.  $\square$

Lemma 4.6 is useful in many ways. It immediately implies our next lemma, which is a strengthening of the submodularity condition in Lemma 4.1, and it also implies Lemmas 4.9 and 4.10.

**Lemma 4.8.** *In  $G$  the function  $\rho$  is subadditive:*

$$\rho_{*,G}(W_1 \cup W_2) \leq \rho_{*,G}(W_1) + \rho_{*,G}(W_2).$$

*Proof.* Since  $\rho_{*,G}(\emptyset) = 0$ , the previous lemma gives  $\rho_{*,G}(W_1 \cap W_2) \geq 0$  for all  $W_1, W_2 \subseteq V(G)$ . So  $\rho_{*,G}(W_1 \cup W_2) \leq \rho_{*,G}(W_1 \cup W_2) + \rho_{*,G}(W_1 \cap W_2) \leq \rho_{*,G}(W_1) + \rho_{*,G}(W_2)$ , by Lemma 4.1.  $\square$

The proof of the following lemma is simple arithmetic, so we omit it.

**Lemma 4.9.** *Both endpoints of a multiedge (for part (A)) or an edge-gadget (for part (B)) are uncolored. Further, when  $G$  is a multigraph, at least one endpoint of each edge is uncolored.*

**Lemma 4.10.** *If  $W \subsetneq V(G)$  and  $w \in V(G)$ , then  $G[W]$  has an nb-coloring  $I, F$  that extends the precoloring  $\emptyset, F_p \cup \{w\}$ .*

*Proof.* Let  $F'_p = F_p \cup \{w\}$  and  $I'_p = \emptyset$ . Because  $G[W]$  is a subgraph of  $G$ , it contains no forbidden subgraphs. By Lemma 4.6, the graph  $G[W]$  with precoloring  $I'_p, F'_p$  satisfies the hypotheses of the Main Theorem. So by minimality,  $G[W]$  has an nb-coloring that extends  $I'_p, F'_p$ .  $\square$

## 4.2 Multigraphs

In this section, we prove part (A) of the Main Theorem. The key step, which we begin with, is to strengthen by 1 the gap lemma we proved in Lemma 4.6 of the previous section. Everything after this stronger gap lemma is a chain of implications that culminates with the fact that  $G$  cannot exist.

**Lemma 4.11.** *If  $W \subsetneq V(G)$  and  $|W| \geq 2$ , then  $\rho_{m,G}(W) \geq 2$ .*

*Proof.* The proof is very similar to that of Lemma 4.6, so we mainly emphasize the differences. Suppose, the lemma is false; that is, some vertex subset  $W$  satisfies  $2 \leq |W| < |V(G)|$  and  $\rho_{m,G}(W) \leq 1$ . Among such  $W$ , choose one to minimize  $\rho_{m,G}(W)$ . By Lemma 4.6 we know that  $\rho_{m,G}(W) = 1$ . First, we note that  $|W| \geq 3$ . Suppose, to the contrary, that  $|W| = 2$ . By Lemma 4.3,  $I_p = \emptyset$ , so each vertex contributes odd weight (1 or 3) and each edge contributes even weight ( $-2$ ), which implies  $\rho_{m,G}(W) \equiv 0 \pmod{2}$ . By Lemma 4.6, we have  $\rho_{m,G}(W) \geq 1$ ; thus  $\rho_{m,G} \geq 2$ . So,  $|W| \geq 3$ , as desired.

As in the previous proof, each  $v \in \overline{W}$  has at most one neighbor in  $W$ . Since  $G$  is connected and  $W \subsetneq V(G)$ , there exists  $w \in W$  with a neighbor not in  $W$ . Let  $G' = G[W]$  with precoloring  $I'_p, F'_p$ , where  $I'_p = \emptyset$  and  $F'_p = (F_p \cap W) \cup \{w\}$ . For each  $X \subseteq W$ , we have  $\rho_{m,G'}(X) \geq \rho_{m,G}(X) - 2 \geq 1 - 2 = -1$ , by Lemma 4.6<sup>6</sup>. Thus, by minimality,  $G'$  has an nb-coloring  $I', F'$  that extends the precoloring  $I'_p, F'_p$ . Now we repeat the construction of graph  $G'$  from the proof of Lemma 4.6.

Suppose  $G'$  has an nb-coloring  $I', F'$  that extends the precoloring  $I'_p, F'_p$ . It is easy to check that  $(I' \setminus \{w_i\} \cup I_W), (F' \setminus \{w_f\} \cup F_W)$  is an nb-coloring of  $G$ . This contradicts that  $G$  is a counterexample. So  $G'$  is not near-bipartite. Recall that  $G'$  is smaller than  $G$ . So to reach a contradiction, we will show that  $G'$ , with precoloring  $I'_p, F'_p$ , satisfies the hypotheses of the Main Theorem.

We must show that  $G'$  does not contain  $K_4$  or  $M_7$ . Recall that (with all vertices uncolored), we have  $\rho_{m,K_4}(V(K_4)) = 0$  and  $\rho_{m,M_7}(V(M_7)) = -1$ . Suppose, to the contrary, that  $G'$  contains a copy of either  $K_4$  or  $M_7$ , and let  $W'$  denote its vertex set. Since  $K_4 \not\subseteq G$  and  $M_7 \not\subseteq G$ , we have  $W' \cap \{w_f, w_i\} \neq \emptyset$ . As in the proof of Lemma 4.6, we have  $\rho_{m,G}(W' \setminus \{w_f, w_i\} \cup W) \leq \rho_{m,G'}(W') + \rho_{m,G}(W) - \rho_{m,G'}(W' \cap \{w_f, w_i\})$ . Note that  $\rho_{m,G}(W') - \rho_{m,G}(W \cap \{w_f, w_i\}) \leq \rho_{m,G}(W) - 3 \leq -3$ . Thus  $\rho_{m,G}(W' \setminus \{w_f, w_i\} \cup W) \leq -3 + 1 = -2$ , which is a contradiction.

Finally, we show that  $\rho_{m,G'}(W') \geq -1$  for all  $W \subseteq V(G)$ . Assume, to the contrary, that there exists  $W'$  with  $\rho_{m,G'}(W') \leq -2$ . Now  $\rho_{m,G}(W' \setminus \{w_i, w_f\} \cup W) \leq \rho_{m,G'}(W') + \rho_{m,G}(W) - \rho_{m,G'}(W' \cap \{w_i, w_f\}) \leq -2 + \rho_{m,G}(W) < \rho_{m,G}(W)$ . By our choice of  $W$ , we know that  $W' \setminus \{w_i, w_f\} \cup W = V(G)$ . If  $w_f \in W'$ , then  $\rho_{m,G}(W' \cap \{w_i, w_f\}) = 1$ . Now  $\rho_{m,G}(W' \setminus \{w_i, w_f\} \cup W) \leq -2 + 1 - 1 = -2$ , a contradiction. So instead, assume  $w_f \notin W'$ . However, now we have  $\rho_{m,G}(W' \setminus \{w_i, w_f\} \cup W) < -2 + 1 - 0 = -1$ . Here the inequality is strict, since the left side counts an edge from  $w_f$  to a neighbor outside of  $W$ , but that edge is not counted on the right (recall from the second paragraph that  $w$  is precolored to be in  $F$  and  $w$  has a neighbor in  $\overline{W}$ ). Again,  $\rho_{m,G}(W' \setminus \{w_i, w_f\} \cup W) \leq -2$ , which is a contradiction. So  $G'$  satisfies the hypotheses of the Main Theorem, which finishes the proof.  $\square$

**Lemma 4.12.**  $\delta(G) \geq 3$ .

<sup>6</sup>This step in the proof is the only place where we actually use Lemma 4.6, and it is why we prove that weaker result before proving this one.

*Proof.* Assume, to the contrary, that  $G$  contains a vertex  $v$  with  $d(v) \leq 2$ . By Lemmas 4.4 and 4.5 we know that  $d(v) = 2$  and  $v \in F_p$ . Let  $w$  and  $x$  denote the neighbors of  $v$ . Form  $G'$  from  $G-v$  by adding edge  $wx$ . (Note that  $wx \notin E(G)$ , since otherwise  $\rho_{m,G}(\{v, w, x\}) = 2(3) + 1(1) - 3(2) = 1$ , which contradicts Lemma 4.11.) Suppose there exists  $W' \subseteq V(G')$  with  $\rho_{m,G'}(W') \leq -2$ . Since  $G'[W'] \not\subseteq G$ , we have  $\{w, x\} \subseteq W'$ . Now  $\rho_{m,G}(W' \cup \{v\}) = \rho_{m,G'}(W') + (\rho_{m,G}(\{v, w, x\}) - \rho_{m,G'}(\{w, x\})) \leq -2 + (-1)$ , which is a contradiction. So assume instead that  $G'$  contains a copy of  $K_4$  or  $M_7$ ; let  $W'$  denote its vertex set. In this case  $\rho_{m,G}(W' \cup \{v\}) = \rho_{m,G'}(W') - 1 \leq 0$ . This contradicts Lemma 4.11 unless  $W' \cup \{v\} = V(G)$ . However, in that case we can easily construct an explicit nb-coloring of  $G$  (when  $G'[W'] = K_4$  we have only a single case, and when  $G'[W'] = M_7$  we have four cases).  $\square$

**Lemma 4.13.**  $F_p = \emptyset$ .

*Proof.* Since  $\delta(G) \geq 3$ , we have  $|E(G)| = \frac{1}{2} \sum_{v \in V(G)} d(v) \geq \frac{3}{2}|V(G)|$ . Now  $\rho_{m,G}(V(G)) = 3|U_p| + |F_p| - 2|E(G)| \leq 3|V(G)| - 2|F_p| - 2(\frac{3}{2}|V(G)|) = -2|F_p|$ . By assumption  $\rho_{m,G}(V(G)) \geq -1$ , so  $F_p = \emptyset$ .  $\square$

**Lemma 4.14.**  $G$  has at most one vertex  $w$  with  $d(w) > 3$ . If  $w$  exists, then  $d(w) = 4$ .

*Proof.* Choose arbitrary vertices  $v, w \in V(G)$ . Since  $\delta(G) \geq 3$ , we have  $2|E(G)| \geq 3(|V(G)| - 2) + d(v) + d(w)$ . Thus  $\rho_{m,G}(V(G)) \leq 3|V(G)| - (3(|V(G)| - 2) + d(v) + d(w)) = 6 - d(v) - d(w)$ . Since  $\rho_{m,G}(V(G)) \geq -1$ , we get  $d(v) + d(w) \leq 7$ . Since  $d(v) \geq 3$ , the lemma holds.  $\square$

**Lemma 4.15.**  $G$  has no multiedges.

*Proof.* Suppose, to the contrary, that  $G$  has a multiedge. By the previous lemma, one of its endpoints has degree 3. So let  $v$  be a 3-vertex with neighborhood  $\{w, x\}$ , and with a multiedge to  $x$ . By Lemma 4.10 there exists an nb-coloring of  $G - v$  with  $w \in F$ . This is a contradiction, as such a coloring can be extended to  $G$  by coloring  $v$  with the color not on  $x$ .  $\square$

**Lemma 4.16.**  $|V(G)| \geq 6$ .

*Proof.* Since  $\delta(G) = 3$ , and  $G$  has no multiedge,  $|V(G)| \geq 4$ . If  $|V(G)| = 4$ , then  $G$  is  $K_4$ , which is a contradiction. So suppose that  $|V(G)| = 5$ . By Lemma 4.14,  $G$  has four 3-vertices and a 4-vertex. Thus,  $G$  is formed from  $K_5$  by deleting two independent edges. So let  $I$  consist of two non-adjacent vertices, and let  $F = V(G) \setminus I$ . This nb-coloring of  $G$  is a contradiction. Thus,  $|V(G)| \geq 6$ , as desired.  $\square$

**Lemma 4.17.**  $G$  has no 3-cycle.

*Proof.* First suppose that  $G$  contains 3-cycles  $vw x$  and  $w x y$ . Form  $G'$  from  $G - \{w, x\}$  by identifying  $v$  and  $y$ ; call this new vertex  $z$ . If there exists  $W' \subseteq V(G')$  with  $\rho_{m,G'}(W') \leq -2$ , then clearly  $z \in W'$ . So  $\rho_{m,G}((W' \setminus \{z\}) \cup \{v, w, x, y\}) \leq -2 + 3(3) - 5(2) = -3$ , a contradiction. Suppose instead that  $G'$  contains a copy of  $M_7$ , and let  $W'$  denote its vertex set. Similar to before,  $\rho_{m,G}((W' \setminus \{z\}) \cup \{v, w, x, y\}) \leq \rho_{m,M_7}(V(M_7)) + 3(3) - 5(2) = -2$ , a contradiction. Finally, suppose that  $G'$  contains a copy of  $K_4$ , and let  $W'$  denote its vertex set. Now  $\rho_{m,G}((W' \setminus \{z\}) \cup \{v, w, x, y\}) \leq 0 + 3(3) - 5(2) = -1$ . This contradicts Lemma 4.11, unless  $V(G) = (W' \setminus \{z\}) \cup \{v, w, x, y\}$ . However, in that case, we can easily check that  $G = M_7$ , a contradiction. Since  $G'$  is smaller than  $G$  and satisfies the hypotheses of the Main Theorem,  $G'$  has an nb-coloring,  $I', F'$ . And we easily extend  $I', F'$  to  $G$ , which is a contradiction.

Now suppose that  $G$  contains a 3-cycle  $vw x$  and none of its edges lie on another 3-cycle. Assume, without loss of generality that  $d(w) = d(x) = 3$ . Let  $y$  denote the third neighbor of  $x$ . Since  $w$  and  $x$  have distinct neighbors off the 3-cycle, we can also assume that  $d(y) = 3$ . Form  $G'$  from  $G - \{v, x\}$  by identifying  $w$  and  $y$ ; call this new neighbor  $z$ . If there exists  $W' \subseteq V(G')$  with  $\rho_{m,G'}(W') \leq -2$ , then also  $\rho_{m,G}(W' \setminus \{z\}) \cup \{w, x, y\} \leq -2 + 2(3) - 2(2) = 0$ , which contradicts Lemma 4.11, since  $(W' \setminus \{z\}) \cup \{w, x, y\} \subsetneq V(G)$ . Note that  $G'$  cannot contain  $K_4$ , since  $G$  does not contain two 3-cycles with a common edge. Suppose instead that  $G'$  contains  $M_7$ . Recall that  $M_7$  contains two edge-disjoint copies of two 3-cycles sharing an edge. Since  $G$  contains no such subgraph, both copies must contain the new vertex  $z$ . But this is impossible: since  $d(w) = d(y) = 3$ , also  $d_{G'}(z) = 3$ .  $\square$



**Lemma 4.18.**  *$G$  does not exist. That is, part (A) of the Main Theorem is true.*

*Proof.* Choose a vertex  $v \in V(G)$  with  $d(v) = 3$ . Let  $\{w, x, y\} = N(v)$ . Form  $G'$  from  $G - v$  by identifying  $w$  and  $x$ ; call this new vertex  $z$ . By Lemma 4.17  $G$  has no 3-cycle, so  $G'$  cannot contain  $K_4$  or  $M_7$ , since neither has a single vertex contained in all of its 3-cycles. For each  $W' \subseteq V(G')$ , Lemma 4.11 implies  $\rho_{m, G'}(W') \geq \rho_{m, G}((W' \setminus \{z\}) \cup \{w, x\}) - 3 \geq 2 - 3 = -1$ . Thus, by minimality,  $G'$  has an nb-coloring  $I', F'$ . And it is easy to extend this to  $G$ . Specifically, remove  $z$  from whichever set contains it and add  $w$  and  $x$  to this set. Now, if both  $y$  and  $z$  were in  $F'$ , then add  $v$  to  $I'$ ; otherwise add  $v$  to  $F'$ .  $\square$

### 4.3 Simple Graphs: More Reducible Configurations

In this section we continue the proof of part (B) of the Main Theorem, which we began in Section 4.1. Our approach mirrors what we did in Section 4.2, where we showed (for part (A)) that a minimal counterexample must be well-behaved. The main results of the section are that  $\delta(G) \geq 3$  and that the subgraph induced by uncolored 3-vertices is a forest. To prove these properties, a key step is strengthening our earlier gap lemma, which we do in Lemma 4.24. In Section 4.4 we will complete the proof of part (B). Using the structural results that we prove here, there we will give a discharging argument to show that  $G$  is very nearly comprised entirely of uncolored 3-vertices that induce a forest, together with uncolored 4-vertices that induce an independent set. (In fact  $G$  can vary slightly from this, but in each case we explicitly construct an nb-coloring.)

We will frequently use our next lemma to extend an nb-coloring from a subgraph of  $G$  to all of  $G$ .

**Lemma 4.19.** *Suppose  $C = x_1 \dots x_k$  is an induced cycle in  $G$  with  $d(x_i) = 3$  for all  $i$ , and let  $\{z_i\} = N(x_i) \setminus \{x_{i-1}, x_{i+1}\}$  for all  $i$ . Fix an nb-coloring  $I', F'$  of  $G - V(C)$ . We can extend  $I', F'$  to  $G$  unless (i)  $z_i \in I'$  for all  $i$  or (ii)  $k$  is odd and  $z_i \in F'$  for all  $i$  and all  $z_i$  are in the same component of  $G[F']$ .*

*Proof.* Fix an nb-coloring  $I', F'$  of  $G - V(C)$ . First suppose that there exist  $z_i \in I'$  and  $z_j \in F'$ . By symmetry, assume that  $z_k \in I'$  and  $z_1 \in F'$ . We iteratively add each  $x_i$  to either  $I'$  or  $F'$ . Let  $I_1 = I' \cup \{x_1\}$  and  $F_1 = F'$ . For each  $j > 1$ , do the following. If  $I_{j-1} \cap \{x_{j-1}, z_j\} = \emptyset$ , then  $I_j = I_{j-1} \cup \{x_j\}$  and  $F_j = F_{j-1}$ ; otherwise  $I_j = I_{j-1}$  and  $F_j = F_{j-1} \cup \{x_j\}$ . It is easy to prove by induction on  $j$  that  $I_k, F_k$  is an nb-coloring of  $G$ .

Now instead assume that  $z_i \in F'$  for all  $i \in [k]$ . If  $k$  is even, then let  $I = I' \cup \bigcup_{i=1}^{k/2} x_{2i-1}$  and  $F = F' \cup \bigcup_{i=1}^{k/2} x_{2i}$ . Now  $I, F$  is an nb-coloring of  $G$ . So assume  $k$  is odd. Suppose, by symmetry, that  $z_{k-1}$  and  $z_k$  are in different components of  $G[F']$ . Let  $I = I' \cup \bigcup_{i=1}^{(k-1)/2} x_{2i-1}$  and  $F = F' \cup \{x_k\} \cup \bigcup_{i=1}^{(k-1)/2} x_{2i}$ . Again  $I, F$  is an nb-coloring of  $G$ .  $\square$

Our next construction is motivated by our desire to avoid the exceptional cases in the previous lemma. Clearly, this is achieved by every nb-coloring of  $G(C, z_1, z_2)$ , which we define next. Ultimately, we will use this construction and lemma after it to show that the uncolored 3-vertices of  $G$  induce a forest. But the proof that  $G(C, z_1, z_2)$  has an nb-coloring is tricky, and we will break it into Lemmas 4.22, 4.27, and 4.29.

**Definition 4.20.** Let  $C = x_1 \dots x_k$  be a  $k$ -cycle in  $G$  induced by 3-vertices, and let  $\{z_i\} = N(x_i) \setminus \{x_{i-1}, x_{i+1}\}$  for all  $i$ . Let  $W = V(G) - C$ . We construct an auxiliary graph  $G(C, z_1, z_2)$  as follows<sup>7</sup>:

- (i) if  $z_1$  and  $z_2$  are the endpoints of an edge-gadget, then  $G(C, z_1, z_2) = G[W]$ ; otherwise
- (ii) if  $z_1 z_2 \in E(G)$ , then  $G(C, z_1, z_2)$  is formed from  $G[W]$  by removing  $z_1 z_2$  and replacing it with an edge-gadget; otherwise
- (iii)  $G(C, z_1, z_2)$  is formed from  $G[W]$  by adding edge  $z_1 z_2$ .

To find an nb-coloring of  $G(C, z_1, z_2)$  by minimality, we must show that  $G(C, z_1, z_2) \notin \mathcal{H}$ . Our next lemma helps us do this.

<sup>7</sup>Part (ii) of Definition 4.20 is the most important place where we construct edge-gadgets. A key consequence of using an edge-gadget is that  $G(C, z_1, z_2)$  is smaller than  $G$ , which is essential for the proof of Lemma 4.22.

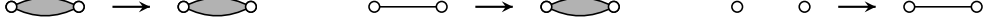


Figure 7: Vertices  $z_1$  and  $z_2$  in the construction of  $G(C, z_1, z_2)$  in Definition 4.20.

**Lemma 4.21.** *We use notation from Definition 3.1. If vertices  $v$  and  $w$  are linked in  $G$  through subgraph  $H'$  with  $|V(H')| < |V(G)|$ , then  $v$  and  $w$  are specially-linked (and the linking graph is in  $\mathcal{H}$ ).*

*Proof.* Suppose, to the contrary, that  $v$  and  $w$  are linked via subgraph  $H'$ , where  $H' + vw \cong H$  for some  $H \notin \mathcal{H}$ . Now  $|E(H')| \leq |E(G)| - 2$  (since  $\delta(G) \geq 2$ ), so  $|E(H)| \leq |E(G)| - 1$ . Thus,  $|V(H)| + |E(H)| < |V(G)| + |E(G)|$ , which implies that  $H$  is smaller than  $G$  in our ordering. By the definition of linked,  $H$  is not near-bipartite. So by the minimality of  $G$ , either  $H$  contains as a subgraph some graph in  $\mathcal{H}$  or else there exists  $W \subseteq V(H)$  such that  $\rho_{s,H}(W) \leq -5$ . In the latter case,  $\rho_{s,G}(W) \leq \rho_{s,H}(W) + 5 \leq 0$ , which contradicts our gap lemma, Lemma 4.6. So  $H$  contains as a subgraph a graph from  $\mathcal{H}$ . Since  $\mathcal{H} \subseteq \mathcal{H}'$ , vertices  $v$  and  $w$  are specially-linked, as desired.  $\square$

**Lemma 4.22.** *If  $C = x_1 \cdots x_k$  is a cycle in  $G$  induced by 3-vertices, then at least one of the following holds:*

- (i)  $k \geq 5$ , or
- (ii) some  $x_i$  is incident to an edge-gadget, or
- (iii)  $V(C) \cap F_p \neq \emptyset$ .

*Proof.* Suppose, to the contrary, that  $k \in \{3, 4\}$ , no  $x_i$  is incident to an edge-gadget, and each  $x_i$  is uncolored. Let  $W = V(G) \setminus V(C)$ , and let  $\{z_i\} = N(x_i) \setminus \{x_{i-1}, x_{i+1}\}$  for all  $i$ .

First suppose  $k = 4$ . Let  $G' = G[W]$  with  $I'_p = I_p$  and  $F'_p = F_p \cup \{z_1\}$ . By Lemma 4.10,  $G'$  has an nb-coloring  $I', F'$  that extends  $I'_p, F'_p$ . By Lemma 4.19, we can extend  $I', F'$  to an nb-coloring of  $G$ .

Now assume  $k = 3$ . By Definition 3.4 and Remark 3.5, there exists  $j$  such that  $z_j \neq z_{j+1}$  and  $z_j, z_{j+1}$  are not specially-linked through a subgraph of  $G[W]$ . By Lemma 4.21, vertices  $z_j$  and  $z_{j+1}$  are not linked through any subgraph of  $G[W]$ . By symmetry, we assume  $j = 1$ . Let  $G' = G(C, z_1, z_2)$ . We first show that we can extend any nb-coloring  $I', F'$  of  $G'$  to  $G$ , by Lemma 4.19. Assume  $z_1, z_2, z_3 \in F'$ . Note that  $z_1$  and  $z_2$  must be in different components of  $G[F']$ , even if they are in the same component of  $G'[F']$ . By Lemma 4.19, we can extend  $I', F'$  to  $G$ , as desired.

Now we must show that  $G'$  does indeed have the desired nb-coloring  $I', F'$ . By construction,  $G'$  is smaller than  $G$ , and  $G'$  contains no forbidden subgraph, since  $z_1$  and  $z_2$  are not linked in  $G$ . By the minimality of  $G$ , if  $\rho_{s,G'}(U) > -5$  for all  $U \subseteq W$ , then  $G'$  is near-bipartite. If  $\{z_1, z_2\} \not\subseteq U$ , then  $\rho_{s,G'}(U) = \rho_{s,G}(U) > -5$ . If  $\{z_1, z_2\} \subseteq U$ , then

$$\rho_{s,G'}(U) \geq \rho_{s,G}(U \cup C) - 8(3) + 5(5) - 6 = \rho_{s,G}(U \cup C) - 5. \quad (4)$$

If there exists  $U$  such that  $\rho_{s,G'}(U) \leq -5$ , then  $\rho_{s,G}(U \cup C) \leq 0$ . By Lemma 4.6, this implies that  $U \cup V(C) = V(G)$ . So  $z_3 \in U$ , and we can add the edge  $x_3z_3$  to the calculation in (4); the new bound claims  $\rho_{s,G}(U \cup C) \leq -5$ , which is a contradiction. Thus,  $G'$  has an nb-coloring  $I', F'$ .  $\square$

A key intermediate result in this section is our improved gap lemma, Lemma 4.24. Our next result is designed to help us prove this gap lemma.

**Lemma 4.23.** *If  $W \subset V(G)$  such that  $|W| = |V(G)| - 2$ , then  $\rho_{s,G}(W) \geq 5$ .*

*Proof.* Assume, to the contrary, that there exists  $W$  satisfying the hypotheses with  $\rho(W) \leq 4$ . Let  $\{v, w\} = V(G) \setminus W$ . If  $v$  and  $w$  are both uncolored and not incident to edge-gadgets, then they each have degree at least 3, by Lemma 4.5; and so together they are incident to at least  $2(3) - 1 = 5$  edges (with equality if  $d(v) = d(w) = 3$  and  $v$  and  $w$  are adjacent). Now  $\rho_{s,G}(V(G)) \leq \rho_{s,G}(W) + 2(8) - 5(5) \leq 4 - 9 = -5$ , which contradicts the hypothesis of the Main Theorem.

Now we assume instead that at least one of  $v$  and  $w$  is either precolored or incident to an edge-gadget. Recall from Lemma 4.9 that both endpoints of each edge-gadget are uncolored. Each precolored vertex has potential 5 less than each uncolored vertex, and is still incident to at least 2 edges, by Lemma 4.4; so the analysis remains the same. Thus, we assume that  $v$  and  $w$  are both uncolored. Suppose that at least one of  $v$  and  $w$  is incident to an edge-gadget, but  $vw$  is not an edge-gadget itself. If  $k$  denotes the total number of edge-gadgets incident to  $v$  and  $w$ , then  $v$  and  $w$  are also incident to at least  $5 - 2k$  more edges. Since each edge-gadget decreases potential more than 2 edges do, the analysis remains the same. Finally, assume that  $vw$  is an edge-gadget and  $v$  and  $w$  are each incident to only one other edge. (If at least one of  $v$  and  $w$  has degree 3, then together they have one incident edge-gadget, and at least three more incident edges, so the analysis is similar to before.) Let  $x$  denote the neighbor of  $w$  other than  $v$ . By Lemma 4.10, we can nb-color  $G[W]$  with precoloring  $I'_p = \emptyset$  and  $F'_p = F_p \cup \{x\}$ . To extend this coloring to  $G$ , add  $v$  to  $F$  and  $w$  to  $I$ .  $\square$

Now we can prove our stronger gap lemma.

**Lemma 4.24.** *If  $W \subset V(G)$  such that  $0 < |W| \leq |V(G)| - 2$ , then  $\rho_{s,G}(W) \geq 4$ .*

*Proof.* Suppose, to the contrary, that some  $W$  satisfies the hypotheses and has  $\rho_{s,G}(W) \leq 3$ . We assume further that  $W$  minimizes  $\rho_{s,G}(W)$  among all such vertex subsets. Let  $\overline{W} = V(G) \setminus W$ .

We first show that if  $v \in \overline{W}$ , then  $|N(v) \cap W| \leq 1$ . Suppose, to the contrary, that  $|N(v) \cap W| \geq 2$ , which gives that  $\rho_{s,G}(W \cup \{v\}) \leq \rho_{s,G}(W) + 8 - 10$ . Lemma 4.23 implies that  $|W| < |V(G)| - 2$ , so  $|W \cup \{v\}| \leq |V(G)| - 2$ , which contradicts the minimality of  $W$ . Thus  $|N(v) \cap W| \leq 1$ , as desired.

By minimality,  $G[W]$  has an nb-coloring  $I_W, F_W$ . We construct a graph  $G'$  with vertex set  $\overline{W} \cup \{w_f, w_i\}$ , similar to the proof of our first gap lemma, Lemma 4.6. We give  $G'$  the precoloring  $I'_p, F'_p$ , where  $I'_p = \{w_i\}$  and  $F'_p = (F_p \cap \overline{W}) \cup \{w_f\}$ . The edge set of  $G'$  is given by

$$E(G') = e(\overline{W}) \cup \{uw_i : ux \in E(G), u \in \overline{W}, x \in I_W\} \cup \{vw_f : vz \in E_G, v \in \overline{W}, z \in F_W\}.$$

If  $w_f$  or  $w_i$  has degree 0, then we delete it. Using Lemma 4.10, we will assume that  $w_f$  is not deleted. Recall that  $|N(v) \cap W| \leq 1$  for each  $v \in \overline{W}$ , so  $G'$  is a simple graph.

If  $G'$  has an nb-coloring  $I', F'$ , then we delete  $\{w_i, w_f\}$  and use the nb-coloring  $I_W, F_W$  on  $G[W]$  to get an nb-coloring of  $G$ . This contradicts that  $G$  is a counterexample, so  $G'$  must not satisfy the hypotheses of the Main Theorem. Thus,  $G'$  contains either a forbidden subgraph or else a vertex set  $U'$  such that  $\rho_{s,G'}(U') \leq -5$ . We start with the latter case. Pick  $U' \subseteq V(G')$  to minimize  $\rho_{s,G'}(U')$ . Now

$$\begin{aligned} \rho_{s,G'}(U') &\geq \rho_{s,G}((U' \setminus \{w_f, w_i\}) \cup W) - \rho_{s,G}(W) + \rho_{s,G'}(U' \cap \{w_f, w_i\}) \\ &\geq \rho_{s,G}((U' \setminus \{w_f, w_i\}) \cup W) - 3 + \rho_{s,G'}(U' \cap \{w_f, w_i\}). \end{aligned}$$

Trivially,  $\rho_{s,G'}(U' \cap \{w_f, w_i\}) \geq 0$ . If  $\overline{W} \not\subseteq U'$ , then  $(U' \setminus \{w_f, w_i\}) \cup W \neq V(G)$ , so Lemma 4.6 implies  $\rho_{s,G}((U' \setminus \{w_f, w_i\}) \cup W) \geq 1$ . Thus,  $\rho_{s,G'}(U') \geq 1 - 3 + 0 = -2$ . If  $\overline{W} \subseteq U'$ , then the minimality of  $\rho_{s,G'}(U')$  implies that  $w_f \in U'$ , so  $\rho_{s,G'}(U' \cap \{w_f, w_i\}) = 3$ . By hypothesis  $\rho_{s,G}((U' \setminus \{w_f, w_i\}) \cup W) \geq -4$ . So  $\rho_{s,G'}(U') \geq -4 - 3 + 3 = -4$ .

Now assume that  $G'$  contains a subgraph  $H' \in \mathcal{H}$ . Because  $G$  is a minimal counterexample,  $G$  is nb-critical, so  $H' \not\subseteq G$ , which implies  $V(H') \cap \{w_i, w_f\} \neq \emptyset$ . Now

$$\begin{aligned} \rho_{s,G}((V(H') \setminus \{w_f, w_i\}) \cup W) &\leq (\rho_{s,H'}(V(H')) - 8) + \rho_{s,G}(W) \\ &\leq \rho_{s,H'}(V(H')) - 5. \end{aligned} \tag{5}$$

**Case 1:  $H' \notin \{K_4, M_7\}$ .** By Corollary 3.8(iii), we know that  $\rho_{s,H}(V(H')) \leq 0$ . This implies that  $\rho_{s,G}((V(H') \setminus \{w_i, w_f\}) \cup W) \leq -5$ , which contradicts that  $G$  is a counterexample.

For Cases 2 and 3, we will use the following fact. Let  $U = (V(H') \setminus \{w_f, w_i\}) \cup W$ . By Corollary 3.8(i),  $\rho_{s,H}(V(H')) \leq 2$ , so inequality (5) gives  $\rho_{s,G}(U) \leq -3$ . Now Lemma 4.6 implies that  $U = V(G)$ .

**Case 2:  $H' = M_7$ .** If  $V(H') \supset \{w_f, w_i\}$ , then inequality (5) improves to  $\rho_{s,G}(U) \leq \rho_{s,H'}(V(H')) - 13$ . So  $\rho_{s,G}(U) \leq 2 - 13 = -11$ , a contradiction. Instead assume that  $|V(H') \cap \{w_f, w_i\}| = 1$ . For ease of notation,

let  $\{w_*\} = V(H') \cap \{w_f, w_i\}$ . Note that each vertex in  $M_7$  is in a copy of  $K_4 - e$ . Let  $x, y, z$  be vertices in  $H'$  such that  $H[\{x, y, z, w_*\}]$  is  $K_4 - e$ . So  $\rho_{s,G}(W \cup \{x, y, z\}) = \rho_{s,G}(W) + 8(3) - 5(5) < \rho_{s,G}(W)$ , Since  $0 < |W \cup \{x, y, z\}| \leq |V(G)| - 3$ , this contradicts the minimality of  $\rho_{s,G}(W)$ .

**Case 3:  $H' = K_4$ .** Because  $w_f$  and  $w_i$  are not adjacent (if they both exist),  $|V(H') \cap \{w_i, w_f\}| = 1$ . So  $G[\overline{W}] = K_3$  and each vertex of  $\overline{W}$  has one edge into  $W$ . By Lemma 4.22, either  $\overline{W}$  contains a precolored vertex or else is incident to an edge-gadget. In each case, the above inequality  $\rho_{s,G}(U) \leq \rho_{s,H'}(V(H')) - 5$  improves to  $\rho_{s,G}(U) \leq \rho_{s,K_4}(V_{K_4}) - 10 \leq -8$ , which contradicts that  $G$  is a counterexample.  $\square$

The previous lemma gives the following three easy corollaries. The first is analogous to Lemma 4.10, but now we can add a vertex to  $I_p$ . The third slightly extends Lemma 4.22.

**Lemma 4.25.** *If  $W \subset V(G)$  such that  $|W| \leq |V(G)| - 2$  and  $w \in W$ , then  $G[W]$  has an nb-coloring that extends the precoloring  $I_p \cup \{w\}, F_p$ .*

*Proof.* Let  $G' = G[W]$  with precoloring  $I_p \cup \{w\}, F_p$ . Each  $U \subseteq W$  satisfies  $\rho_{G',s}(U) = \rho_{G,s}(U) - 8 \geq 4 - 8 = -4$ , so  $G'$  has the desired coloring by the Main Theorem.  $\square$

**Lemma 4.26.** *Each vertex in  $G$  is incident to at most one edge-gadget.*

*Proof.* If, to the contrary, some  $v$  is incident to edge-gadgets with endpoints  $w$  and  $x$ , then  $\rho_{s,G}(\{v, w, x\}) \leq 8(3) - 11(2) = 2$ , which contradicts Lemma 4.24. (A short case analysis shows that  $|V(G)| \geq 5$ .)  $\square$

**Lemma 4.27.** *If  $C = x_1 \cdots x_k$  is a cycle in  $G$  induced by 3-vertices, then at least one of the following holds:*

- (i)  $k \geq 6$ , or
- (ii) some  $x_i$  is incident to an edge-gadget, or
- (iii)  $V(C) \cap F_p \neq \emptyset$ .

*Proof.* The proof is nearly identical to the case  $k = 3$  in the proof of Lemma 4.22. Let  $\{z_i\} = N(x_i) \setminus \{x_{i-1}, x_{i+1}\}$  for all  $i$ . By Remark 3.5 and symmetry, assume  $z_1 \neq z_2$ . If we let  $G' = G(C, z_1, z_2)$ , then the only difference is in proving that  $G'$  has an nb-coloring. For each  $U \subseteq V(G')$  with  $|U| \geq 2$ , Lemma 4.24 gives  $\rho_{s,G'}(U) \geq \rho_{s,G}(U) - 6 \geq 4 - 6 = -2$ . So  $G'$  has an nb-coloring by the Main Theorem.  $\square$

We now prove that  $\delta(G) \geq 3$ , which will be helpful for our discharging argument in the next subsection.

**Lemma 4.28.**  $\delta(G) \geq 3$ .

*Proof.* Suppose, to the contrary, that some  $v \in V(G)$  has  $d(v) \leq 2$ . Lemma 4.4 implies that  $d(v) = 2$ . Lemma 4.5 shows that either  $v \in F_p$  or  $v$  is incident to an edge-gadget, and Lemma 4.9 implies that  $v$  cannot satisfy not both. Let  $N(v) = \{w_1, w_2\}$ .

**Case 1:  $v \in U_p$  and  $vw_1$  is an edge-gadget.** By Lemma 4.26,  $vw_2$  is an edge and not an edge-gadget. Let  $G' = G - v$ . By Lemma 4.10,  $G'$  has an nb-coloring  $I', F'$  with  $w_2 \in F'$ . To extend  $I', F'$  to  $G$ , we color  $v$  with the color unused on  $w_1$ . This contradicts that  $G$  is a counterexample.

So now assume that  $v \in F_p$ , and both  $vw_1, vw_2$  are edges and not edge-gadgets. Note that  $w_1$  and  $w_2$  are both uncolored, since otherwise  $\rho_{s,G}(\{v, w_i\}) = 3(2) - 5 = 1$ , which contradicts Lemma 4.24.

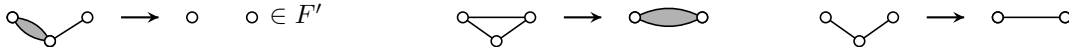


Figure 8: Constructing  $G'$  from  $G$  for Cases 1, 2, and 3 in the proof of Lemma 4.28.

**Case 2:  $w_1 \in N(w_2)$ .** We form  $G'$  from  $G - v$  by replacing  $w_1 w_2$  with an edge-gadget (if it is not already an edge-gadget); This is analogous to our earlier construction of  $G(C, z_1, z_2)$ . To extend any nb-coloring  $I', F'$  of  $G'$  to  $G$ , we simply add  $v$  to  $F'$ . Because  $G'$  is smaller than  $G$ , by minimality  $G'$  must contain a forbidden subgraph or a vertex set  $W'$  such that  $\rho_{s,G'}(W') \leq -5$ .

By hypothesis,  $G$  contains no forbidden subgraph, and by construction graphs in  $\mathcal{H}$  have no edge-gadgets. So  $G'$  contains no forbidden subgraph. To reach a contradiction, we show that  $\rho_{s,G'}(W') \geq -4$  for all  $W' \subseteq V(G')$ . If  $\{w_1, w_2\} \not\subseteq W'$ , then  $\rho_{s,G'}(W') = \rho_{s,G}(W')$ ; and if  $\{w_1, w_2\} \subseteq W'$ , then

$$\rho_{s,G'}(W') \geq \rho_{s,G}(W' \cup \{v\}) - 6 - 3 + 5(2) \geq \rho_{s,G}(W' \cup \{v\}) + 1 \geq -3.$$

**Case 3:  $w_1 \notin N(w_2)$ .** We form  $G'$  from  $G - v$  by adding edge  $w_1w_2$ . If  $G'$  has an nb-coloring  $I', F'$ , then we can extend it to  $G$  by adding  $v$  to  $F'$ . So we assume  $G'$  has no nb-coloring. By construction,  $G'$  is smaller than  $G$ . So by minimality  $G'$  has a forbidden subgraph or contains a vertex subset  $W'$  such that  $\rho_{s,G'}(W') \leq -5$ . Similar to Case 2,

$$\rho_{s,G'}(W') \geq \rho_{s,G}(W' \cup \{v\}) - 5 - 3 + 10 \geq \rho_{s,G}(W' \cup \{v\}) + 2 \geq -2.$$

So  $G'$  must contain a forbidden subgraph.

By definition, this implies that  $w_1$  and  $w_2$  are linked via some subgraph  $H$ . By Lemma 4.21 they are specially-linked. Corollary 3.8(i) implies that

$$\rho_{s,G}(V(H) \cup \{v\}) \leq \rho_{s,H}(V(H)) + 3 - 5 \leq 0. \quad (6)$$

Lemma 4.6 shows that  $V(G) = V(H) \cup \{v\}$ . Further,  $H$  is an induced subgraph and no vertex in  $V(G) \setminus \{v\}$  is precolored; otherwise inequality (6) can be strengthened by 5, which gives an outright contradiction.

It is straightforward to check that if  $H \in \{K_4, W_5, J_7, J_{12}\}$  and  $w_1w_2 \in E(H)$ , then  $H - w_1w_2$  has an nb-coloring  $I', F'$  with  $\{w_1, w_2\} \subseteq I'$ . So  $H$  must contain a cycle  $C = x_1, \dots, x_k$  as in Definition 3.4. By Lemma 4.27,  $G$  contains no instance of  $C$  as in Definition 3.4. So there exists  $j$  such that either  $w_1w_2 = x_jz_j$  or else  $w_1w_2 = x_jx_{j+1}$ . Thus,  $H - C$  is an induced subgraph of  $G$ , so it has an nb-coloring  $I', F'$ .

**Case 3.a:  $w_1w_2 = x_jz_j$ .** By symmetry, assume  $j = 1$ . By Lemma 4.25,  $H - C$  has an nb-coloring  $I', F'$  with  $z_1 \in I'$ . To extend  $I', F'$  to  $G$ , let  $I = I' \cup \{x_1\}$  and  $F' = F' \cup \{v\} \cup \{x_2, \dots, x_k\}$ .

**Case 3.b:  $w_1w_2 = x_jx_{j+1}$ .** By symmetry, assume  $j = k - 1$ . By Lemma 4.10, we assume  $z_1 \in F'$ . By Lemma 3.2 and Definition 3.4(ii.c), we assume that  $\{z_1, \dots, z_k\} \subseteq F'$ . To extend  $I', F'$  to  $G$ , if  $k$  is even, then let  $I = I' \cup \bigcup_{i=1}^{k/2} x_{2i}$  and  $F = F' \cup \{v\} \cup \bigcup_{i=1}^{k/2} x_{2i-1}$ . If  $k$  is odd, then let  $I = I' \cup \{x_k\} \cup \bigcup_{i=1}^{(k-1)/2} x_{2i}$  and  $F = F' \cup \{v\} \cup \bigcup_{i=1}^{(k-1)/2} x_{2i-1}$ .  $\square$

Now we can show that the uncolored 3-vertices, with no incident edge-gadgets, induce a forest. We extend the ideas of Lemma 4.27 to all finite  $k$ .

**Lemma 4.29.** *If  $C = x_1 \cdots x_k$  is a cycle in  $G$  induced by 3-vertices, then at least one of the following holds:*

- (i) *some  $x_i$  is incident to an edge-gadget or*
- (ii)  *$V(C) \cap F_p \neq \emptyset$ .*

*Proof.* Suppose, to the contrary, that  $x_1 \cdots x_k$  satisfies the hypotheses, but both possible conclusions fail. By Lemma 4.27,  $k \geq 6$ . Let  $W = V(G) - C$  and let  $\{z_i\} = N(x_i) - C$ . If  $k$  is even, then  $G[W]$  has an nb-coloring by minimality, and we can extend it to  $G$  by Lemma 4.19. Thus, we assume that  $k$  is odd; so  $k \geq 7$ .

If  $z_1 = \cdots = z_k$ , then  $\rho_{s,G}(V(C) \cup \{z_1\}) = 8(k+1) - 5(2k) = 8 - 2k \leq -6$ , which is a contradiction. Thus, the set  $\{z_1, \dots, z_k\}$  contains at least two distinct vertices. Our plan for the rest of the proof is similar to the first sentence of this paragraph. We will find a subset  $V_{J_\ell^*}$  of  $W$  that contains all  $z_i$  and such that  $\rho_{s,G}(V_{J_\ell^*}) \leq 7$ . (We will show that each distinct pair  $z_i, z_{i+1}$  is linked, and let  $V_{J_\ell^*}$  be the vertices of the union of their linking subgraphs.) This implies that  $\rho_{s,G}(V_{J_\ell^*} \cup C) \leq \rho_{s,G}(V_{J_\ell^*}) + 8k - 5(2k) \leq 7 - 2k \leq -7$ , which is a contradiction. So it remains to find this  $V_{J_\ell^*}$  and prove that  $\rho_{s,G}(V_{J_\ell^*}) \leq 7$ .

Suppose there exists  $j \in \{1, \dots, k\}$  such that  $z_j \neq z_{j+1}$  and  $z_j$  and  $z_{j+1}$  are not linked. Let  $G' = G(C, z_j, z_{j+1})$ . Note that  $\rho_{s,G'}(U) \geq \rho_{s,G}(U) - 6 \geq 4 - 6 = -2$  for all  $U \subseteq W$ . Since  $z_j$  and  $z_{j+1}$  are not linked,  $G'$  contains no forbidden subgraphs. So, by minimality,  $G'$  has an nb-coloring,  $I', F'$ . And by Lemma 4.19, we can extend  $I', F'$  to  $G$ . Thus, for each  $j$  with  $z_j \neq z_{j+1}$ , we know that  $z_j$  and  $z_{j+1}$  are linked. By Lemma 4.21, in fact they are specially-linked. Let  $L = \{j : 1 \leq j < k, z_j \neq z_{j+1}\}$ . As shown above,  $L \neq \emptyset$ . For each  $j \in L$ , let  $H_j$  denote the subgraph of  $G[W]$  that links  $z_j$  with  $z_{j+1}$ .

**Claim 4.30.** For each  $U \subseteq V(H_j)$ , we have  $\rho_{s,G}(V(H_j)) \leq \rho_{s,G}(U)$ .

Proof. Let  $\tilde{H}_j$  be the graph in  $\mathcal{H}$  formed from  $H_j$  by adding edge  $z_j z_{j+1}$ . We note that  $\rho_{s,G}(H_j) = \rho_{s,\tilde{H}_j}(V_{\tilde{H}_j}) + 5$ , and we consider the possibilities for  $\rho_{s,\tilde{H}_j}(V_{\tilde{H}_j})$ . If  $\rho_{s,\tilde{H}_j}(V_{\tilde{H}_j}) \leq -1$ , then  $\rho_{s,G}(H_j) \leq 4$ . By the gap lemma,  $\rho_{s,G}(U) \geq 4$ . If  $\rho_{s,\tilde{H}_j}(V_{\tilde{H}_j}) \in \{0, 1\}$ , then  $\rho_{s,G}(H_j) \leq 6$ . By Corollary 3.8(iv),  $\rho_{s,G}(U) \geq 6$ . Finally, assume that  $\rho_{s,\tilde{H}_j}(V_{\tilde{H}_j}) \geq 2$ . This means that  $\tilde{H}_j = K_4$ . The proper, induced, non-trivial subgraphs of  $K_4$  are  $K_1, K_2, K_3$ , which have potentials 8, 11, 9. This proves the claim.  $\diamond$

Denote the elements of  $L$  by  $j_1^*, \dots, j_\ell^*$ . Let  $J_0 = \{z_1\}$ , and for each  $j_i^*$ , let  $J_j = \cup_{j' \leq j, j' \in L} H_{j'}$ . Since  $z_j \in V(H_j \cap J_{j-1})$ , it is a non-empty subset of  $V(H_j)$ . So the previous claim implies that  $\rho_{s,G}(V(J_{t-1}) \cap V(H_t)) \geq \rho_{s,G}(V(H_t))$  for all  $t$ . Now Lemma 4.1 implies

$$\rho_{s,G}(V(J_t)) = \rho_{s,G}(V(J_{t-1}) \cup V(H_t)) \leq \rho_{s,G}(V(J_{t-1})) + \rho_{s,G}(V(H_t)) - \rho_{s,G}(V(J_{t-1}) \cap V(H_t)) \leq \rho_{s,G}(V(J_{t-1})).$$

By applying this inequality for each  $t \in \{2, \dots, \ell\}$ , we get

$$\rho_{s,G}(V(J_\ell)) \leq \rho_{s,G}(V(J_{j_1^*})) = \rho_{s,G}(V(H_{j_1^*})) \leq 7,$$

which completes the proof.  $\square$

## 4.4 Simple Graphs: Discharging and Finishing the Coloring

In this section we continue our proof that our counterexample  $G$  is “well-behaved”; we ultimately construct an nb-coloring of  $G$ , which contradicts that  $G$  is a counterexample.

### 4.4.1 Discharging to Force Structure

Let  $d'(v)$  denote the degree of vertex  $v$ , when we count each edge-gadget as contributing 2 to the degree of each endpoint. *Throughout this section whenever we write degree we mean  $d'$ .*  $d'(v)$

Let  $L$  denote the set of vertices in  $G$  that are uncolored, degree 3, and not incident to any edge-gadget ( $L$  is for low degree, or little risk). Let  $B = V(G) \setminus L$  (here  $B$  is for bigger degree, or bigger risk). Let  $B_j \subset B$  denote the set of vertices in  $G$  that are uncolored, degree  $j$ , and not incident to any edge-gadget.  $L$   
 $B_j$   
Let  $B_j^{(eg)} \subset B$  denote the set of vertices in  $G$  that are degree  $j$  and incident to an edge-gadget. Let  $B_j^{(f)} \subset B$  denote the set of vertices in  $G$  that are degree  $j$  and in  $F_p$ . By Proposition 4.9 each vertex  $v$  is incident to at most one edge-gadget, and not incident to an edge-gadget at all when  $v \in F_p$ . That is,  $B_j^{(eg)} \cap B_j^{(f)} = \emptyset$   $B_j^{(eg)}$   
 $B_j^{(f)}$   
for each  $j$ . Let  $B_* = B \setminus (B_4 \cup B_5 \cup B_4^{(eg)} \cup B_5^{(eg)} \cup B_3^{(f)})$ . We will use discharging to show that nearly all of  $V(G)$  is contained in  $L \cup B_4$  and that  $G[B_4]$  has very few edges. In particular, we will show that  $B_* = \emptyset$ .  $B_*$   
Our idea is to assign charges to  $V(G) \cup E(G)$  that sum to at most 4, and to discharge so that every vertex and edge has nonnegative charge, but each vertex outside  $L \cup B_4$  has positive charge.

We recall a few useful facts. By Lemma 4.28,  $\delta(G) \geq 3$ . By Lemma 4.29,  $G[L]$  is a forest. By Lemma 4.3,  $I_p = \emptyset$ , and by hypothesis  $\rho_{s,G}(V_g) \geq -4$ .

We assign to each vertex  $v$  and edge  $e$  a charge, denoted  $\text{ch}(v)$  or  $\text{ch}(e)$  as follows. For each vertex  $v \in U_p$ , let  $\text{ch}(v) = 2.5d'(v) - 8$ , and for each  $v \in F_p$ , let  $\text{ch}(v) = 2.5d'(v) - 3$ . For each edge-gadget  $e$ , let  $\text{ch}(e) = 1$ . (Each edge  $e$  that is not an edge-gadget has  $\text{ch}(e) = 0$ .) The sum of these initial charges is  $\text{ch}(v)$   
 $\text{ch}(e)$

$$\begin{aligned} \sum_{v \in U_p} 5d'(v)/2 - 8 + \sum_{v \in F_p} 5d'(v)/2 - 3 + e''(V(G)) &= -8|U_p| - 3|F_p| + 5e'(V(G)) + 11e''(V(G)) \\ &= -\rho_{s,G}(V(G)) \\ &\leq 4. \end{aligned}$$

We use only a single discharging rule, and write  $\text{ch}^*$  for the charges after applying it.  $\text{ch}^*$

(R1) Each vertex  $v \in B$  gives  $1/2$  to each incident 3-vertex and gives  $1/2$  to each edge with its other endpoint in  $B$  (which means giving  $2/2$  to each incident edge-gadget).

Now we show that each vertex and edge ends with nonnegative charge. Note that each edge-gadget  $e$  has  $\text{ch}^*(e) = 1 + 4(1/2) = 3$  since, by definition, both its endpoints are in  $B$ . Further, each edge  $e$  induced by  $B$  has  $\text{ch}^*(e) = 0 + 2(1/2) = 1$ . For each tree  $T$  of  $G[L]$ , we compute the charge of the entire tree (the sum of the charges of its vertices), showing it is at least 1. Let  $k = |V(T)|$ . The number of edges with exactly one endpoint in  $T$  is  $3k - 2(k - 1) = k + 2$ . Note that  $\text{ch}(T) = k((5/2)3 - 8) = -k/2$ . So  $\text{ch}^*(T) = -k/2 + (k + 2)/2 = 1$ .

Now we consider vertices in  $B$ . If  $v \in F_p$ , then  $\text{ch}^*(v) = 5d'(v)/2 - 3 - d'(v)/2 = 2d'(v) - 3 \geq 3$ . Recall that  $\delta(G) \geq 3$ ; that is, each vertex has at least 3 neighbors (excluding multiplicity for edge-gadgets). So if  $v$  is incident to an edge-gadget, then  $d'(v) \geq 4$ . Thus, if  $v \in U_p \cap B$ , then  $d'(v) \geq 4$ . Hence, if  $v \in U_p \cap B$ , then  $\text{ch}^*(v) = 5d'(v)/2 - 8 - d'(v)/2 = 2d'(v) - 8 \geq 0$ .

Let  $\ell$  denote the number of components in  $L$ . Recall that  $e'(B)$  and  $e''(B)$  denote, respectively, the number of edges in  $G[B]$  that are not edge-gadgets, and are edge-gadgets. Our observations imply that

$$\ell + e'(B) + 3e''(B) + 2|B_5| + 2|B_5^{(eg)}| + 3|B_3^{(f)}| + 4|B_*| \leq 4. \quad (7)$$

In Lemma 4.31 we use (7) to greatly restrict the structure of  $G$ . For the proof we will use a key lemma about extending nb-colorings of  $G[B]$  to all of  $G$ . To keep the flow of our presentation, we state the lemma now, but defer its proof a bit longer.

**Lemma 4.34 (Rephrased).** For a graph  $G$ , let  $\varphi'$  be a coloring of some  $W \subseteq V(G)$  such that  $\varphi'$  is an nb-coloring of  $G[W]$ , and such that  $G - W$  is a forest in which each vertex has degree 3 in  $G$ . We can extend  $\varphi'$  to an nb-coloring of  $G$  whenever each component  $T$  of the forest has either (i) a leaf with no neighbors in  $W$  colored  $F$  or (ii) an odd number of incident edges leading to neighbors in  $W$  colored  $F$ .

**Lemma 4.31.**  $\ell + e'(B) \leq 4$ ,  $\ell \geq 1$ ,  $e'(B) \geq 1$ , and  $V(G) = L \cup B_4$ .

*Proof.* The first inequality follows directly from (7). Next we recall that  $\delta(G) \geq 3$ , which implies  $|V(G)| \geq 4$ ; combining these inequalities yields  $|E(G)| \geq 6$ . Since (7) implies  $e(B) \leq 4$ , we must have  $L \neq \emptyset$ . That is,  $\ell \geq 1$ . Since  $\ell \geq 1$ , note that (7) implies  $B_* = \emptyset$ . Further, if  $|B_5^{(eg)}| \geq 1$ , then (7) fails, since  $e''(B) \geq 1$ , so  $1 + 3 + 2 \not\leq 4$ ; thus,  $B_5^{(eg)} = \emptyset$ .

All that remains is to show that  $V(G) = L \cup B_4$ . This will imply  $e'(B) \geq 1$ , since otherwise we can color  $B$  with  $I$  and color  $L$  with  $F$ . Since  $B_* = \emptyset$  and  $B_5^{(eg)} = \emptyset$ , to show that  $V(G) = L \cup B_4$ , we will show that  $B_3^{(f)} = \emptyset$ ,  $B_4^{(eg)} = \emptyset$ , and  $B_5 = \emptyset$ .

Suppose that  $B_3^{(f)} \neq \emptyset$ . Inequality (7) implies that  $e'(B) + e''(B) = 0$ , and  $|B_3^{(f)}| = 1$ . Let  $w$  denote the vertex in  $B_3^{(f)}$ , and let  $v_1, v_2, v_3$  denote the neighbors of  $w$ . Since  $e'(B) + e''(B) = 0$ , each  $v_i$  is in  $L$ . Let  $T'$  denote the subgraph of  $T$  that is the union of the three paths with endpoints in  $\{v_1, v_2, v_3\}$ . Either  $T'$  is a subdivision of  $K_{1,3}$  or else  $T'$  is a path. In the first case, let  $x$  denote the vertex of degree 3 in  $T'$ . Now we let  $I = B \cup \{x\} \setminus \{w\}$  and  $F = L \cup \{w\} \setminus \{x\}$ . In the second case, some  $v_i$  has degree 2 in  $T'$ ; by symmetry, say it is  $v_2$ . Now let  $I = B \cup \{v_2\} \setminus \{w\}$  and  $F = L \cup \{w\} \setminus \{v_2\}$ . Thus, we must have  $B_3^{(f)} = \emptyset$ .

Suppose that  $B^{(eg)} \neq \emptyset$ , which implies that  $e''(B) \geq 1$ . Now (7) implies  $e''(B) = 1$ ,  $e'(B) = 0$ ,  $\ell = 1$ , and  $V(G) = L \cup B_4 \cup B_4^{(eg)}$ . Let  $\tilde{B}$  denote the 2 endpoints of the edge-gadget. Since  $\ell = 1$ , let  $T = G[L]$ . If  $T$  has at least three leaves, then one of them, call it  $v$ , has a neighbor not in  $\tilde{B}$ . Choose  $w \in \tilde{B}$  such that  $v \notin N(w)$ . Let  $F = \{w\}$  and  $I = B \setminus \{w\}$ . Since  $v$  has two neighbors in  $B$  colored  $I$ , we can extend the coloring to  $G$  by Lemma 4.34 (Rephrased), part (i). Thus, we assume  $T$  has only two leaves; that is,  $T$  is a path. Further, we assume that each leaf of  $T$  is adjacent to both vertices in  $\tilde{B}$ , since otherwise the argument above still works. Since  $G$  has no copy of  $K_4$ , the path  $T$  is longer than a single edge. So  $B_4 \supsetneq \tilde{B}$ . Let  $v$  denote a vertex of  $\tilde{B}$  and  $w$  a vertex in  $B_4 \setminus \tilde{B}$ . Let  $I = B \setminus \{v, w\}$ .

Let  $z_1, \dots, z_4$  denote the neighbors of  $w$  along the path  $T$  (in order). Let  $I = (B \setminus \{v, w\}) \cup \{z_1, z_3\}$  and  $F = (L \setminus \{z_1, z_3\}) \cup \{v\}$ . It is easy to check that  $I, F$  is an nb-coloring of  $G$ . Thus,  $e''(B) = 0$ , which implies  $B_4^{(eg)} = \emptyset$ .

Finally, suppose  $B_5 \neq \emptyset$ . Now (7) implies  $|B_5| = 1$ ,  $e'(B) = 1$ , and  $\ell = 1$ . So let  $T = G[L]$ . Let  $e$  denote the edge induced by  $B$  and let  $x$  denote an endpoint of  $e$  with  $d(x) = 4$ . The only edges incident to  $T$  with an endpoint colored  $F$  are the 3 edges incident to  $x$  (other than  $e$ ). So we can extend the nb-coloring of  $B$  to  $V(G)$  by Lemma 4.34(ii). This shows that  $B_5 = \emptyset$ , which completes the proof of the lemma.  $\square$

#### 4.4.2 Why the Theorem We Prove Must Be Sharp

In Section 4.4.4 we will show that if a graph  $G$  satisfies  $\delta(G) = 3$ ,  $\Delta(G) = 4$ , has its vertices of degree 3 induce a forest with  $\ell$  components, and has at most  $4 - \ell$  edges with both endpoints of degree 4, then either  $G$  (i) is near-bipartite, (ii) contains a subgraph isomorphic to  $M_7$ , or (iii) is  $J_7$  or  $J_{12}$ . In Section 4.4.3 we prove several lemmas that help us find nb-colorings. Even with these tools, Section 4.4.4 consists of a long, technical case analysis. So, before we continue, we should explain why Section 4.4.4 is essential.

Our case analysis would be greatly reduced if we could instead assume that  $\ell + e'(B) \leq 3$ , and it would be nearly trivial if  $\ell + e'(B) \leq 2$ . These assumptions correspond to the moderately weaker result that  $G$  is near-bipartite whenever all  $W \subseteq V(G)$  satisfy  $\rho_{s,G}(W) \geq -3$  (respectively  $\rho_{s,G}(W) \geq -2$ ). The work in Section 4.4.4 is necessary because such modifications would make our work up to this point more difficult, bordering on impossible.

The technique that we use—letting  $G$  be a minimum counterexample—is akin to a proof by induction. A weaker theorem provides a weaker inductive hypothesis<sup>8</sup>. The gaps in the gap lemmas ( $1 - (-4) = 5$  and  $4 - (-4) = 8$ ) correspond to the decreases in potential resulting from precoloring a single vertex ( $8 - 3 = 5$  and  $8 - 0 = 8$ ). The latter values would not change by altering the statement of the Main Theorem. If we merely had the weaker inductive hypothesis that graphs smaller than  $G$  with potential at least  $-3$  are near-bipartite, then our first gap lemma (Lemma 4.6) would be insufficient to precolor a vertex (Lemma 4.10). But we cannot delay proving Lemma 4.10 until after a larger gap is proved *precisely because* Lemma 4.10 is used in the proofs of the stronger gap lemmas (Lemmas 4.11 and 4.24).

#### 4.4.3 Coloring Lemmas

In the previous lemma we showed that  $V(G) = L \cup B_4$ . Further,  $\ell \geq 1$ ,  $e'(B) \geq 1$ , and  $\ell + e'(B) \leq 4$ . In Section 4.4.4, we will show how to color  $G$ . Our main tools will be Lemmas 4.34 and 4.35, which allow us to extend partial nb-colorings to components of  $G[L]$ . To prove the first of these, we use the following technical result. Let  $S_1 \uplus S_2$  denote the disjoint union of sets  $S_1$  and  $S_2$ . When vertices  $v$  and  $w$  are adjacent we write  $v \leftrightarrow w$ , and otherwise  $v \not\leftrightarrow w$ .

$S_1 \uplus S_2$   
 $v \leftrightarrow w$   
 $v \not\leftrightarrow w$

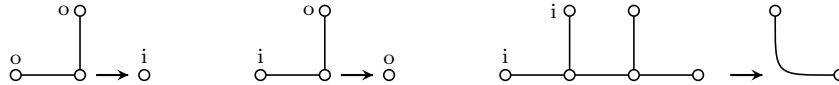


Figure 9: The induction step in the proof of Lemma 4.32.

**Lemma 4.32.** *Let  $T$  be a tree in which each non-leaf vertex has degree 3. Let  $S_{in} \uplus S_{out}$  be a partition of the leaves of  $T$ . If  $|S_{out}|$  is odd, then  $T$  has an independent set  $S$  such that  $S_{in} \subseteq S$  and  $S_{out} \cap S = \emptyset$ , and also each component of  $T - S$  contains at most one leaf of  $T$ .*

*Proof.* Let  $k$  denote the number of leaves in  $T$ . Our proof is by induction on  $k$ . The base case is when  $k = 3$ , so  $T = K_{1,3}$ . If all leaves are in  $S_{out}$ , then we take  $S$  to be the center vertex. Otherwise, one leaf is in  $S_{out}$  and the other two are in  $S_{in}$ , so we take  $S$  to consist of the two leaves in  $S_{in}$ .

Now suppose that  $k \geq 4$ . The number of non-leaf vertices in  $T$  is  $k - 2$ , and each of these has at most two leaf neighbors. By Pigeonhole, some non-leaf vertex  $v$  has exactly two leaf neighbors, say  $w_1$  and  $w_2$ . If  $w_1, w_2 \in S_{out}$ , then we apply induction to  $T - \{w_1, w_2\}$ , with leaf partition  $S'_{in} \uplus S'_{out}$ , where  $S'_{in} = S \cup \{v\}$  and  $S'_{out} = S_{out} \setminus \{w_1, w_2\}$ . If  $|\{w_1, w_2\} \cap S_{out}| = 1$ , then we assume  $w_1 \in S_{out}$  (by symmetry) and let

<sup>8</sup>Leading to a dictum of Douglas West, “If you can’t prove something, try proving something harder!”



$T' = T - \{w_1, w_2\}$ . We apply induction to  $T'$  with  $S'_{out} = (S_{out} \setminus \{w_1\}) \cup \{v\}$  and  $S'_{in} = S_{in} \setminus \{w_2\}$ , and let  $S'$  be the guaranteed independent set. Let  $S = S' \cup \{w_2\}$ . Finally, suppose that  $w_1, w_2 \in S_{in}$ . Now let  $x$  denote the third neighbor of  $v$ . Form  $T'$  from  $T - \{v, w_1, w_2\}$  by suppressing  $x$ . Let  $S'_{out} = S_{out}$  and  $S'_{in} = S_{in} \setminus \{w_1, w_2\}$ . Given the independent set  $S'$  for  $T'$  by induction, let  $S = S' \cup \{w_1, w_2\}$ .  $\square$

Recall, from Lemma 4.31, that  $V(G) = L \cup B_4$  and that  $G[L]$  is a forest.

**Definition 4.33.** Fix an nb-coloring  $\varphi'$  of  $G[B]$ . Now each edge from a vertex of  $L$  to a vertex of  $B$  colored  $F$  is an  $F$ -edge (an  $I$ -edge is defined analogously). We say that the  $F$ -edges incident to a component  $T$  of  $G[L]$  are  $F$ -edges belonging to  $T$ . A component  $T$  of  $G[L]$  is  $F$ -odd (resp.  $F$ -null) if its number of  $F$ -edges is odd (resp. 0). Further,  $T$  is  $F$ -leaf-good if some leaf of  $T$  has two neighbors in  $B$  colored  $I$ . (If  $T$  is  $F$ -null, then clearly  $T$  is  $F$ -leaf-good.)

$F$ -edge  
 $F$ -odd  
 $F$ -null  
  
 $F$ -leaf-good

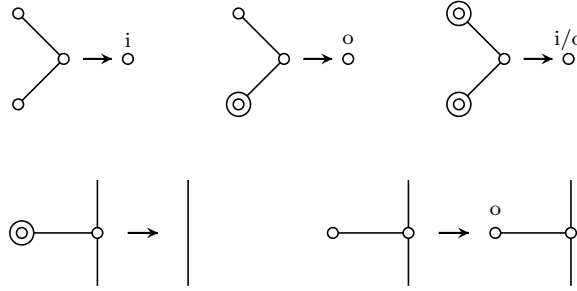


Figure 10: The construction of tree  $T'$  in the proof of Lemma 4.34.

**Lemma 4.34.** For a graph  $G$ , let  $\varphi'$  be a coloring of some  $W \subseteq V(G)$  such that  $\varphi'$  is an nb-coloring of  $G[W]$ , and such that  $G \setminus W$  is a forest in which each vertex has degree 3 in  $G$ . We can extend  $\varphi'$  to an nb-coloring of  $G$  whenever each component  $T$  of the forest is either (i)  $F$ -odd or (ii)  $F$ -leaf-good.

*Proof.* Suppose that  $G$ ,  $W$ , and  $\varphi'$  satisfy the hypotheses. Let  $T$  be a (tree) component of  $G - W$ . We show how to extend  $\varphi'$  to  $V(T)$  so that no two of its vertices with incident  $F$ -edges are linked by a path in  $T$  entirely colored  $F$ .

From  $T$  we form a new tree  $T'$ , and leaf partition  $S_{in} \uplus S_{out}$ , as follows. When a non-leaf  $v$  of  $T$  has an incident  $I$ -edge, we suppress  $v$ . When a non-leaf  $v$  of  $T$  has an incident  $F$ -edge, we add a leaf  $w_v$  incident to  $v$  and add  $w_v$  to  $S_{out}$ . When a leaf  $v$  of  $T$  has two incident  $F$ -edges, we add  $v$  to  $S_{in}$ . When a leaf  $v$  has both an incident  $I$ -edge and an incident  $F$ -edge, we add  $v$  to  $S_{out}$ . Now consider leaves of  $T$  with two incident  $I$ -edges (if such leaves exist). For all but one of these, say  $w$ , we add them to  $S_{in}$  or  $S_{out}$  arbitrarily. Finally, we add  $w$  to either  $S_{in}$  or  $S_{out}$  so that  $|S_{out}|$  is odd. Under both hypotheses (i) and (ii), we get that  $|S_{out}|$  is odd.

Now we invoke Lemma 4.32, to find an independent set  $S$  such that  $S_{in} \subseteq S$  and  $S_{out} \cap S = \emptyset$ , and also each component of  $T - S$  contains at most one leaf of  $T'$ . We color each vertex of  $S$  with  $I$ , except for leaves of  $T$  with two incident  $I$ -edges. It is easy to check that no two vertices of  $T$  with incident  $F$ -edges are linked by a path in  $T$  all colored  $F$ .  $\square$

**Lemma 4.35.** Let  $G$  be a tree or else be connected and have a single cycle  $C$ , which is not a 3-cycle. Form  $G'$  from  $G$  by adding a new vertex  $v$  and making  $v$  adjacent to at most four vertices in  $V(G)$ , at least one of which is on  $C$ , if  $C$  exists. Now  $G'$  has a near bipartite coloring  $I, F$  with  $I \subseteq N_{G'}(v)$ .

*Proof.* First suppose that  $G$  is a tree. If at least  $d_{G'}(v) - 1$  neighbors of  $v$  induce an independent set, then we color them with  $I$  and color the rest of  $G'$  with  $F$ . If this is not the case, then  $d_{G'}(v) = 4$  and the four neighbors of  $v$  induce either 2 or 3 edges. In each case, we can color two of these neighbors with  $I$  and the rest of  $G'$  with  $F$ .

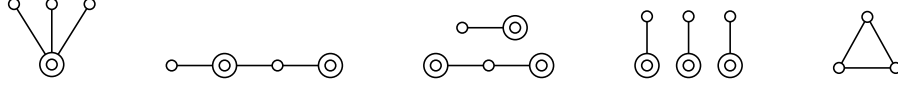


Figure 11: The five possibilities for  $G[\tilde{B}]$  when  $e(B) = 3$ .

So assume instead that  $G$  has a cycle,  $C$ . Let  $S = N_{G'}(v)$ . Our goal is again to use color  $I$  on some independent set  $S' \subseteq S$ . As before  $S'$  must intersect every cycle in  $G'$  through  $v$ , but now we also require that some vertex in  $S'$  lies on  $C$ . If some independent  $S' \subseteq S$  has size at least  $d_{G'}(v) - 1$  and intersects  $C$ , then we are done. This includes the case when  $S$  induces at most one edge, specifically when  $d_{G'}(v) \leq 2$ . If  $d_{G'}(v) = 3$ , but the case above does not apply, then  $S$  induces  $P_3$  with only the center vertex on  $C$ ; so we let  $S'$  consist of this center vertex. Thus, we assume that  $d_{G'}(v) = 4$ , and that  $S$  induces 2, 3, or 4 edges.

First suppose that  $S$  induces 4 edges. Since  $G$  has no 3-cycle,  $G[S] = C_4$ . Now all vertices of  $S$  lie on  $C$ , so we take  $S'$  to be either independent subset of size 2.

Suppose instead that  $S$  induces 3 edges; so  $G[S] \in \{P_4, K_{1,3}\}$ . When  $G[S] = K_{1,3}$ , let  $S'$  be the independent subset of size 3 unless it does not intersect  $C$ ; in that case, let  $S'$  be the other vertex. If  $G[S] = P_4$ , then denote the vertices of  $S$  by  $w_1, \dots, w_4$  in order along the path. We either let  $S' = \{w_1, w_3\}$  or let  $S' = \{w_2, w_4\}$ . (If each choice for  $S'$  misses some cycle in  $G'$ , then  $G$  contains at least two distinct cycles, contradicting the hypothesis.)

Finally, assume  $S$  induces two edges; so  $G[S] \in \{2K_2, P_3 + K_1\}$ . Suppose  $G[S] = P_3 + K_1$ . If the independent set  $S' \subset S$  of size 3 has a vertex on  $C$ , then we are done. Otherwise, let  $S'$  consist of the center vertex of the  $P_3$  and its nonneighbor. So assume instead that  $G[S] = 2K_2$ . Now it is straightforward to check that we can use as  $S'$  one of the independent sets of size 2 (the general idea is to use one with as many vertices on  $C$  as possible, though not all such sets will work).  $\square$

#### 4.4.4 Coloring the Graph

Recall that  $B = B_4$ . Let  $\tilde{B}$  denote the subset of  $B$  incident to edges in  $G[B]$ . (Since  $e'(B) \leq 3$ , we have  $|\tilde{B}| \leq 6$ .) We form  $\tilde{G}$  from  $G$  by deleting all vertices of  $B \setminus \tilde{B}$  and suppressing all of their neighbors that were not leaves in  $G[L]$ . We call this process *shrinking*  $G$ . (Later we also use the notation  $\tilde{T}$ . In each case, the reader should think of  $\sim$  as meaning ‘shrinking down to the most important part’.) If  $\tilde{G}$  has an nb-coloring  $I, F$ , then we can extend this coloring to  $G$  by adding the deleted vertices of  $B$  to  $I$  and the suppressed vertices of  $L$  to  $F$ . Our goal is to color  $\tilde{G}$ . If we can’t, then we try “unshrinking” a deleted vertex and its 4 suppressed neighbors. If no vertex exists to unshrink, then we show that  $G$  contains a forbidden subgraph, contradicting our hypothesis.

We often use Lemma 4.35 to extend an nb-coloring of  $\tilde{B}$  to a tree  $T$  of  $G[L]$ , specifically when  $F \cup V(T)$  induces a cycle. The idea is to find a vertex  $x \in B \setminus \tilde{B}$  and add it to  $F$ . This allows us to add neighbors of  $x$  in  $T$  to  $I$  (as long as they are not leaves in  $T$ ). When we do this, we call  $x$  the *helper* and say that we *color*  $T$  by Lemma 4.35, with  $x$  as *helper*.

When we describe an nb-coloring of  $B$ , we often specify only the vertices in  $B \cap F$ , implying that  $B \setminus F$  is colored  $I$ . We extend this coloring to each component of  $G[L]$  using Lemmas 4.34 and 4.35.

**Lemma 4.36.** *If  $e'(B) = 3$ , then  $G$  is near-bipartite.*

*Proof.* Suppose that  $e'(B) = 3$ . Now Lemma 4.31 implies  $\ell = 1$ ; hence we write  $T$  for  $G[L]$ . Note that  $G[\tilde{B}] \in \{K_{1,3}, P_4, P_2 + P_3, 3K_2, K_3\}$ . (Here  $K_{1,3}$  denotes a tree on 4 vertices with three leaves,  $P_t$  denotes a path on  $t$  vertices,  $P_2 + P_3$  denotes the disjoint union of  $P_2$  and  $P_3$ , and  $3K_2$  denotes  $K_2 + K_2 + K_2$ .) All cases but the last can be handled quickly (as we show below) by coloring  $\tilde{B}$  so that we can extend the coloring to  $T$  using Lemma 4.34.

In each case we describe  $F$  and implicitly let  $I = B \setminus F$ . If  $G[\tilde{B}] = K_{1,3}$ , then let  $F$  consist of the leaves in the  $K_{1,3}$ . Since  $T$  has 9  $F$ -edges, it is  $F$ -odd, so we can extend the coloring by Lemma 4.34. If  $G[\tilde{B}] = P_4$ , then let  $F = \{v, w\}$ , where  $v$  and  $w$  are at distance two along the  $P_4$ . Now  $T$  has 5  $F$ -edges. If

$G[\tilde{B}] = P_2 + P_3$ , then let  $F = \{v, w\}$ , where  $v$  is a leaf of the  $P_2$  and  $w$  is the center vertex of the  $P_3$ . Now,  $T$  has 5  $F$ -edges, so is  $F$ -odd. Finally, suppose  $G[\tilde{B}] = 3K_2$ . Let  $F$  consist of one vertex from each  $K_2$ . Again,  $T$  has 9  $F$ -edges, so is  $F$ -odd.

Now suppose  $G[\tilde{B}] = K_3$ . If  $T$  has at least 4 leaves, then  $G[B]$  also has some isolated vertices, one of which is adjacent to a leaf  $w$  of  $T$ . Let  $F = \{v_1, v_2\}$ , where the  $v_i$  are two vertices of  $G[\tilde{B}]$  not adjacent to  $w$ . Now we can extend the coloring to  $T$ , since it is  $F$ -leaf-good. So assume that  $T$  has at most 3 leaves. Further, we assume that each leaf has two neighbors in  $\tilde{B}$ , since otherwise the argument above still works. Form  $\tilde{T}$  from  $T$  by suppressing each vertex  $w$  with  $d_T(w) = 2$  that has a neighbor in  $B \setminus \tilde{B}$ . Now  $\tilde{T}$  has six incident edges to  $\tilde{B}$ , so  $\tilde{T} \in \{K_{1,3}, P_4\}$ .  $\tilde{T}$

Suppose that  $\tilde{T} = K_{1,3}$ , and let  $v_1, v_2, v_3$  denote the vertices of  $\tilde{B}$ . So  $\tilde{G} = J_7$ , as shown in Figure 1. Let  $w$  denote a leaf of  $T$  that is not adjacent to  $v_3$ , and pick  $x \in B \setminus \tilde{B}$ ; vertex  $x$  exists since  $J_7$  is forbidden as a subgraph, so  $G \neq \tilde{G}$ . Let  $F = \{v_1, v_2, x\}$ , and color  $w$  with  $I$ . The subgraph induced by  $(V(T) \setminus \{w\}) \cup \{v_1, v_2\}$  has a single cycle. We assume that  $x$  has a neighbor on this cycle; if not, then we repeat the argument with  $v_1$  or  $v_2$  in place of  $v_3$ . Thus, we can extend the coloring to  $V(T) \setminus \{w\}$  by Lemma 4.35, using  $x$  as helper.

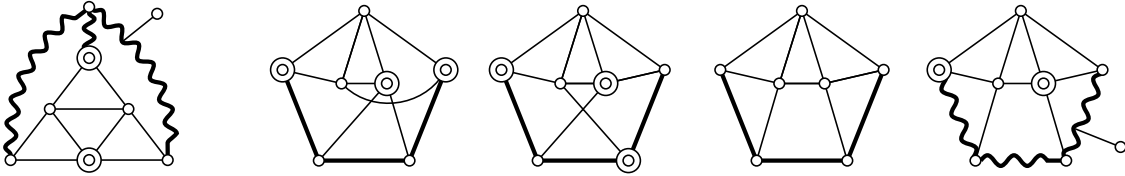


Figure 12: Either  $\tilde{T} = K_{1,3}$  (left) or  $\tilde{T} = P_4$  (right).

Assume instead that  $\tilde{T} = P_4$ . Suppose that  $T = P_4$ . By Pigeonhole at least one vertex in  $\tilde{B}$  is adjacent to both leaves of the  $P_4$ . Now we have three ways for the remaining two vertices of  $\tilde{B}$  to attach. Thus, we have three possibilities for  $G$ , each with 7 vertices. Two of these are non-planar (one has a  $K_{3,3}$ -minor and the other a  $K_5$ -minor). Each non-planar case has an independent set of size 3, which we take as  $I$ . In fact, this approach works whenever  $\tilde{G}$  is either of these non-planar graphs; since  $\tilde{G}$  has an  $I, F$  coloring, so does  $G$ . So assume instead that  $\tilde{G}$  is the other possibility; it is planar and contains  $M_7$  as a (non-induced) subgraph. This implies that  $G \neq \tilde{G}$ , so  $T \neq \tilde{T}$ . Let  $w$  denote a leaf of  $T$  and  $v_1, v_2$  its neighbors in  $\tilde{B}$ . Since  $T \neq \tilde{T}$ , tree  $T$  has a helper vertex  $x$ . Note that  $G[(L \setminus \{w\}) \cup \{v_1, v_2\}]$  is unicyclic, and let  $C$  denote its cycle. We assume that  $x$  has neighbors on  $C$ , since if not, then we repeat the argument with  $w$  replaced by the other leaf of  $T$ . Let  $F = \{v_1, v_2, x\}$ . Now we can extend the coloring to  $G$  by Lemma 4.35, using  $x$  as the helper. This finishes the case  $e'(B) = 3$ .  $\square$

**Lemma 4.37.** *If  $e'(B) = 2$ , then  $G$  is near-bipartite.*

*Proof.* If  $e'(B) = 2$ , then  $G[\tilde{B}] \in \{P_3, 2K_2\}$  and  $\ell \leq 2$ . Suppose  $G[\tilde{B}] = P_3$ . If  $\ell = 1$ , then we color  $G$  as follows. Let  $v_1$  denote a leaf of  $G[\tilde{B}]$  and let  $F = \tilde{B} \setminus \{v_1\}$ . We are done by Lemma 4.34(i), since  $\tilde{T}$  (and therefore also  $T$ ) has exactly 5  $F$ -edges. Thus, we assume  $\ell = 2$ .

We denote the two trees of  $G[L]$  by  $T_1$  and  $T_2$ . Let  $v_1$  and  $v_2$  denote the leaves of  $G[\tilde{B}]$ , and let  $w$  denote its non-leaf vertex. If  $w$  has a single neighbor in each of  $T_1$  and  $T_2$ , then let  $F = \{w\}$ . Each  $T_i$  is  $F$ -odd, so we are done. Thus, we assume  $w$  has two neighbors in  $T_1$  (by symmetry). Further,  $T_1$  is a path with  $w$  adjacent to both endpoints, since otherwise letting  $F = \{w\}$  makes both  $T_1$  and  $T_2$  be  $F$ -leaf-good. Note that the numbers of edges incident to  $v_1$  and  $v_2$  that lead to  $T_1$  must have the same parity. If not, then we let  $F = \{v_1, v_2, w\}$  and both  $T_1$  and  $T_2$  are  $F$ -odd. So the possibilities for the numbers of edges from  $v_1$  and  $v_2$  to  $T_1$  are 0,0; 0,2; 2,0; 2,2; 1,1; 1,3; 3,1; and 3,3. We refer to these as Case 0,0; Case 0,2; etc.

The easiest to handle are Cases 3,3 and 1,3 (and 3,1, by symmetry). Let  $F = \{w, v_2\}$ , which makes  $T_1$  to be  $F$ -odd and  $T_2$  to be  $F$ -null. So now assume that  $v_1$  and  $v_2$  each have at least one neighbor in  $T_2$ .

Before considering the other cases, we prove the following claim.

**Claim 4.38.** *No vertex in  $B \setminus \tilde{B}$  has a neighbor in  $T_1$ .*

Proof. Suppose, to the contrary, that such a vertex exists; call it  $x$ . If  $x$  has an odd number of edges to  $T_1$  and  $T_2$ , then we let  $F = \{w, x\}$ . Both  $T_1$  and  $T_2$  are  $F$ -odd, so we are done. If  $N(x) \subseteq V(T_1)$ , then let  $F = \{w, x\}$ ; since  $T_2$  is  $F$ -null, we color it by Lemma 4.34, and we color  $T_1 \cup \{w\}$  by Lemma 4.35, using  $x$  as helper. So assume that  $x$  has two edges to each of  $T_1$  and  $T_2$ . If a leaf of  $T_2$  is not incident to  $x$ , then let  $F = \{w, x\}$  so that  $T_2$  is  $F$ -leaf-good and colorable by Lemma 4.34, while  $T_1$  is colorable by Lemma 4.35, using  $x$  as helper. Assume instead that  $T_2$  is a path whose endpoints are adjacent to  $x$ . If  $v_1$  has no neighbors in  $T_1$ , then let  $F = \{w, v_1, x\}$ ; now  $T_2$  is  $F$ -odd, so colorable by Lemma 4.34, and  $T_1$  is colorable by Lemma 4.35 using  $x$  as helper. If  $v_1$  has one neighbor in  $T_1$ , then let  $F = \{w, v_1, x\}$  so that  $T_1$  is  $F$ -odd, and thus colorable by Lemma 4.34, while  $T_2$  is colorable by Lemma 4.35, using  $v_1$  as helper. Because we have already ruled out cases 3,1 and 3,3; it follows that  $v_1$  must have exactly two neighbors in  $T_1$ . By symmetry,  $v_2$  also has exactly two neighbors in  $T_1$ .

Let  $y_1, \dots, y_\ell$  be the vertices of  $T_1$  in order. Let  $z_1, z_2, z_3, z_4$  be the four neighbors of  $v_1$  or  $x$  in  $T_1$  in order; note that these  $z_i$  are distinct, since  $w \leftrightarrow \{y_1, y_\ell\}$ . If  $x \leftrightarrow \{y_1, y_\ell\}$ , then let  $F = \{w, x, v_1\}$  and color  $T_2$  with Lemma 4.34 since  $T_2$  is  $F$ -odd. To color  $T_1$ , contract  $wv_1$  into a vertex  $z$ , and then color  $T_1 \cup \{x\}$  by Lemma 4.35 using  $z$  as helper. By symmetry, we assume that  $x \not\leftrightarrow y_1$ . By symmetry between  $v_1$  and  $v_2$ , let us assume that  $v_1 \not\leftrightarrow y_1$ , and thus  $y_1 \neq z_1$ . Under these assumptions, color  $G \setminus T_2$  with  $F = \{w, x, v_1\} \cup (T_1 \setminus \{y_1, z_2, z_4\})$  and  $I = V(G) \setminus (F \cup V(T_2))$  and extend this coloring to all of  $G$  via Lemma 4.34 since  $T_2$  is  $F$ -odd. Therefore  $N(T_1) \subseteq \tilde{B}$ .  $\diamond$

This claim shows that Case 0,0 is impossible. Since  $G$  contains no copy of  $K_4$ , Cases 2,0 and 0,2 are also impossible. So all that remain are Case 1,1 and Case 2,2.

Suppose we are in Case 1,1. That is,  $v_1$  and  $v_2$  each send a single edge to  $T_1$ . By the claim, this implies that  $T_1 = K_2$ . Now  $T_2$  must be a path with  $v_1$  and  $v_2$  each adjacent to both endpoints of  $T_2$  (otherwise we let  $F = \{v_1, w\}$  or  $F = \{v_2, w\}$ , so  $T_1$  is  $F$ -odd and  $T_2$  is  $F$ -leaf-good). If  $T_2 \neq K_2$ , then let  $F = \{v_1, v_2\}$ . We extend this coloring to  $G$  as follows. Color the endpoints of  $T_2$  with  $I$  and the rest of  $T_2$  with  $F$ , and color all of  $T_1$  with  $F$ . (Now  $T_1$  has a  $v_1, v_2$ -path in  $F$ , but it does not extend to a cycle in  $F$ .) But if  $T_2 = K_2$ , then  $G$  contains the Moser Spindle (in fact  $G - v_1w$  is the Moser Spindle), which is a contradiction. This completes Case 1,1.

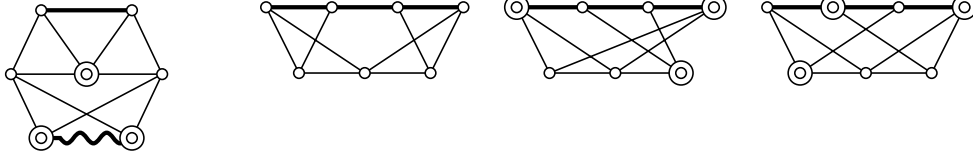


Figure 13: Left: Case 1,1. Right: Case 2,2 (with  $T_2$  undrawn).

Suppose we are in Case 2,2. That is, both  $v_1$  and  $v_2$  have two edges to  $T_1$  and so  $T_1$  is a path on four vertices. Let  $z_1, z_2, z_3, z_4$  denote the vertices of  $T_1$  in order. If  $z_1$  and  $z_4$  are neighbors of the same  $v_i$ , then by symmetry assume it is  $v_1$ . Now instead let  $F = \{v_1, w\}$ . To color  $T_1$ , use  $I$  on  $z_1$  and  $z_4$  and use  $F$  on the rest of  $T_1$ . Finally, we can color  $T_2$ , since it is  $F$ -odd. So instead  $z_1$  and  $z_4$  must be neighbors of distinct  $v_i$ . By symmetry, we have only two cases: either (a)  $v_1 \leftrightarrow \{z_1, z_2\}$  and  $v_2 \leftrightarrow \{z_3, z_4\}$  or (b)  $v_1 \leftrightarrow \{z_1, z_3\}$  and  $v_2 \leftrightarrow \{z_2, z_4\}$ . In (a) subset  $\tilde{B} \cup V(T_1)$  induces the Moser Spindle  $M_7$ , a contradiction. In (b), let  $F = \{w, v_2\}$  and color  $z_1, z_2, z_3, z_4$  as  $F, I, F, I$ . Finally, color  $T_2$  by Lemma 4.34, since it has exactly 1  $F$ -edge. This completes the case  $G[\tilde{B}] = P_3$ .

Now suppose that  $G[\tilde{B}] = 2K_2$ . Denote the vertices of  $\tilde{B}$  by  $v_1, v_2, v_3, v_4$ , where  $v_1 \leftrightarrow v_2$  and  $v_3 \leftrightarrow v_4$ .

**Claim 4.39.**  $G[L]$  consists of two trees,  $T_1$  and  $T_2$ . We may assume that  $|T_1| \geq 2$  and each leaf of  $T_1$  has both neighbors (outside of  $T_1$ ) in the same component of  $G[\tilde{B}]$ . So  $T_1$  has at most 4 leaves.

Proof. If  $G[L]$  has only a single component, then let  $F$  consist of three vertices in  $\tilde{B}$ . Now the tree has 9  $F$ -edges, so it is  $F$ -odd, and we are done by Lemma 4.34(i). Instead assume the forest has two trees,  $T_1$  and  $T_2$ . For each  $i \in [4]$ , let  $a_i$  denote the parity of the number of edges from  $v_i$  to  $T_1$ . Suppose  $a_1 \neq a_3$ . Now

let  $F = \{v_1, v_3\}$ . We are done, since each  $T_i$  is  $F$ -odd. Thus  $a_1 = a_3$ . By swapping the roles of  $v_1$  and  $v_2$ , and also  $v_3$  and  $v_4$ , we get  $a_1 = a_3 = a_2 = a_4$ . By symmetry between  $T_1$  and  $T_2$ , we assume that each  $v_i$  has an even number of edges to  $T_1$ . Suppose there exists a leaf  $w$  of  $T_1$  with at most one neighbor in  $\tilde{B}$ . (This includes the case that  $|T_1| = 1$ , since  $a_1 = a_3 = a_2 = a_4$ .) Let  $F$  consist of three vertices in  $\tilde{B}$ , excluding any neighbor of  $w$ . Now we are done, since  $T_1$  is  $F$ -leaf-good, by  $w$ , and  $T_2$  is  $F$ -odd. Thus each leaf  $w$  of  $T_1$  must have both neighbors (outside of  $T_1$ ) in  $\tilde{B}$ . Since  $\tilde{B}$  sends at most 8 edges to  $T_1$ , we conclude that  $T_1$  has at most four leaves. If some leaf  $w$  of  $T_1$  has neighbors in two components of  $G[\tilde{B}]$ , then we are also done, as follows. Let  $F$  consist of three vertices in  $\tilde{B}$ , including both neighbors of  $w$ . Again  $T_2$  is  $F$ -odd, so we can color it by Lemma 4.34(i). We can also color  $T_1$ , by treating  $w$  like a vertex with its two neighbors in  $\tilde{B}$  colored  $I$ . Now  $T_1$  may contain a path colored  $F$  linking these neighbors of  $w$ , but it will not extend to a cycle colored  $F$ , since the neighbors of  $w$  are in different components of  $G[\tilde{B}]$ .  $\diamond$

Form  $\tilde{T}_1$  from  $T_1$  by suppressing each vertex  $w$  such that  $d_{T_1}(w) = 2$  and  $w$  has no neighbor in  $\tilde{B}$ . It suffices to color  $\tilde{T}_1$ , since we can extend the coloring to  $T_1$  by coloring each suppressed vertex with  $F$ . We show that each vertex of  $\tilde{B}$  has 2 edges to  $T_1$ . (This number is always either 0 or 2, as shown above.) Recall that each leaf of  $T_1$  has both neighbors (outside  $T_1$ ) in the same component of  $G[\tilde{B}]$ . Since  $T_1$  has a leaf, its two neighbors in  $\tilde{B}$  each send two edges to  $T_1$ . First suppose they are the only two such vertices in  $\tilde{B}$ . Recall that each leaf of  $T_1$  has its two neighbors in  $\tilde{B}$  in the same component of  $G[\tilde{B}]$ ; so assume that  $v_1$  and  $v_2$  both have two edges to  $T_1$  and  $v_3$  and  $v_4$  have none. Note that  $T_1 \neq K_2$ , since  $K_4 \not\subset G$ . So there exists  $x \in B \setminus \tilde{B}$  with a neighbor in  $T_1$ . If  $x$  sends an odd number of edges to each  $T_i$ , then we let  $F = \{v_1, v_3, x\}$ , and we are done since each  $T_i$  is  $F$ -odd. So assume  $x$  sends an even number of edges to each  $T_i$ . Now let  $F = \{v_2, v_3, v_4, x\}$ . Again  $T_2$  is  $F$ -odd. And we can color  $T_1$  by Lemma 4.35, with  $x$  as helper.

Now instead suppose that exactly three vertices in  $\tilde{B}$  each have two edges to  $T_1$ ; by symmetry, say  $v_1, v_2, v_3$ . As is true for  $T_1$ , each leaf of  $\tilde{T}_1$  has both neighbors in the same component of  $G[\tilde{B}]$ , so  $\tilde{T}_1$  has only two leaves (that is,  $T_1$  and  $\tilde{T}_1$  are paths). Denote the vertices of  $\tilde{T}_1$  by  $z_1, z_2, z_3, z_4$ . So  $\{z_1, z_4\} \leftrightarrow \{v_1, v_2\}$  and  $\{z_2, z_3\} \leftrightarrow v_3$ . Let  $F = \{v_2, v_3\}$ . Now  $T_2$  is  $F$ -odd. To color  $T_1$ , use  $I$  on  $z_3$  and use  $F$  on  $V(T_1) \setminus \{z_3\}$ . Thus, we conclude that each of the four vertices of  $\tilde{B}$  sends two edges to  $T_1$ , so  $|\tilde{T}_1| = 6$ .



Figure 14: When  $\tilde{T}_1$  is a 6-vertex path.

Suppose that  $\tilde{T}_1$  is a path; label its vertices  $z_1, \dots, z_6$  and let  $w_i$  denote the neighbor of  $z_i$  in  $B$ , for each  $i \in \{2, \dots, 5\}$  (possibly the  $w_i$  are not distinct). If  $w_2 \neq w_3$ , then color  $\tilde{B}$  so that  $w_2$  uses  $F$ ,  $w_3$  uses  $I$ , and in each component of  $G[\tilde{B}]$  one vertex uses  $F$  and the other one uses  $I$ . This implies that  $|F \cap \{w_4, w_5\}| = |I \cap \{w_4, w_5\}| = 1$ , since each leaf has both neighbors in  $\tilde{B}$  in the same component of  $G[\tilde{B}]$ . To extend the coloring to  $\tilde{T}_1$ , we use  $I$  on the vertices  $z_i$  and  $z_j$  such that  $w_i, w_j \in F$  (and color the other  $z_t$  with  $F$ ). By symmetry, assume that  $v_1, v_3 \in F$ . Because the neighbors of  $z_1$  and  $z_6$  are in the same component of  $\tilde{B}$ , the above coloring of  $T_1$  satisfies the conclusion of Lemma 4.34; in particular there is no path between  $v_1$  and  $v_3$  in  $F$ . Thus we can color all of  $V(T_2)$  with  $F$ .

So assume  $w_2 = w_3$  and (by symmetry)  $w_4 = w_5$ . Since  $z_1$  and  $z_6$  have both neighbors in the same component of  $G[\tilde{B}]$  (and  $G$  is simple), we have  $w_2 = w_3 \leftrightarrow w_4 = w_5$ . So say  $v_1 = w_2 = w_3$ ,  $v_2 = w_4 = w_5$ , and  $\{z_1, z_6\} \leftrightarrow \{v_3, v_4\}$ . Let  $F = \{v_2, v_3, v_4\}$ . To extend this coloring to  $T_1$ , color  $z_1, z_4, z_6$  with  $I$  and color  $z_2, z_3, z_5$  with  $F$ . Since  $T_2$  is  $F$ -odd, we can extend the coloring to  $T_2$  by Lemma 4.34. Thus, we conclude that  $T_1$  is not a path.

Suppose  $\tilde{T}_1$  has exactly 3 leaves. Now  $\tilde{T}_1$  is formed from a 5-vertex path by adding a pendant edge at one internal vertex. Denote the vertices of the path by  $z_1, \dots, z_5$  and the new leaf by  $z^*$ . By symmetry, we assume either  $z^* \leftrightarrow z_4$  or  $z^* \leftrightarrow z_3$ . In the first case, color one vertex in each component of  $G[\tilde{B}]$  with  $I$  and the other with  $F$ , so that the neighbor of  $z_3$  is colored  $I$ . Now each leaf of  $\tilde{T}_1$  has one neighbor colored  $I$  and one colored  $F$ , so  $z_2$  has a neighbor colored  $F$ . To extend the coloring to  $\tilde{T}_1$ , color  $z_2, z_4$  with  $I$  and color  $z_1, z_3, z_5, z^*$  with  $F$ . Because the neighbors of the leaves are in the same component of  $\tilde{B}$ , the above

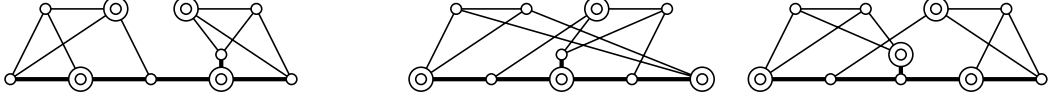


Figure 15: Examples when  $\tilde{T}_1$  is a tree with 3 leaves. Left:  $z^* \leftrightarrow z_4$ . Right:  $z^* \leftrightarrow z_3$ .

coloring of  $T_1$  satisfies the conclusion of Lemma 4.34; in particular there is no path between vertices of  $\tilde{B}$  in  $F$ . Thus, we can color all of  $V(T_2)$  with  $F$ . This finishes the case when  $z^* \leftrightarrow z_4$ . So instead assume  $z^* \leftrightarrow z_3$ . Since each leaf has both neighbors in the same component of  $G[\tilde{B}]$ , also  $z_2$  and  $z_4$  have their neighbors in the same component of  $G[\tilde{B}]$ . By symmetry between  $z_1$  and  $z_5$ , assume this is not the component with vertices adjacent to  $z_1$ . Now color the neighbor of  $z_2$  in  $\tilde{B}$  with  $I$  and the rest of  $\tilde{B}$  with  $F$ . We extend this coloring to  $T_2$  using Lemma 4.34, since  $T_2$  is  $F$ -odd. If  $z^*$  has a neighbor colored  $I$ , then we extend the coloring to the  $z_i$ 's by coloring  $z_1, z_3, z_5$  with  $I$  and coloring  $z_2, z_4, z^*$  with  $F$ . Otherwise, only  $z_2$  and  $z_5$  have neighbors colored  $I$ , so we color  $z_1, z^*, z_4$  with  $I$  and color  $z_2, z_3, z_5$  with  $F$ . This completes the case that  $\tilde{T}_1$  has three leaves.

Finally, suppose  $\tilde{T}_1$  has exactly 4 leaves. Recall that all internal vertices of  $\tilde{T}_1$  have degree 3, so  $\tilde{T}_1$  has two adjacent 3-vertices. Let  $z_1, z_2, z_3, z_4$  denote the leaves of  $\tilde{T}_1$  with  $\{z_1, z_2\} \leftrightarrow \{v_1, v_2\}$  and  $\{z_3, z_4\} \leftrightarrow \{v_3, v_4\}$ ; this follows from Claim 4.39. By symmetry between  $v_3$  and  $v_4$ , we assume  $\text{dist}_T(z_1, z_4) = 3$ . Let  $z_5$  and  $z_6$  denote (respectively) the neighbors in  $T$  of  $z_1$  and  $z_4$ . Either  $z_5 \leftrightarrow \{z_1, z_2\}$  or else  $z_5 \leftrightarrow \{z_1, z_3\}$ . In the first case, let  $F = \{v_2, v_3, v_4\}$ . To extend the coloring to  $\tilde{T}_1$ , use  $F$  on  $z_1, z_2, z_6$  and use  $I$  on  $z_3, z_4, z_5$ . (Again  $T_2$  is  $F$ -odd.) So assume we are in the second case:  $z_5 \leftrightarrow \{z_1, z_3\}$ . Suppose some pendant edge of  $\tilde{T}_1$  corresponds to a path of length at least 2 in  $T$ ; by symmetry, say it is  $z_1 z_5$ . Let  $F = \{v_1, v_2, v_3\}$ . To color  $T_1$ , use  $I$  on  $z_1, z_2, z_5$  and use  $F$  on  $z_3, z_4, z_6$ . (Again  $T_2$  is  $F$ -odd.) Similarly, suppose  $z_5 z_6$  corresponds to a path of length at least 2. Now let  $F = \{v_1, v_3\}$ . Color  $z_5, z_6$  with  $I$  and color  $z_1, z_2, z_3, z_4$  with  $F$ . Because there is no path in  $F$  from  $v_1$  to  $v_3$ , we may color  $V(T_2)$  with  $F$ . Thus, we conclude that  $\tilde{T}_1 = T_1$ .

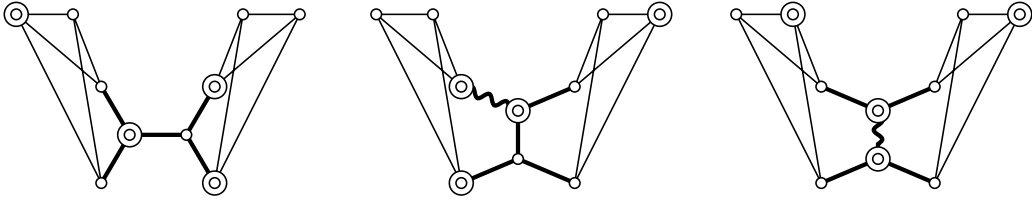


Figure 16: If  $\tilde{T}_1$  has 4 leaves, then  $\tilde{T}_1 = T_1$  with  $z_5 \leftrightarrow \{z_1, z_3\}$  and  $z_6 \leftrightarrow \{z_2, z_4\}$ .

Suppose some leaf  $w$  of  $T_2$  has a neighbor in  $B \setminus \tilde{B}$ . In each component of  $G[\tilde{B}]$ , color one vertex  $F$  and the other  $I$ ; do this so that any neighbor of  $w$  in  $\tilde{B}$  is colored  $I$ . Now  $T_2$  is  $F$ -leaf-good. By symmetry, we assume that  $v_1, v_3 \in F$  and  $v_2, v_4 \in I$ . For  $T_1$ , color  $z_6$  with  $I$  and  $z_1, \dots, z_5$  with  $F$ . This does create a  $v_1, v_3$ -path in  $F$  through  $T_1$ , but this is okay, since no such path exists in  $T_2$ . Thus, each leaf of  $T_2$  has no neighbors in  $B \setminus \tilde{B}$ . Since  $\tilde{B}$  has only 4 edges to  $T_2$ , we see that  $T_2$  is a path. Suppose a leaf  $w$  of  $T_2$  has neighbors in distinct components of  $G[\tilde{B}]$ , by symmetry say  $v_1$  and  $v_3$ . Now we color  $\tilde{B} \cup V(T_1)$  as in the immediately previous case. We color  $w$  with  $I$  and  $T_2 \setminus \{w\}$  with  $F$ . Thus, no such  $w$  exists. Suppose  $T_2 \neq K_2$ . Color all of  $\tilde{B}$  with  $F$ , color  $N(\tilde{B}) \cap (T_1 \cup T_2)$  with  $I$ , and color  $(T_1 \cup T_2) \setminus N(\tilde{B})$  with  $F$ . Thus, we conclude that  $T_2 = K_2$ . So  $G$  is the 12-vertex graph below, which is nb-critical. It is forbidden by the hypothesis, which is a contradiction. This completes the case that  $G[\tilde{B}] = 2K_2$ .  $\square$

**Lemma 4.40.** *If  $e'(B) = 1$ , then  $G$  is near bipartite.*

*Proof.* Suppose  $G[\tilde{B}] = K_2$ . Denote  $\tilde{B}$  by  $\{v_1, v_2\}$ . If  $G[L]$  has only a single component, then color  $v_1$  with  $F$  and color  $v_2$  with  $I$ . We can color  $G[L]$ , since it is  $F$ -odd. Suppose instead that  $G[L]$  has two components; call them  $T_1$  and  $T_2$ . Suppose  $v_1$  has 3 edges to  $T_1$  (and none to  $T_2$ ). Let  $F = \{v_1\}$ . Now  $T_1$  is  $F$ -odd and  $T_2$  is  $F$ -null, so we are done. Thus, by symmetry we assume  $v_1$  and  $v_2$  each have 1 edge to one tree and 2 edges

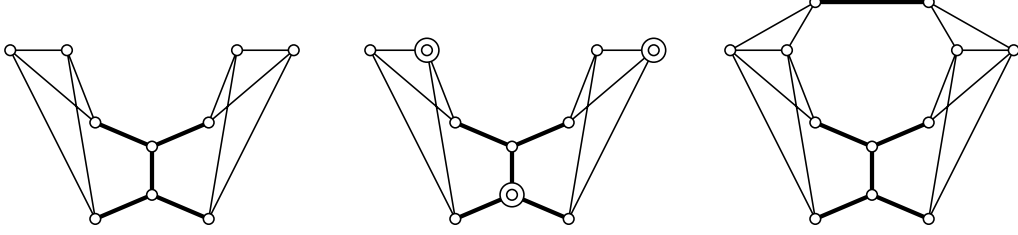


Figure 17: If  $\tilde{T}_1$  has 4 leaves and  $G$  has no nb-coloring, then  $G = J_{12}$ .

to the other. If  $v_1$  and  $v_2$  have (respectively) 1 and 2 edges to  $T_1$ , then let  $F = \{v_1, v_2\}$ ; now both  $T_1$  and  $T_2$  are  $F$ -odd. So assume that  $v_1$  and  $v_2$  each have 1 edge to  $T_1$  and 2 edges to  $T_2$ . Suppose some leaf  $w$  of  $T_2$  has a neighbor  $x \in B \setminus \tilde{B}$ . Let  $F$  consist of a single vertex of  $\tilde{B}$  that is not adjacent to  $w$ . Now  $T_2$  is  $F$ -leaf-good and  $T_1$  is  $F$ -odd. Thus, all leaves of  $T_2$  have no neighbors in  $B \setminus \tilde{B}$ . So  $T_2$  is a path. Since  $K_4 \not\subset G$ , we know  $T_2 \neq K_2$ . So there exists  $x \in B \setminus \tilde{B}$  with a neighbor in  $T_2$ . If  $x$  sends an odd number of edges to both  $T_1$  and  $T_2$ , then we let  $F = \{v_1, v_2, x\}$ , and both  $T_1$  and  $T_2$  are  $F$ -odd. Otherwise, let  $F = \{v_1, x\}$ . Again,  $T_1$  is  $F$ -odd. Also, we can color  $T_2$  by Lemma 4.35, with  $x$  as the helper. Thus, we conclude that  $G[L]$  has three components; we call these  $T_1, T_2, T_3$ .

We say that  $x \in B$  splits as  $a_1/a_2/a_3$  if  $x$  has  $a_i$  edges to  $T_i$ , for each  $i \in [3]$ . For  $x \in \tilde{B}$  we have  $a_1 + a_2 + a_3 = 3$  and for  $x \in B \setminus \tilde{B}$ , we have  $a_1 + a_2 + a_3 = 4$ . If we care only about the parities of the  $a_i$ , we say, for example, that  $x$  splits as  $e/o/o$  (to denote that  $a_1$  is even, while  $a_2$  and  $a_3$  are odd). If  $v_1$  splits as  $1/1/1$  or as some permutation of  $3/0/0$ , then let  $F = v_1$ . Now we are done, since  $T_1$  is  $F$ -odd, while  $T_2$  and  $T_3$  are both either  $F$ -odd or  $F$ -null. So assume that  $v_1$  (and  $v_2$ , by symmetry) splits as some permutation of  $2/1/0$ . By symmetry between the  $T_i$ , we assume that  $v_1$  splits as  $2/1/0$ . A priori we have 6 cases for how  $v_2$  splits: (a)  $0/1/2$ , (b)  $0/2/1$ , (c)  $1/0/2$ , (d)  $1/2/0$ , (e)  $2/0/1$ , (f)  $2/1/0$ . Before considering these cases, we prove an easy claim.

splits as  
 $a_1/a_2/a_3$   
 $e/o/o$

**Claim 4.41.** *If  $v_i$  has 2 edges to  $T_j$ , then  $T_j$  is a path with each endpoint adjacent to  $v_i$ .*

Proof. Suppose not. By symmetry we assume that  $v_1$  has 2 edges to  $T_1$ , 1 edge to  $T_2$ , and 0 edges to  $T_3$ , but  $T_1$  has a leaf  $w$  such that  $w \not\sim v_1$ . Let  $F = \{v_1\}$ . Now  $T_1$  is  $F$ -leaf-good (by  $w$ ),  $T_2$  is  $F$ -odd, and  $T_3$  is  $F$ -null. So we can extend the coloring of  $B$  to all of  $G$ , a contradiction.  $\diamond$

Now we consider cases (a)–(f). For (d), let  $F = \{v_1, v_2\}$ . Now  $T_1$  and  $T_2$  are  $F$ -odd, while  $T_3$  is  $F$ -null. For (e), Claim 4.41 implies that  $T_1$  is a path with each endpoint adjacent to both  $v_1$  and  $v_2$ . Note that  $T_1 \neq K_2$ , since  $K_4 \not\subset G$ . Let  $F = \{v_1, v_2\}$  and note that  $T_2$  and  $T_3$  are both  $F$ -odd. To color  $T_1$ , use  $I$  on both leaves and  $F$  everywhere else. This finishes (e). Note that (b) and (c) are the same case, by symmetry between both the  $v_i$ 's and the  $T_j$ 's. Thus, we must consider cases (a), (c), and (f).

**Case (a):  $v_1$  splits as  $2/1/0$  and  $v_2$  splits as  $0/1/2$ .** By Claim 4.41,  $T_1$  is a path with both endpoints adjacent to  $v_1$ ; similarly,  $T_3$  is a path with both endpoints adjacent to  $v_2$ . If some  $x \in B \setminus \tilde{B}$  splits as  $o/e/o$ , then let  $F = \{v_1, x\}$ . Now each  $T_i$  is  $F$ -odd, so we are done. Suppose some  $x \in B \setminus \tilde{B}$  splits as  $e/e/e$ ; we consider the possibilities. If  $x$  splits as  $0/0/4$ , then let  $F = \{v_2, x\}$ . Now  $T_1$  is  $F$ -null,  $T_2$  is  $F$ -odd, and we can color  $T_3$  by Lemma 4.35, with  $x$  as helper. So  $x$  cannot split as  $0/0/4$ ; similarly,  $x$  cannot split as  $4/0/0$ . If  $x$  splits as  $0/2/2$ , then let  $F = \{v_2, x\}$ . Now  $T_1$  is  $F$ -null and  $T_2$  is  $F$ -odd. To color  $T_3$ , use  $I$  on one neighbor of  $x$  and color the rest of  $T_3$  with  $F$ . So assume no vertex splits as  $0/2/2$ ; similarly, no vertex splits as  $2/2/0$ . Thus each vertex that splits as  $e/e/e$  splits as  $2/0/2$  or  $0/4/0$ . If instead there exist  $x, y \in B \setminus \tilde{B}$  that split (respectively) as  $o/o/e$  and  $e/o/o$ , then let  $F = \{v_1, x, y\}$ . Again, each  $T_i$  is  $F$ -odd, so we are done. By symmetry (between  $T_1$  and  $T_3$ ) we assume that no vertex in  $B \setminus \tilde{B}$  splits as  $e/o/o$ . Hence, every vertex splits as  $o/o/e$  or  $2/0/2$  or  $0/4/0$ .

We consider the possibilities for a vertex  $x \in B \setminus \tilde{B}$  that splits as  $o/o/e$ . If  $x$  splits as  $1/1/2$ , then let  $F = \{v_1, v_2, x\}$ . Trees  $T_1$  and  $T_2$  are both  $F$ -odd, and we can color  $T_3$  by Lemma 4.35, with  $x$  as helper. So each  $x \in B \setminus \tilde{B}$  must split as  $1/3/0$ ,  $3/1/0$ ,  $0/4/0$ , or  $2/0/2$ . Since  $T_3$  has a neighbor in  $B \setminus \tilde{B}$ , some  $x \in B \setminus \tilde{B}$

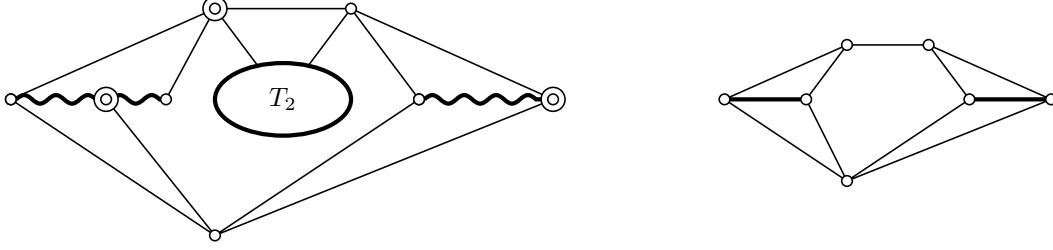


Figure 18: Case (a).

splits as  $2/0/2$ . Suppose some  $y$  splits as  $1/3/0$  or  $3/1/0$ . Let  $F = \{v_1, v_2, x, y\}$ . Trees  $T_1$  and  $T_2$  are  $F$ -odd, and we can color  $T_3$  by Lemma 4.35, with  $x$  as helper. So assume no such  $y$  exists. That is, each vertex splits as  $2/0/2$  or  $0/4/0$ . Recall that  $x$  splits as  $2/0/2$ , and suppose that  $x$  has a neighbor  $z$  that is not a leaf of  $T_1$  or  $T_3$ . By symmetry, say  $z \in T_1$ . Let  $F = \{v_2, x\}$ . To color  $T_1$ , use  $I$  on  $z$  and  $F$  on the rest of  $T_1$ . To color  $T_3$ , use  $I$  on a neighbor of  $x$  (and  $F$  on the rest of  $T_3$ ). Finally,  $T_2$  is  $F$ -odd. So assume that no such  $z$  exists. This implies that  $x$  is unique. So  $T_1 = K_2$  and  $T_3 = K_2$ . But now  $\{v_1, v_2, x\} \cup V(T_1) \cup V(T_2)$  induces a Moser spindle, which is a contradiction. This finishes case (a).

**Case (c):  $v_1$  splits as  $2/1/0$  and  $v_2$  splits as  $1/0/2$ .** By Claim 4.41  $T_1$  is a path with both endpoints adjacent to  $v_1$  and  $T_3$  is a path with both endpoints adjacent to  $v_2$ . Consider some vertex  $x \in B \setminus \tilde{B}$  and the parities of edges that  $x$  has to  $T_1$ ,  $T_2$ , and  $T_3$ . A priori, the options are  $o/o/e$ ,  $o/e/o$ ,  $e/o/o$ , and  $e/e/e$ . If  $x$  splits as  $e/o/o$ , then let  $F = \{v_2, x\}$ . Now each  $T_i$  is  $F$ -odd, so we are done. Similarly, if  $x$  splits as  $o/e/o$ , then let  $F = \{v_1, x\}$ . So assume each vertex in  $B \setminus \tilde{B}$  splits as  $o/o/e$  or  $e/e/e$ .

Suppose  $T_3 \neq K_2$ , and let  $x \in B \setminus \tilde{B}$  be a neighbor of some internal vertex  $y$  of  $T_3$ . Suppose  $x$  splits as  $e/e/e$ . Let  $F = \{v_1, v_2, x\}$ . Note that  $T_1$  and  $T_2$  are  $F$ -odd. To color  $T_3$ , we use Lemma 4.35, with  $x$  as helper. So assume instead that  $x$  splits as  $o/o/e$ . (Since  $x$  sends edges to  $T_3$ , it splits as  $1/1/2$ .) Let  $F = \{v_1, x\}$ , and note that  $T_1$  is  $F$ -odd. Color  $y$  with  $I$  and color the rest of  $T_3$  with  $F$ . Finally,  $T_2$  is  $F$ -even. We color all of  $T_2$  with  $F$ . This creates a single  $v_1, x$ -path colored  $F$  in  $T_2$ , but this is okay since neither  $T_1$  nor  $T_3$  has such a path. This implies that  $T_3 = K_2$ .

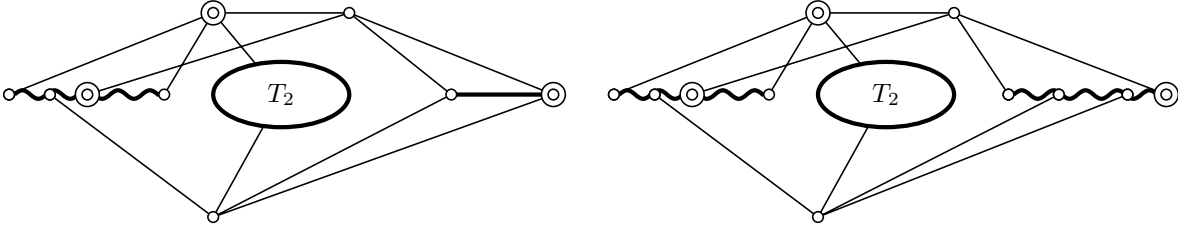


Figure 19: Case (c), part 1.

Let  $x$  be a neighbor of  $T_3$  other than  $v_2$ . If  $x$  splits as  $e/e/e$ , then the argument in the previous paragraph still works. So assume  $x$  splits as  $o/o/e$ , that is, as  $1/1/2$ .

Suppose either  $x$  or  $v_2$  has a neighbor  $z$  in  $T_1$  that is not a leaf of  $T_1$ . Let  $F = \{v_2, x\}$ . Note that  $T_2$  is  $F$ -odd. To color  $T_1$ , we use  $I$  on  $z$  and use  $F$  on the rest of  $T_1$ . (Note that  $x$  and  $v_2$  each have only a single neighbor in  $T_1$ , and one of these neighbors,  $z$ , is colored  $I$ , so  $T_1$  has no  $v_2, x$ -path in  $F$ .) To extend to  $T_3$ , we color one of its vertices with  $I$  and the other with  $F$ . Thus, no such  $z$  exists. That is,  $N_{T_1}(v_1, x)$  is simply the two leaves of  $T_1$ .

Suppose that  $T_1 \neq K_2$ , and let  $y$  be a neighbor of  $T_1$  in  $B \setminus (\tilde{B} \cup \{x\})$ . Recall that each vertex in  $B \setminus \tilde{B}$  splits as  $o/o/e$  or  $e/e/e$ . If  $y$  splits as  $o/o/e$ , then let  $F = \{v_1, v_2, x, y\}$ . Note that  $T_1$  and  $T_2$  are both  $F$ -odd. Although  $T_3$  is  $F$ -even, we simply color one of its vertices with  $I$  and the other with  $F$ . So instead assume that  $y$  splits as  $e/e/e$ . Since  $T_3$  is  $K_2$ , vertex  $y$  sends no edges to  $T_3$ . We let  $F = \{v_2, x, y\}$ . Now  $T_2$  is  $F$ -odd, and  $T_3$  is again easy to color. Since  $T_1$  is  $F$ -even, we color it by Lemma 4.35, with  $y$  as helper. So we conclude



that no such  $y$  exists. That is,  $T_1 = K_2$ . Now  $\{v_1, v_2, x\} \cup V(T_1) \cup V(T_3)$  induces a Moser spindle, which is a contradiction. This finishes case (c).

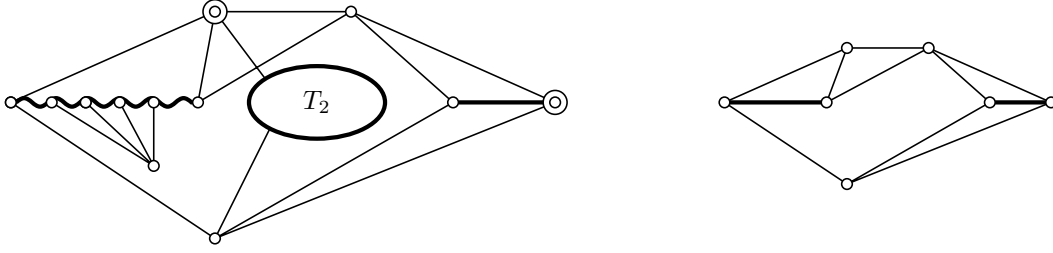


Figure 20: Case (c), part 2.

**Case (f):  $v_1$  splits as 2/1/0 and  $v_2$  splits as 2/1/0.** By Claim 4.41,  $T_1$  is a path each endpoint adjacent to both  $v_1$  and  $v_2$ . Note that  $T_1 \neq K_2$ , since  $K_4 \not\subset G$ . If  $T_2$  has a leaf adjacent to neither  $v_1$  nor  $v_2$ , then let  $F = \{v_1, v_2\}$ . Now  $T_2$  is  $F$ -leaf-good and  $T_3$  is  $F$ -null. Since  $T_1 \neq K_2$ , we can color both leaves of  $T_1$  with  $I$  and its internal vertices with  $F$ . So  $T_2$  is a path with each leaf adjacent to one of  $\{v_1, v_2\}$ . We consider a vertex  $x \in B \setminus \tilde{B}$  and the possible ways it splits. If  $x$  splits as o/e/o, then let  $F = \{v_1, x\}$ . Now each  $T_i$  is  $F$ -odd, so we are done. The other possibilities for the way that  $x$  splits are 1/3/0, 3/1/0, 0/1/3, 0/3/1, 1/1/2, 2/1/1, 4/0/0, 0/4/0, 0/0/4, 2/2/0, 2/0/2, 0/2/2. If  $x$  splits as 1/3/0 or 3/1/0, then let  $F = \{v_1, v_2, x\}$ . Now  $T_1$  and  $T_2$  are  $F$ -odd, and  $T_3$  is  $F$ -null. If  $x$  splits as 0/1/3 or 0/3/1, then let  $F = \{v_1, v_2, x\}$ . Now  $T_2$  and  $T_3$  are  $F$ -odd. To color  $T_1$ , use  $I$  on its two leaves and use  $F$  elsewhere. If  $x$  splits as 4/0/0, then let  $F = \{x, v_1\}$ . Now  $T_2$  is  $F$ -odd and  $T_3$  is  $F$ -null. We color  $T_1$  by Lemma 4.35, with  $x$  as helper. If  $x$  splits as 0/4/0, then let  $F = \{v_1, v_2, x\}$ . Now  $T_3$  is  $F$ -null. To color  $T_1$ , use color  $I$  on its leaves and use  $F$  elsewhere. To color  $T_2$ , use Lemma 4.35, with  $x$  as helper. If  $x$  splits as 2/2/0, then let  $F = \{v_1, x\}$ . Note that  $T_3$  is  $F$ -null and  $T_2$  is  $F$ -odd. To color  $T_1$ , use  $I$  on one neighbor of  $x$  in  $T_1$ , and use  $F$  on the rest of  $T_1$ . Suppose that  $x$  splits as 2/1/1. By symmetry between  $v_1$  and  $v_2$ , assume that  $v_1$  and  $x$  do not dominate all leaves in  $T_2$ . Now let  $F = \{v_1, x\}$ . Clearly,  $T_3$  is  $F$ -odd, and  $T_2$  is  $F$ -leaf-good. For  $T_1$ , color one neighbor of  $x$  in  $T_1$  with  $I$  and color the rest of  $T_1$  with  $F$ . We have handled all possibilities for the way  $x$  splits except 1/1/2, 2/0/2, 0/2/2, and 0/0/4.

Suppose  $T_3$  is not a path (so it has at least three leaves). Since  $T_1 \neq K_2$ , there exists  $x \in B \setminus \tilde{B}$  that splits as either 1/1/2 or else 2/0/2. In the first case, let  $F = \{v_1, v_2, x\}$ . Trees  $T_1$  and  $T_2$  are both  $F$ -odd. And  $T_3$  is  $F$ -leaf-good, so we are done. In the second case, let  $F = \{v_1, x\}$ . Again  $T_3$  is  $F$ -leaf good, and  $T_2$  is  $F$ -odd. We color  $T_1$  by Lemma 4.35, with  $x$  as helper. So assume  $T_3$  is a path. Suppose some  $y$  splits as 0/0/4. Since  $T_1 \neq K_2$ , some  $x$  splits as 1/1/2 or 2/0/2. If  $x$  is not adjacent to both leaves of  $T_3$ , then we can ignore  $y$  and repeat the argument that starts this paragraph. If  $x$  splits as 1/1/2, then let  $F = \{v_1, v_2, x, y\}$ , so that  $T_1$  and  $T_2$  are each  $F$ -odd, and color  $T_3$  by Lemma 4.35, with  $y$  as helper. If  $x$  splits as 2/0/2, then let  $F = \{v_1, x, y\}$ , so that  $T_2$  is  $F$ -odd,  $T_1$  can be handled by coloring one neighbor of  $x$  with  $I$  (and the rest with  $F$ ), and  $T_3$  can be colored by Lemma 4.35, with  $y$  as helper. Thus, no such  $y$  exists. Now we are down to three ways that vertices in  $B \setminus \tilde{B}$  split: 1/1/2, 2/0/2, 0/2/2.

Suppose some  $x$  splits as 2/0/2 and some  $y$  splits as 1/1/2. By the previous paragraph, they must both be adjacent to both leaves of  $T_3$ . Now let  $F = \{v_1, v_2, x, y\}$ . Trees  $T_1$  and  $T_2$  are both  $F$ -odd. For  $T_3$ , we

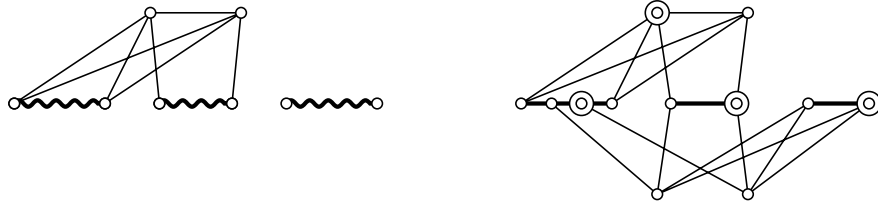


Figure 21: Case (f).

color one leaf with  $I$  and the rest of  $T_3$  with  $F$ . This implies that vertices split as exactly one of the ways  $2/0/2$  and  $1/1/2$  (since  $T_1 \neq K_2$ ). Suppose  $x$  splits as  $2/0/2$ . Since  $T_2$  has more than two incident edges, some  $y$  splits as  $0/2/2$ . Let  $F = \{v_1, x, y\}$ . Note that  $T_2$  is  $F$ -odd. Use  $I$  to color a neighbor of  $x$  in  $T_1$  and a neighbor of  $y$  in  $T_3$ . So no vertex splits as  $2/0/2$ .

Since  $T_1 \neq K_2$ , some vertex  $x$  splits as  $1/1/2$ . If  $x$  is not adjacent to both leaves of  $T_3$ , then let  $F = \{v_1, v_2, x\}$ . Now  $T_1$  and  $T_2$  are  $F$ -odd, and  $T_3$  is  $F$ -leaf-good. So assume  $x$  is adjacent to both leaves of  $T_3$ . Suppose there exists  $y$  of type  $0/2/2$ . Let  $F = \{v_1, v_2, x, y\}$ . Again,  $T_1$  and  $T_2$  are  $F$ -odd. To color  $T_3$ , use  $I$  on a neighbor of  $y$ , and use  $F$  elsewhere. So no such  $y$  exists. That is, all vertices in  $B \setminus \tilde{B}$  are type  $1/1/2$ . Further, each is adjacent to both leaves of  $T_3$ , so exactly two such vertices exist. Thus,  $T_3 = K_2$  and  $T_2 = K_2$  and  $T_1 = P_4$ . This implies that  $|G| = |T_1| + |T_2| + |T_3| + |\tilde{B}| + 2 = 4 + 2 + 2 + 2 + 2 = 12$ . There is exactly one possibility for  $G$ . It is shown on the right in Figure 21, along with an nb-coloring. This finishes Case (f), finishes the larger case that  $G[\tilde{B}] = K_2$ , and completes the proof of (B) in our Main Theorem.  $\square$

## 5 Algorithmic Details

Section 4 contains two types of assertions: (i) graphs of a certain form are near-bipartite and (ii) graphs of a certain form do not satisfy the assumptions of the Main Theorem. To prove each assertion of type (i), we find an nb-coloring. So our proof is constructive, and naturally yields an algorithm. In this section we detail the efficiency of this algorithm. We assume the graph is stored as a list of vertices, and that each vertex stores a list of incident edges, multiedges, edge-gadgets, and its precoloring (if this exists).

Let  $T_m(n)$  denote the maximum running time of the algorithm on a multigraph with  $n$  vertices, and let  $T_s(n)$  denote the corresponding function for simple graphs. As before we write  $T_*(n)$  in statements that hold for both  $T_m(n)$  and  $T_s(n)$ . Our algorithm is recursive, so our upper bound on  $T_*(n)$  is in terms of  $T_*(n-1)$ . We use the crude estimate  $\sum_{i=1}^n i^d \leq \int_1^{n+1} x^d dx \leq n^{d+1}$  for sufficiently large  $n$  and  $d > 1$ . Thus, to prove  $T_*(n) \leq O(n^{d+1})$  it suffices to show that  $T_*(n) \leq T_*(n-1) + O(n^d)$ . When  $G$  contains a vertex set  $W$  with  $\rho_*(W)$  small, we first color  $G[W]$  and second color  $G'$ , formed from  $G$  by contracting  $W$  down to two vertices. That is, the algorithm recurses on two graphs  $G[W]$  and  $G'$ , which satisfy  $|V(G[W])| + |V(G')| = |V(G)| + 2$ . (This case arises in the proofs of our gap lemmas.) Simple calculus shows that  $(n+2-k)^d + k^d$ , with  $3 \leq k \leq n-1$ , is maximized when  $k \in \{3, n-1\}$ . So if  $T_*(j) \leq cj^{d+1}$  for all  $j < n$  and some fixed  $c$ , then  $\max_{3 \leq k \leq n-1} \{T_*(n+2-k) + T_*(k)\} \leq c(n-1)^{d+1} + O(n^d)$ . Hence, to prove  $T_*(n) \leq O(n^{d+1})$  it also suffices to prove that  $T_*(n) \leq O(n^d) + \max_{3 \leq k \leq n-1} \{T_*(n+2-k) + T_*(k)\}$ . So in the individual steps below we focus on the time to construct the recursive calls, and extend the colorings afterward. Only after listing all steps do we account for the time spent on the recursive calls.

We assume that every graph with at most 30 vertices can be nb-colored in time  $O(1)$ , if it has an nb-coloring. We also assume that we can iterate through each graph in  $\mathcal{H}$  in time  $O(1)$ . Since each graph in  $\mathcal{H}$  has at most 22 vertices, we can determine whether a given pair of vertices is linked in a graph with order  $n$  by a graph in  $\mathcal{H}$  in time  $O(n^{20})$ . In practice this can be done much faster, since we only need to consider connected subgraphs.

We start with Part (A) of the Main Theorem. Let  $G$  be an input graph with  $n$  vertices. We assume that  $G$  satisfies the hypotheses of the Main Theorem, so  $|E(G)| = O(n)$ . We list in order the steps of the algorithm. Each step except the last describes how to color the graph if it satisfies certain conditions. Each step assumes that the conditions of the previous steps fail to hold. We will show that  $T_m(n) = O(n^6)$ .

1.  $G$  is disconnected. We recurse on each component.
2.  $G$  contains a vertex  $v$  satisfying at least one of the following conditions:  $d(v) = 1$ ,  $v$  is precolored  $I$ ,  $|N(v)| = 1$ , or  $d(v) = 2$  and  $v$  is uncolored. Each of these criteria can be tested in time  $O(n)$ . If any criterion is satisfied, then we apply the proof of Lemma 4.2, 4.3, 4.4, or 4.5. Constructing the graph to recurse on takes time  $O(n)$ ; extending the coloring takes time  $O(1)$ .
3.  $G$  contains a proper non-trivial vertex subset  $W$  with  $\rho_{*,G}(W) \leq 0$ . We find a subset  $W$  with smallest potential, and among them choose one with largest order (so  $W \neq \emptyset$ ). By Corollary 2.4 with  $m_1 = 0$

and  $m_2 = 1$ , this takes time  $O(n^3 \log(n))$ . We recurse on  $G[W]$ , and then construct  $G'$  as in the proof of Lemma 4.6. Constructing  $G'$  takes time  $O(n)$ . Merging the two colorings takes time  $O(n)$ . So the total time for these steps is  $O(n) + O(n) + O(n^3 \log(n)) \leq O(n^4)$ .

4.  $G$  contains a vertex subset  $W$  with  $\rho_{m,G}(W) = 1$  and  $1 \leq |W| \leq n - 1$ . We use the same operations as in the previous step, but apply Corollary 2.4 with  $m_1 = 1, m_2 = 1$ . Our running time is now  $O(n^5)$ .
5.  $G$  contains a vertex  $v$  satisfying at least one of the following conditions:  $d(v) = 2$ ,  $v$  is precolored  $F$ ,  $v$  is incident to a multiedge, or  $v$  has neighbors that are adjacent. The first three criteria can be tested in time  $O(n)$ ; the last in time  $O(n^3)$ . We apply the proof of Lemma 4.12, 4.13, 4.15, or 4.17. Constructing the graph to recurse on takes time  $O(1)$ ; extending the coloring also takes time  $O(1)$ .
6. We apply the proof of Lemma 4.18. Constructing the graph to recurse on takes time  $O(1)$ ; extending the coloring also takes time  $O(1)$ .

In each step above, the time spent on pre- and post-processing the recursive calls is  $O(n^5)$ , and the time for the recursion is  $\max\{T_m(n-1), \max_{3 \leq k \leq n-1}\{T_m(n+2-k) + T_m(k)\}\}$ . Thus, we have  $T_m(n) \leq O(n^5) + \max\{T_m(n-1), \max_{3 \leq k \leq n-1}\{T_m(n+2-k) + T_m(k)\}\}$ . So  $T_m(n) \leq O(n^6)$ .

We now consider Part (B) of the Main Theorem. Since we merged arguments in Section 4.1, the first three steps are the same; so we omit them below. Before we list the algorithm's steps, we note that by the start of Section 4.3 (where we begin after skipping the common three steps), we have proved Lemma 4.21: If two vertices in  $G$  are linked, then they are specially-linked (and the linking graph is in  $\mathcal{H}$ ). So we can decide if a given pair of vertices is linked in time  $O(n^{20})$ . Also note that  $G(C, z_1, z_2)$  can be constructed in time  $O(|C|)$ . Let  $L$  denote the set of uncolored vertices of degree 3 with no incident edge-gadgets. Note that applying the arguments of Section 4.4.3 takes time  $O(n)$ . As above, for each step we focus on the pre- and post-processing time. Only at the end do we consider the time for the recursion. We will show that  $T_s(n) \leq O(n^{22})$ .

4.  $G[L]$  contains an induced cycle  $C$  of length 3 or 4. We can find  $C$  in time  $O(|L|^4) \leq O(n^4)$ . If  $C$  has length 4, then we apply Lemma 4.22. Constructing the graph to recurse on takes time  $O(1)$ ; extending the coloring also takes time  $O(1)$ . If  $C$  has length 3, then we must find a pair of vertices  $z_1, z_2$  in  $N(C)$  that are not linked. We check  $\binom{3}{2}$  pairs, which takes total time  $O(n^{20})$ . Constructing  $G(C, z_1, z_2)$  as in Lemma 4.22 (the graph we recurse on) takes time  $O(1)$ ; extending the coloring also takes time  $O(1)$ .
5.  $G$  contains a vertex subset  $W$  with  $\rho_{m,G}(W) \leq 4$  and  $1 \leq |W| \leq n - 2$ . We perform the same operations as in step 3 above, but apply Corollary 2.4 with  $m_1 = 1, m_2 = 2$ . The running time is now  $O(n^6)$ .
6.  $G[L]$  contains an induced cycle  $C$  of length 5. We can find  $C$  in time  $O(|L|^5) \leq O(n^5)$ . We perform the same operations as in step 4 above, but now we check  $\binom{5}{2}$  vertex pairs.
7.  $G$  contains a vertex  $v$  with  $d(v) = 2$ . We can find  $v$  in time  $O(n)$ . We apply the proof of Lemma 4.28. Note that Case 3 of Lemma 4.28 (where the neighbors of  $v$  are linked) implies that  $V(G) = V(H) \cup \{v\}$ . So  $|V(G)| \leq 23$ . We assumed above that  $n \geq 30$ , so we can construct the graph to recurse on in time  $O(1)$ ; extending the coloring also takes time  $O(1)$ .
8.  $G[L]$  contains an induced cycle  $C$ . Now  $C$  can be found in time  $O(|L|) \leq O(n)$ . We perform the same operations as in step 4 above, but with Lemma 4.29 instead of Lemma 4.22. We only need to check for non-linked pairs of vertices among neighbors of consecutive members of  $C$ , so we only check  $|C| - 1$  pairs. Since  $|C| \leq n - 1$ , this step runs in time  $O(n * n^{20}) = O(n^{21})$ .
9.  $G$  contains a vertex  $v$  that satisfies at least one of the following:  $d(v) = 5$ ,  $v$  is precolored, or  $v$  is incident to an edge-gadget. Each of these criteria can be tested in time  $O(n)$ . We apply the proof of Lemma 4.31; finding the coloring takes time  $O(n)$ .
10. We apply the arguments of Section 4.4.4. Finding the coloring takes time  $O(n)$ .

Thus,  $T_s(n) \leq O(n^{21}) + \max\{T_s(n-1), \max_{3 \leq k \leq n-1}\{T_s(n+2-k) + T_s(k)\}\}$ , so  $T_s(n) \leq O(n^{22})$ .

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