On the Boundedness of Positive Solutions of the Reciprocal Max-Type Difference Equation $x_n = \max\left\{\frac{A^1_{n-1}}{x_{n-1}}, \frac{A^2_{n-1}}{x_{n-2}}, \ldots, \frac{A^t_{n-1}}{x_{n-t}}\right\}$ with Periodic Parameters

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Abstract

We investigate the boundedness of positive solutions of the reciprocal max-type difference equation

$$x_n = \max\left\{\frac{A^1_{n-1}}{x_{n-1}}, \frac{A^2_{n-1}}{x_{n-2}}, \ldots, \frac{A^t_{n-1}}{x_{n-t}}\right\}, \quad n = 1, 2, \ldots,$$

where, for each value of i, the sequence $\{A^i_n\}_{n=0}^{\infty}$ of positive numbers is periodic with period p_i. We give both sufficient conditions on the p_i's for the boundedness of all solutions and sufficient conditions for all solutions to be unbounded. This work essentially complements the work by Biddell and Franke, who showed that as long as every positive solution of our equation is *bounded*, then every positive solution is eventually periodic, thereby leaving open the question as to when solutions are bounded.

Keywords: reciprocal max-type equation; unbounded solutions; eventually periodic solutions; neuroscience; morphogenesis

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1 Introduction

Difference equations with the maximum function and reciprocal arguments, or reciprocal max-type difference equations, were conceived circa 1994, when G. Ladas at the University of Rhode Island in a seminar on difference equations and applications took a variant of the Lozi map,

$$y_n = |y_{n-1}| - y_{n-2}, \quad n = 1, 2, \ldots,$$

made a change of variables, $y_n = \ln x_n$, and produced the following equation:

$$x_n = \frac{\max\{x_{n-1}^2, 1\}}{x_{n-1}x_{n-2}}, \quad n = 1, 2, \ldots .$$

This equation was extended to the equation

$$x_n = \frac{\max\{x_{n-1}^k, A\}}{x_{n-1}^\ell x_{n-2}}, \quad n = 1, 2, \ldots, \tag{1}$$

where $k, \ell \in \mathbf{Z}$, $A \in (0, \infty)$, and initial conditions are positive (cf. [19]). Investigation of (1) with $A = 1$ led to interesting results about the periodicity of solutions, e.g., every solution is periodic with period 9 if $k = 2$ and $\ell = 1$.

From 1994 to the present, difference equations with the maximum (or minimum) function and reciprocal arguments have rapidly evolved into a diverse family of equations. In 1998, Al-Amleh, Hoag, and Ladas [2] investigated one of the earliest autonomous reciprocal max-type equations,

$$x_n = \max\left\{\frac{a}{x_{n-1}}, \frac{A}{x_{n-2}}\right\}, \quad n = 1, 2, \ldots, \tag{2}$$

where $a, A \in \mathbf{R} - \{0\}$. One major result they obtained was that when $a = 1$, $A \in (0, \infty)$, and initial conditions are positive, every solution is periodic with

1. period 2 if $A \in (0, 1)$;
2. period 3 if $A = 1$;
3. period 4 if $A \in (1, \infty)$.

However, they also showed that every solution is *unbounded* when $a \neq A$, $a, A \in (-\infty, 0)$, and $x_{-1}, x_0 \in \mathbf{R} - \{0\}$.

Replacing the constant coefficient A by the variable coefficient A_n (and replacing a by 1) in (2), in 1997 Briden et al. [7] studied the resulting nonautonomous equation

$$x_n = \max\left\{\frac{1}{x_{n-1}}, \frac{A_{n-1}}{x_{n-2}}\right\}, \quad n = 1, 2, \dots, \tag{3}$$

where $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of positive numbers with period 2 and initial conditions are positive. They showed that every positive solution is eventually periodic with

1. period 2 if $A_0A_1 \in (0,1)$;
2. period 6 if $A_0A_1 = 1$;
3. period 4 if $A_0A_1 \in (1,\infty)$.

At this point, no unbounded solutions were known. A few years later, Briden et al. [6] and Grove et al. [13] changed the period of $\{A_n\}_{n=0}^{\infty}$ in (3) to period 3, and unbounded solutions made their first appearance with a positive periodic parameter and positive initial conditions in a reciprocal max-type equation. Specifically, letting $\{A_n\}_{n=0}^{\infty}$ be positive and periodic with period 3, they showed that every solution is

1. eventually periodic with period 2, if $A_n \in (0,1)$ for all $n \geq 0$;
2. eventually periodic with period 12, if $A_n \in (1,\infty)$ for all $n \geq 0$;
3. *unbounded*, if $A_{i+1} < 1 < A_i$ for some $i \in \{0,1,2\}$;
4. eventually periodic with period 3 in all other cases.

Upon the discovery that unbounded solutions could indeed occur with the reciprocal max-type equation, Kent and Radin [16] in 2003 sought necessary and sufficient conditions for boundedness with the equation

$$x_n = \max\left\{\frac{A_{n-1}}{x_{n-1}}, \frac{B_{n-1}}{x_{n-2}}\right\}, \quad n = 1, 2, \dots, \tag{4}$$

where $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ are positive periodic sequences with minimal periods p and q, respectively, and initial conditions are positive. They showed that every positive solution is:

1. bounded, if neither p nor q is a multiple of 3;
2. unbounded, if $p = 3k$ for some $k \in \mathbf{Z}^+$, such that for some $i \in \{1, 2, 3\}$ and for all $j = 1, 2, \dots, k$,

$$A_{i+3j} < B_1, B_2, \dots, B_q < A_{i+3j+1};$$

3. unbounded, if $q = 3k$ for some $k \in \mathbf{Z}^+$, such that for some $i \in \{1, 2, 3\}$ and for all $j = 1, 2, \dots, k$,

$$B_{i+3j} < A_1, A_2, \dots, A_q < B_{i+3j+1}.$$

In contrast, Kerbert and Radin [17] showed in 2008 that every positive solution of the equation

$$x_n = \max\left\{\frac{A_{n-1}}{x_{n-1}}, \frac{B_{n-1}}{x_{n-3}}\right\}, \quad n = 1, 2, \dots, \tag{5}$$

is unbounded if $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ are positive periodic sequences with minimal periods p and q, respectively, and p or q is a multiple of 4 (together with certain other conditions on A_n and B_n which we omit for the sake of brevity). Furthermore, for the equation

$$x_n = \max\left\{\frac{A_{n-1}}{x_{n-2}}, \frac{B_{n-1}}{x_{n-3}}\right\}, \quad n = 1, 2, \dots, \tag{6}$$

where again $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ are positive periodic sequences with minimal periods p and q, respectively, the methods in this paper can be used to show that every positive solution is unbounded if either p or q is a multiple of 5 (together with certain other conditions which we again omit for brevity).

Around the same time as Kerbert and Radin's investigation, Bidwell and Franke in a landmark paper [5] considered the following equation:

$$x_n = \max\left\{\frac{A^1_{n-1}}{x_{n-1}}, \frac{A^2_{n-1}}{x_{n-2}}, \dots, \frac{A^t_{n-1}}{x_{n-t}}\right\}, \quad n = 1, 2, \dots, \tag{7}$$

where $t \in \{1, 2, \dots\}$, $\{A^i_n\}_{n=0}^{\infty}$, for all $i = 1, \dots, t$, is a nonnegative periodic sequence with period $p_i \in \mathbf{Z}^+$, and initial conditions are positive. They showed that if every solution of (7) is *bounded*, then every solution is eventually periodic. The question then remained: *Under what conditions on the nonnegative periodic parameters is every solution bounded?*

In Sections 3 and 4, we largely answer this question when the periodic parameters, $\{A_n^i\}_{n=1}^{\infty}$ for all $i = 1, \dots, t$, are positive. Specifically, we find sufficient conditions on the parameters' periods such that every solution is bounded and also sufficient conditions on the parameters' periods and on the parameters' values, in comparison with each other, such that every solution is unbounded.

The past decade has seen investigation of difference equations that are extensions and generalizations of (7), as well as difference equations with maxima that have been inspired by differential equations with maxima and automatic control theory (cf. [3]). All such difference equations have added an order of great complexity. For a sampling of this work, see the papers by Çinar, Stević, and Yalçinkaya [10]; Iričan and Elsayad [15]; Liu, Yang, and Stević [20]; Qin, Sun, and Xi [21]; Sauer [22] and [23]; Stević [25]; Sun [26]; Touafek and Halim [27]; and Yang, Liu, and Lin [28].

2 Preliminaries

For convenience, we write $[k]$ to denote the set $\{1, \dots, k\}$. Let $\{A_n^i\}_{n=0}^{\infty}$, for each $i \in [t]$, be a periodic sequence of positive real numbers with prime period p_i. The following preliminaries will be useful in the sequel.

Definition 1 (Boundedness and Persistence)**.** A positive sequence $\{x_n\}_{n=-t}^{\infty}$ is *bounded* if there exists a positive constant M with

$$0 < x_n \leq M \quad \text{for all } n = -t, \dots, 0, 1, \dots$$

and it *persists* (or is *persistent*) if there exists a positive constant m with

$$m \leq x_n \quad \text{for all } n = -t, \dots, 0, 1, \dots.$$

Remark 1. To show that a positive solution of a difference equation

$$x_n = f(x_{n-1}, x_{n-2}, \dots, x_{n-t}), \quad n = 0, 1, \dots,$$

is unbounded and does not persist, we exhibit a subsequence of the solution which diverges to infinity and a subsequence which converges to zero.

Definition 2 (Eventual Periodicity)**.** A positive sequence $\{x_n\}_{n=-t}^{\infty}$ is *eventually periodic* if there exists $N \geq -t$ such that $\{x_n\}_{n=N}^{\infty}$ is periodic.

Definition 3 (Extended Periodicity)**.** A positive sequence $\{x_n\}_{n=-t}^{\infty}$ is *extended periodic with period* p if for all $i \in [p]$ either $\lim_{n\to\infty} x_{pn+i} = \infty$ or $\lim_{n\to\infty} x_{pn+i} = 0$.

Lemma 4. *If $\{x_n\}_{n=-t}^{\infty}$ is a positive solution of (7), then $\{x_n\}_{n=-t}^{\infty}$ is bounded if and only if it persists.*

Proof. We show that persistence implies boundedness. The proof of the converse is similar, so we omit it.

Let $\{x_n\}_{n=-t}^{\infty}$ be a persistent positive solution of equation (7); since it is persistent, there exists $\epsilon > 0$ such that $x_i > \epsilon$ for all i. Let $\alpha = \max_j A^j$. Now we get $x_n = \max_{1\leq j\leq t}\left\{\frac{A^j}{x_{n-j}}\right\} \leq \frac{\alpha}{\epsilon}$. Thus, $\{x_n\}_{n=-t}^{\infty}$ is bounded. □

In the following definition, the sequences $\{A_n^i\}_{n=0}^{\infty}$ need not be periodic.

Definition 5 (Hypothesis (H))**.** A set of t sequences of positive real numbers $\{A_n^i\}_{n=0}^{\infty}$, $i \in [t]$, satisfies Hypothesis (H) if there exists $j \in [t+1]$ such that for all $i \in [t]$ we have

$$\begin{aligned} S_{A^i} &= \sup\{A^i_{(t+1)n+(t+1)+j} :\ n = 0, 1, \ldots\} \\ &< I_{A^{t+1-i}} = \inf\{A^{t+1-i}_{(t+1)n+(t+1-i)+j} :\ n = 0, 1, \ldots\}. \end{aligned}$$

Remark 2. If $\{A_n^i\}_{n=0}^{\infty}$, $i \in [t]$, is periodic with period $t+1$, then Hypothesis (H) becomes the following:

$$A^i_{(t+1)n+(t+1)+j} < A^{t+1-i}_{(t+1)n+(t+1-i)+j}$$

for some $j \in [t+1]$ and for all $i \in [t]$.

The number $t+1$ is "special" and plays a central role in the results of the sequel. This number is based on an *ansatz*, derived from various observations. It is easy to show that every positive solution of the difference equation

$$x_n = \frac{1}{x_{n-t}}, \quad n = 1, 2, \ldots$$

where $t \in \mathbf{Z}^+$ and initial conditions are positive, is periodic with period $2t$. Now observe the following:

1. For (4), the period of either $\{A_n\}_{n=0}^{\infty}$ or $\{B_n\}_{n=0}^{\infty}$ must be a multiple of 3 in order for every solution to be unbounded. Now 3 is the average of 2 and 4, the respective periods of every positive solution of the equations

$$x_n = \frac{1}{x_{n-1}} \quad \text{and} \quad x_n = \frac{1}{x_{n-2}}.$$

The right sides of these two equations make up the arguments of (4).

2. Similarly, for (5), the period of either $\{A_n\}_{n=0}^{\infty}$ or $\{B_n\}_{n=0}^{\infty}$ must be a multiple of 4 for every solution to be unbounded. Now 4 is the average of 2 and 6, the respective periods of every positive solution of the equations

$$x_n = \frac{1}{x_{n-1}} \quad \text{and} \quad x_n = \frac{1}{x_{n-3}}.$$

The right sides of these two equations make up the arguments of (5).

3. Finally, for (6), the period of either $\{A_n\}_{n=0}^{\infty}$ or $\{B_n\}_{n=0}^{\infty}$ must be a multiple of 5 for every solution to be unbounded. Now 5 is the average of 4 and 6, the respective periods of every positive solution of the equations

$$x_n = \frac{1}{x_{n-2}} \quad \text{and} \quad x_n = \frac{1}{x_{n-3}}.$$

The right sides of these two equations make up the arguments of (6).

Now we note that the number $t+1$ is the average of the respective periods of every positive solution of the equations

$$x_n = \frac{1}{x_{n-1}}, \quad x_n = \frac{1}{x_{n-2}}, \quad \ldots, \quad x_n = \frac{1}{x_{n-t}},$$

where

$$\frac{2\sum_{\ell=1}^{t} \ell}{t} = \frac{2t(t+1)}{2t} = t+1.$$

The right sides of these t equations make up the arguments of (7).

3 Sufficient Conditions for Boundedness

In this section, we find sufficient conditions on the periods p_i of the sequences $\{A^i_n\}_{n=0}^{\infty}$ for all $i \in [t]$ such that every positive solution of (7) is bounded (and persists). The bulk of the work falls into Lemmas 6 and 7. In the first lemma, we prove that for any $M \in \mathbf{Z}^+$, we can find sufficiently small ϵ so that if some $x_n < \epsilon$, then we know the exact value of nearly all of the $(t+1)M$ terms preceeding x_n. In the second lemma, we show that if, in addition, certain of the t sequences $\{A^i_n\}_{n=0}^{\infty}$ have period relatively prime to $t+1$, then some specific term further along in the sequence must be at least as large as this x_n. Finally, we show that every such sequence must be bounded.

Lemma 6. *Given $M \in \mathbf{Z}^+$, there exist $r > 0$ and $\epsilon > 0$ such that if there exists $N \in \mathbf{N}$ with $x_N < \epsilon$, then we have*

$$x_{N-k(t+1)-i} = \frac{A^{t+1-i}_{N-k(t+1)-i-1}}{x_{N-(k+1)(t+1)}} \tag{8}$$

$$x_{N-(k+1)(t+1)} < \epsilon r^{k+1} \tag{9}$$

for all $i \in [t]$ and all k with $0 \le k \le M-1$.

Proof. Intuitively, we want to know the exact value of x_i for many of the values of i in some large range, namely a range of $M(t+1)$ successive values. Given M, we find a small value ϵ such that if $x_N < \epsilon$, then we know the exact value of each x_{N-j} when $j \in [M(t+1)]$ and $(t+1) \not| j$. Roughly speaking, we show that if $j \in [M(t+1)]$ and $(t+1)|j$, then x_{N-j} must be very small; so small, in fact, that for each $i \in [t]$ the maximum in the definition of x_{N-j+i} is achieved by the term that divides by x_{N-j}. Throughout the proof, we need a number of inequalities that bound ϵ from above. We do not list these explicitly. Rather, we note only that we have a finite number of inequalities that bound ϵ from above, and the upper bounds are strictly positive. Thus, we can choose positive ϵ that satisfies them all.

By possibly relabeling, we assume $N = 0$. Let $\alpha = \max_{i,j} A^i_j$ and $\beta = \min_{i,j} A^i_j$ and let $r = \alpha/\beta$. We primarily want to prove (8), which will satisfy the hypothesis of our next lemma. However, for the proof, we need to prove (9) as well. Our proof is by induction on k. When invoking the induction hypothesis, we will only assume (9). Thus, our base case $k = 0$ is simply a special case of the general induction step, since by hypothesis we have $x_0 < \epsilon$.

To prove (8) it suffices to show that

$$x_{-k(t+1)-j} > x_{-(k+1)(t+1)} \frac{A^{j-i}_{-k(t+1)-i-1}}{A^{t+1-i}_{-k(t+1)-i-1}} \tag{10}$$

for all $i \in [t]$ and $i < j < i + (t+1)$, such that $j \neq t+1$. This inequality may look daunting, but it is simply saying that when computing $x_{-k(t+1)-i}$, the term that divides by $x_{-(k+1)(t+1)}$ is larger than the term that divides by $x_{-k(t+1)-j}$. We first prove (9). We consider for $x_{-k(t+1)-1}$ which argument is maximum, and we show this is the argument that divides by $x_{-(k+1)(t+1)}$. If instead the maximum argument divides by $x_{-k(t+1)-j}$, then we get

$$\begin{aligned} x_{-k(t+1)-1} &= \frac{A^{j-1}_{-k(t+1)-2}}{x_{-k(t+1)-j}} \\ &< \frac{A^{j-1}_{-k(t+1)-2}}{A^{j}_{-k(t+1)-1}} \epsilon r^k \\ &< \frac{A^{1}_{-k(t+1)-1}}{\epsilon r^k} \\ &< x_{-k(t+1)-1}, \end{aligned}$$

which is a contradiction. The first inequality holds because $x_{-k(t+1)} < \epsilon r^k$ and $A^j_{-k(t+1)-1}/x_{-k(t+1)-j} \leq x_{-k(t+1)}$, the second holds because ϵ is sufficiently small, and the third holds because $x_{-k(t+1)} < \epsilon r^k$. This contradiction implies that $x_{-k(t+1)-1} = \frac{A^t_{-k(t+1)-2}}{x_{-(k+1)(t+1)}}$. Rewriting, we have

$$\begin{aligned} x_{-(k+1)(t+1)} &= \frac{A^{t}_{-k(t+1)-2}}{x_{k(t+1)-1}} \\ &< \frac{A^{t}_{-k(t+1)-2}}{A^{1}_{-k(t+1)-1}} \epsilon r^k \\ &< \epsilon r^{k+1}. \end{aligned}$$

As above, the first inequality holds because $x_{-k(t+1)} < \epsilon r^k$; the second inequality comes from the definition of r. Thus (9) holds.

Now we prove (8). To do so, we first prove (10) for $i < j < t+1$, and then prove (10) for $t+1 < j < i+(t+1)$. By hypothesis, $x_{-k(t+1)} < \epsilon r^k$.

For each $j \in [t]$, we get $\epsilon r^k > x_{-k(t+1)} \geq \frac{A^j_{-k(t+1)-1}}{x_{-k(t+1)-j}}$. Cross-multiplying gives

$$\begin{aligned} x_{-k(t+1)-j} &> \frac{A^j_{-k(t+1)-1}}{\epsilon r^k} \\ &> \epsilon r^{k+1} \frac{A^{j-i}_{-k(t+1)-i-1}}{A^{t+1-i}_{-k(t+1)-i-1}} \\ &> x_{-(k+1)(t+1)} \frac{A^{j-i}_{-k(t+1)-i-1}}{A^{t+1-i}_{-k(t+1)-i-1}}, \end{aligned}$$

where the second inequality holds because ϵ is sufficiently small and the third holds by (9), which we proved above. So we have proved (10) for $i < j < t+1$. Now we prove it for $t+1 < j < i+(t+1)$. The argument is quite similar.

By (9) we have $x_{-(k+1)(t+1)} < \epsilon r^{k+1}$. By transitivity, for each j with $t+1 < j < i+(t+1)$, we get $\epsilon r^{k+1} > x_{-(k+1)(t+1)} \geq \frac{A^j_{-(k+1)(t+1)-1}}{x_{-(k+1)(t+1)-j}}$. Rewriting this, we get

$$\begin{aligned} x_{-(k+1)(t+1)-j} &> \frac{A^j_{-(k+1)(t+1)-1}}{\epsilon r^{k+1}} \\ &> \epsilon r^{k+1} \frac{A^{j-i}_{-k(t+1)-i-1}}{A^{t+1-i}_{-k(t+1)-i-1}} \\ &> x_{-(k+1)(t+1)} \frac{A^{j-i}_{-k(t+1)-i-1}}{A^{t+1-i}_{-k(t+1)-i-1}}. \end{aligned}$$

As above, the second inequality holds because ϵ is sufficiently small and the third holds by (9). So we have proved (10) for all $i < j < i+(t+1)$. Together with the case above, this proves (8), and thus completes the proof. $\square$

Lemma 7. *Let $\{x_n\}_{n=-t}^{\infty}$ be a solution of (7). Suppose there exists i with* $\gcd(t+1, p_i p_{t+1-i}) = 1$, *and let $P = p_i p_{t+1-i}$. If there exists $N \geq 0$ such that for all $j \in [t]$ and $k \in [P]$*

$$\begin{aligned} x_{N+k(t+1)-j} &= \max\left\{ \frac{A^1_{N+k(t+1)-j-1}}{x_{N+k(t+1)-j-1}}, \frac{A^2_{N+k(t+1)-j-1}}{x_{N+k(t+1)-j-2}}, \dots, \frac{A^t_{N+k(t+1)-j-1}}{x_{N+k(t+1)-j-t}} \right\} \\ &= \frac{A^{t+1-j}_{N+k(t+1)-j-1}}{x_{N+k(t+1)-(t+1)}}, \end{aligned} \tag{11}$$

then $x_{N+P(t+1)} \geq x_N$.

Proof. First note that for any choice of P, we can satisfy the second hypothesis by (8) of Lemma 6. Thus, to apply the present lemma, we will only need to demonstrate that there exists i with $\gcd(t+1, p_i p_{t+1-i}) = 1$. By definition

$$x_{N+k(t+1)} = \max_{1 \leq j \leq t} \left\{ \frac{A^j_{N+k(t+1)-1}}{x_{N+k(t+1)-j}} \right\}. \tag{12}$$

By substituting (11) into (12), we get

$$x_{N+k(t+1)} = x_{N+(k-1)(t+1)} \max_{1 \leq j \leq t} \left\{ \frac{A^j_{N+k(t+1)-1}}{A^{t+1-j}_{N+k(t+1)-j-1}} \right\}. \tag{13}$$

By repeated application of recurrence (13) for all $k \in [P]$, we get

$$x_{N+P(t+1)} = x_N \prod_{k=1}^{P} \max_{1 \leq j \leq t} \left\{ \frac{A^j_{N+k(t+1)-1}}{A^{t+1-j}_{N+k(t+1)-j-1}} \right\}. \tag{14}$$

Recall that $\gcd(t+1, p_i p_{t+1-i}) = 1$. As a result, $t+1$ is an additive generator of $\mathbf{Z}/P\mathbf{Z}$. Applying this fact to subscripts, we get

$$\prod_{k=1}^{P} A^i_{N+k(t+1)-1} = \prod_{k=1}^{P} A^i_k, \tag{15}$$

$$\prod_{k=1}^{P} A^{t+1-i}_{N+k(t+1)-1} = \prod_{k=1}^{P} A^{t+1-i}_k. \tag{16}$$

From (14), we get that

$$x_{N+P(t+1)} \geq x_N \prod_{k=1}^{P} \frac{A^i_{N+k(t+1)-1}}{A^{t+1-i}_{N+k(t+1)-i-1}} = x_N \frac{\prod_{k=1}^{P} A^i_k}{\prod_{k=1}^{P} A^{t+1-i}_k}, \tag{17}$$

where the inequality follows from the definition of maximum, and the equality follows from substituting (15) and (16). An analogous argument gives that $x_{N+P(t+1)} \geq x_N \frac{\prod_{k=1}^{P} A^{t+1-i}_k}{\prod_{k=1}^{P} A^i_k}$. Combining this inequality with (17), we get that

$$x_{N+P(t+1)} \geq x_N \max\left\{\frac{\prod_{k=1}^{P} A_k^i}{\prod_{k=1}^{P} A_k^{t+1-i}}, \frac{\prod_{k=1}^{P} A_k^{t+1-i}}{\prod_{k=1}^{P} A_k^i}\right\} \geq x_N(1) = x_N. \quad (18)$$

Here the second inequality holds because the arguments to max are reciprocals of each other (hence one of them is at least 1). $\square$

Lemma 8. *Let $\{x_n\}_{n=-t}^{\infty}$ be a solution of (7). If $\{x_n\}_{n=-t}^{\infty}$ does not persist, then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=-t}^{\infty}$ which possesses the following properties:*

(i) $x_{n_{k+1}} < x_{n_k}$ *for all* $k = 1, 2, \ldots$.

(ii) If $n_{k+1} > 1 + n_k$, then $x_{n_k} \leq x_n$ and $x_{n_{k+1}} < x_n$ for each $k = 1, 2, \ldots$ and for all $n_k < n < n_{k+1}$.

(iii) $\lim_{k\to\infty} x_{n_k} = 0$.

Sketch. This lemma was proved for the case $t = 2$ in [16]; however, that proof also holds for general t. For completeness, we sketch the proof here. Since $\{x_n\}_{n=-t}^{\infty}$ does not persist, it contains a subsequence $\{x_{n_k}\}_{n=0}^{\infty}$ with $\lim_{k\to\infty} x_{n_k} = 0$. We greedily take a strictly decreasing subsequence of $\{x_{n_k}\}_{n=0}^{\infty}$; it will evidently satisfy all three desired properties. $\square$

Theorem 9 (Bounded Solutions)**.** *Let $\{x_n\}_{n=-t}^{\infty}$ be a solution of (7). If there exists $i \in [t]$ with $\gcd(t+1, p_i p_{t+1-i}) = 1$, then $\{x_n\}_{n=-t}^{\infty}$ is bounded and persists.*

Sketch. Assume that the hypothesis holds. Suppose to the contrary that $\{x_n\}_{n=-t}^{\infty}$ does not persist. By Lemma 8, we have a subsequence $\{x_{n_k}\}_{k=0}^{\infty}$ for which the three properties of Lemma 8 hold. Property (iii) states that $\lim_{k\to\infty} x_{n_k} = 0$, so we can apply Lemma 6. The conclusion of Lemma 6 satisfies the hypothesis of Lemma 7. Now by Lemma 7, there exists n_k such that $x_{n_k} \geq x_{n_k - P(t+1)}$ (where $P = p_i p_{t+1-i}$ as in Lemma 7). This contradicts the properties of Lemma 8 as follows. If $x_{n_k - P(t+1)}$ is an element of the subsequence $\{x_{n_k}\}_{k=0}^{\infty}$, then it contradicts Property (i). Otherwise, Property (ii) implies that there exist an integer s such that $n_s < n_k - P(t+1)$ and $x_{n_s} \leq x_{n_k - P(t+1)}$. But now $x_{n_s} \leq x_{n_k - P(t+1)} \leq x_{n_k}$, which again contradicts Property (i). $\square$

4 Sufficient Conditions for Every Solution to Be Unbounded

In this section, we present the second of our two main results. We initially show that if the sequences $\{A_n^i\}_{n=0}^{\infty}$, $i = 1, \ldots, t$, of positive real numbers, which are not necessarily periodic, satisfy Hypothesis (H), then every positive solution of (7) is unbounded (and does not persist). We then show that Hypothesis (H) is satisfied when $\{A_n^i\}_{n=0}^{\infty}$ are periodic, and certain of them have period a multiple of $t+1$.

Theorem 10. *If $\{A_n^i\}_{n=0}^{\infty}$, $i = 1, 2, \ldots, t$, is a set of sequences of positive real numbers satisfying Hypothesis (H), then every positive solution of* (7) *is unbounded.*

Proof. Let $\{x_n\}_{n=-t}^{\infty}$ be a positive solution of (7) and let $j \in \{0, 1, \ldots, t\}$. If the constants S_{A^i}, I_{A^i} as defined in Hypothesis (H) are postive for all $i = 1, 2, \ldots, t$, then for all $n \geq 0$, we have the following:

$$\begin{aligned}
x_{(t+1)n+(t+2)+j} &= \max_{1\leq i\leq t}\left\{\frac{A^i_{(t+1)n+(t+1)+j}}{x_{(t+1)n+(t+2)-i+j}}\right\} \\
&= \max_{1\leq i\leq t}\left\{\frac{A^i_{(t+1)n+(t+1)+j}}{\max_{1\leq k\leq t}\left\{\frac{A^k_{(t+1)n+(t+1)-i+j}}{x_{(t+1)n+(t+2)-k-i+j}}\right\}}\right\} \\
&= \max_{1\leq i\leq t}\left\{\min_{1\leq k\leq t}\left\{\frac{A^i_{(t+1)n+(t+1)+j}x_{(t+1)n+(t+2)-k-i+j}}{A^k_{(t+1)n+(t+1)-i+j}}\right\}\right\} \\
&\leq \max_{1\leq i\leq t}\left\{\frac{A^i_{(t+1)n+(t+1)+j}}{A^{t+1-i}_{(t+1)n+(t+1-i)+j}}\right\}x_{(t+1)n+1+j} \\
&\leq \max_{1\leq i\leq t}\left\{\frac{S_{A^i}}{I_{A^{t+1-i}}}\right\}x_{(t+1)n+1+j}.
\end{aligned}$$

The first inequality comes from letting $k = t+1-i$ in the min, and the second comes from the definitions of S_{A^i} and $I_{A^{t+1-i}}$. Now let $\alpha = \max_{1\leq i\leq t}\left\{\frac{S_{A^i}}{I_{A^{t+1-i}}}\right\}$. Since the sequences $\{A_n^i\}_{n=0}^{\infty}$ satisfy Hypothesis (H), we get $\alpha < 1$. We have just shown that $x_{(t+1)n+(t+2)+j} \leq \alpha x_{(t+1)n+1+j}$ for all

$n \geq 0$, so $x_{(t+1)n+1+j} \leq \alpha^n x_{1+j}$. It follows that $\lim_{n\to\infty} x_{(t+1)n+1+j} = 0$. Furthermore, $\lim_{n\to\infty} x_{(t+1)n+k+j} = \infty$ for all $2 \leq k \leq t+1$. Below we give the proof for $k = 2$; the other proofs are analogous.

$$\begin{aligned} x_{(t+1)n+2+j} &= \max_{1\leq i\leq t+1} \left\{ \frac{A^i_{(t+1)n+1+j}}{x_{(t+1)n+(2-i)+j}} \right\} \\ &\geq \frac{A^1_{(t+1)n+1+j}}{x_{(t+1)n+1+j}} \\ &\geq \frac{I_{A^1}}{x_{(t+1)n+1+j}}. \end{aligned}$$

□

Remark 3. Observe that the solution in Theorem 10 is extended periodic.

Corollary 11 (Periodic Coefficients)**.** Let $\{A^i_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with period $p_i \in \mathbf{Z}^+$ for all $i \in [t]$. Let $d = t/2$ if t is even and let $d = (t-1)/2$ if t is odd. If t is odd, then assume also that for $i = (t+1)/2$ we have $k_i \in \mathbf{Z}^+$ such that $p_i = (t+1)k_i$ and for all $l_i \in [k_i]$:

$$A^i_{(t+1)l_i+(t+1)+j} < A^i_{(t+1)l_i+i+j}.$$

Every positive solution of (7) is unbounded if either of the following holds:

1. For each $i \in [d]$, we have (a) $p_i = (t+1)k_i$, where $k_i \in \mathbf{Z}^+$, and (b) for some $j \in [t+1]$ and for all $l_i \in [k_i]$ we have:

$$\begin{aligned} A^i_{(t+1)l_i+(t+1)+j} &< \min\{A^{t+1-i}_1, \ldots, A^{t+1-i}_{p_{t+1-i}}\} \\ &< \max\{A^{t+1-i}_1, \ldots, A^{t+1-i}_{p_{t+1-i}}\} \\ &< A^i_{(t+1)l_i+i+j}. \end{aligned}$$

2. For each $i \in [d]$, we have (a) $p_{t+1-i} = (t+1)k_{t+1-i}$, where $k_{t+1-i} \in \mathbf{Z}^+$, and (b) for some $j \in [t+1]$ and for all $l_{t+1-i} \in [k_{t+1-i}]$ we have:

$$\begin{aligned} A^{t+1-i}_{(t+1)l_{t+1-i}+(t+1)+j} &< \min\{A^i_1, \ldots, A^i_{p_i}\} \\ &< \max\{A^i_1, \ldots, A^i_{p_i}\} \\ &< A^{t+1-i}_{(t+1)l_{t+1-i}+(t+1-i)+j}. \end{aligned}$$

Proof. $\{A^i_n\}_{n=0}^{\infty}$, for all $i \in \{1, 2, \ldots, t\}$, satisfies Hypothesis (H). □

5 Future Goals

We conclude with an open problem and with suggestions for two potential biological applications that could be considered novel for max-type difference equations. First, our open problem.

Open Problem. *Consider the difference equation*

$$x_n = \max\left\{\frac{A_{n-1}^{k_1}}{x_{n-k_1}}, \frac{A_{n-1}^{k_2}}{x_{n-k_2}}, \dots, \frac{A_{n-1}^{k_t}}{x_{n-k_t}}\right\}, \quad n = 1, 2, \dots,$$

where $t \in \{2, 3, \dots\}$*,* $k_1, k_2, \dots, k_t \in \{1, 2, \dots\}$*,* $\{A_n^{k_i}\}_{n=0}^{\infty}$ $(i = 1, 2, \dots, t)$ *is a periodic sequence of* nonnegative *numbers with period* $p_{k_i} \in \{1, 2, \dots\}$*, and initial conditions are positive. Find necessary and sufficient conditions on the periods and periodic parameters such that every solution is bounded.*

Second, max-type difference equations in essence belong to a larger group of difference equations, called piecewise-defined difference equations [1, 4, 12, 14, 18]. Nice features of these difference equations which make them especially suited to serve as models of biological processes and systems include their "decision-making" properties with the incorporation of thresholds and their sometimes eventually periodic or unbounded behavior. Max-type difference equations such as (7) have the additional attractive feature of allowing an arbitrary number of variable parameters.

Piecewise-defined difference equations have been used as models for *neural networks* [8, 9], as well as differential equations with maxima (cf. [3]), the counterparts to max-type difference equations. Less frequently, they have been applied to *morphogenesis* [11, 24], which investigates the origins of growth and shape from the embryo to the full adult. The study of morphogenesis includes analyzing the occurrence of repetitive patterns of development, for example, zebra stripes. There is also abnormal morphogenesis, which is seen in the development of cancer in which there is excessive, almost unbounded, growth of tissues.

We propose that in the future one might consider max-type equations such as (7) or modifications of (7) as candidates for the modeling of neural networks and/or morphogenesis.

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