List colorings of K_5-minor-free graphs with special list assignments

Daniel W. Cranston[*], Anja Pruchnewski[†], Zsolt Tuza[‡], Margit Voigt[§]

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Abstract

A *list assignment* L of a graph G is a function that assigns a set (list) $L(v)$ of colors to every vertex v of G. Graph G is called L*-list colorable* if it admits a vertex coloring φ such that $\varphi(v) \in L(v)$ for all $v \in V(G)$ and $\varphi(v) \neq \varphi(w)$ for all $vw \in E(G)$.

The following question was raised by Bruce Richter. Let G be a planar, 3-connected graph that is not a complete graph. Denoting by $d(v)$ the degree of vertex v, is G L-list colorable for every list assignment L with $|L(v)| = \min\{d(v), 6\}$ for all $v \in V(G)$?

More generally, we ask for which pairs (r, k) the following question has an affirmative answer. Let r and k be integers and let G be a K_5-minor-free r-connected graph that is not a Gallai tree (i.e., at least one block of G is neither a complete graph nor an odd cycle). Is G L-list colorable for every list assignment L with $|L(v)| = \min\{d(v), k\}$ for all $v \in V(G)$?

We investigate this question by considering the components of $G[S_k]$, where $S_k := \{v \in V(G) \mid d(v) < k\}$ is the set of vertices with small degree in G. We are especially interested in the minimum distance $d(S_k)$ in G between the components of $G[S_k]$.

[*]Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, Richmond, Virginia, USA

[†]Faculty of Mathematics and Natural Sciences, Ilmenau University of Technology, Ilmenau, Germany

[‡]Computer and Automation Institute, Hungarian Academy of Sciences, Budapest; and Department of Computer Science and Systems Technology, University of Pannonia, Veszprém, Hungary. Research supported in part by the Hungarian Scientific Research Fund, OTKA grant 81493.

[§]Faculty of Information Technology and Mathematics, University of Applied Sciences, Dresden, Germany

1 Introduction

In the seventies of the last century, the concept of list colorings was introduced independently by Erdős, Rubin and Taylor [2] and by Vizing [11]. Since then this topic has been studied extensively by many authors, including [1]-[12]. In particular, list colorings of planar graphs have received and continue to receive enormous amounts of attention; see, e.g., the surveys [9, 6].

Let $G = (V, E)$ be a graph, let $f : V \longrightarrow \mathbb{N}$ be a function, and let $k \geq 0$ be an integer. A *list assignment* L of G is a function that assigns to every vertex v of G a set (list) $L(v)$ of colors (usually each color is a positive integer). We say that L is an *f-assignment* or a *k-assignment* if $|L(v)| = f(v)$ for all $v \in V$ or $|L(v)| = k$ for all $v \in V$, respectively. A *coloring* of G is a function φ that assigns a color to each vertex of G so that $\varphi(v) \neq \varphi(w)$ whenever $vw \in E$. An *L-coloring* of G is a coloring φ of G such that $\varphi(v) \in L(v)$ for all $v \in V$. If G admits an L-coloring, then G is *L-colorable.* When $L(v) = \{1 \ldots, k\}$ the corresponding terms become *k-coloring* and *k-colorable*, respectively. The graph G is said to be *f-list colorable* if G is L-colorable for every f-assignment L of G. When $f(v) = k$ for all $v \in V$, the corresponding term becomes *k-list colorable.*

Erdős, Rubin and Taylor [2] asked, among other problems, the following two questions. Are there planar graphs that are not 4-list colorable? Is every planar graph 5-list colorable? Both questions were answered in 1993. In [12] Voigt gave the first example of a non 4-list colorable planar graph. In [8] Thomassen answered the second question with a beautiful proof of the following result.

Theorem 1 ([8]) *Every planar graph is 5-list colorable.*

Škrekovski [7] extended this result to K_5-minor-free graphs.

Theorem 2 ([7]) *Every K_5-minor-free graph is 5-list colorable.*

In 2008 Hutchinson [3] published results on list colorings of subclasses of planar graphs, where, for every $v \in V(G)$, the function $f(v)$ is the minimum of the vertex degree $d(v)$ and a given integer.

Theorem 3 ([3]) *If G is a 2-connected outerplanar bipartite graph and $f(v) = \min\{d(v), 4\}$ for all $v \in V$, then G is f-list colorable.*

Theorem 4 ([3]) *If G is a 2-connected outerplanar near-triangulation and $f(v) = \min\{d(v), 5\}$ for all $v \in V$, then G is f-list colorable except when the graph is K_3 with identical 2-lists.*

In the same paper Hutchinson mentioned the following problem posed by Bruce Richter.

Problem 1 ([3]) *Let G be a planar, 3-connected graph that is not a complete graph and let $f(v) = \min\{d(v), 6\}$ for all $v \in V$. Is G f-list colorable?*

In this paper we give partial results concerning the above problem. Here we study the class of K_5-minor-free graphs, which contains the class of planar graphs as a subset. We also investigate the analogous question for non 3-connected K_5-minor-free graphs. In that case both the complete graphs and the so-called Gallai trees play a special role. A *Gallai tree* is a graph G such that every block of G is either a complete graph or an odd cycle. Let $\kappa(G)$ denote the *connectivity* of G, that is, the cardinality of a smallest vertex cut set of G.

Problem 2 *Let $r \geq 1$ and $k \geq 5$ be integers. Let G be a K_5-minor-free graph with $\kappa(G) = r$, such that G is not a Gallai tree. Is G f-list colorable when $f(v) = \min\{d(v), k\}$?*

An important tool for our investigations is the following theorem.

Theorem 5 ([1, 2, 5]) *Let G be a connected graph and let L be a list assignment with $|L(v)| \geq d(v)$ for all $v \in V$. If G has no L-coloring, then the following three conditions hold.*

(a) $|L(v)| = d(v)$ for every vertex $v \in V(G)$.

(b) G is a Gallai tree.

(c) Let $\mathcal{B}$ be the set of blocks of G and $\mathcal{B}(v) \subseteq \mathcal{B}$ the set of blocks of G containing a specified vertex v. There exist color sets $L(B)$ for all $B \in \mathcal{B}$ such that $L(B_1) \cap L(B_2) = \emptyset$ whenever B_1 and B_2 have a common vertex and for all $v \in V(G)$ we have $L(v) = \bigcup_{\mathcal{B}(v)} L(B)$.

In this paper we investigate Problem 2, considering subsets of planar graphs that fulfill special requirements. Let

$$S_k := \{v \in V(G) \mid d(v) < k\} \quad \text{and} \quad B_k := \{v \in V(G) \mid d(v) \geq k\}$$

be the sets of vertices with small degree in G and with big degree in G, respectively. The smallest distance between components of $G[S_k]$ in G is denoted by $d(S_k)$, where $G[S_k]$ is the subgraph of G induced by S_k. If $G[S_k]$ has at most one component, then let $d(S_k) = 0$. We may always assume that G is connected since otherwise we can consider seperately each component of G. We will answer the question of Problem 2 for many cases. Our results for K_5-minor-free graphs are summarized in the following tables.

Table 1: $\kappa(G) \in \{1, 2\}$

$k \backslash d(S_k)$	2	3	4	≥ 5
5	$-$	$-$	$-$	?
6	$-$	$-/?$	?	$+$
7	$-$	$-/?$	?	$+$
≥ 8	$-$	$+$	$+$	$+$

Table 2: $\kappa(G) \in \{3, 4\}$

$k \backslash d(S_k)$	2	3	4	≥ 5
5	$-$	$-$	$-$	?
6	$-$	?	?	$+$
7	$-$	$+$	$+$	$+$
≥ 8	$-$	$+$	$+$	$+$

In Section 2 we give some results for $k \geq 6$ that can be obtained by simple observations including the solution for $\kappa(G) \geq 5$. Section 3 contains our main results for connected graphs, whereas Section 4 deals with graphs with $\kappa(G) \in \{3, 4\}$. In Section 5 we consider the case $k = 5$ and in Section 6 we mention open problems.

Note that the original problem asked for planar graphs with $\kappa(G) \in \{3, 4\}$. In that case the answer for $d(S_k) = 2$ and $k \geq 6$ is still unknown. All other entries of the above tables are valid also for planar graphs.

2 Observations

In this section we collect some immediate results. Let $k = 6$, that is, let

$$f(v) = \min\{d(v), 6\} \quad \text{for all} \quad v \in V.$$

Observation 1 *If G is a K_5-minor-free graph and $d(v) \geq 5$ for all $v \in V$, then G is f-list colorable.*

In this case we have $|L(v)| \geq 5$ for all $v \in V$ and we are done since every K_5-minor-free graph is 5-list colorable [7]. If G is 5-connected, then the degree of each vertex is at least 5. So our next observation follows immediately.

Observation 2 *If G is a 5-connected K_5-minor-free graph, then G is f-list colorable.*

The next observation follows from Theorem 5.

Observation 3 *If G is a K_5-minor-free graph that is not a Gallai tree and $d(v) \leq 6$ for all $v \in V$, then G is f-list colorable.*

Observation 4 *Let G be a K_5-minor-free graph. If the vertices of degree at most 5 have pairwise distance at least 3 in G, then G is f-list colorable.*

To prove Observation 4, we color each vertex $v \in S_6$ with an arbitrary color from its list. We delete the color used on each v from the lists of the neighbors of v and remove the colored vertices from G, obtaining G^*. Because of the hypothesis, every remaining vertex has at most one neighbor $v \in S_6$ in G. Thus we have $|L^*(v)| \geq 5$ for the reduced lists of the vertices of G^*, so G^* is L^*-list colorable by Theorem 2.

Proposition 5 *If G is a K_5-minor-free graph with $d(S_6) \geq 5$, then G is f-list colorable.*

Proof. Let G be a K_5-minor-free graph with $d(S_6) \geq 5$ and let L be a list assignment with $|L(v)| = \min\{d(v), 6\}$ for all $v \in V$. Let $G_1, G_2, \ldots$ be the components of $G[S_6]$. For each G_i, choose a vertex $w_i \in V \setminus S_6$ that has at least one neighbor v_i in G_i. Color each w_i by $\varphi_i \in L(w_i) \setminus L(v_i)$. This is always possible since $|L(w_i)| = d(w_i) \geq 6 > d(v_i) = |L(v_i)|$. Delete φ_i from the lists of the neighbors of w_i in G, obtaining a list assignment L'. Form subgraph G' from G, by removing the vertices of S_6 and the vertices w_i.

Since the vertices w_i have distance at least 3 to each other in G, each vertex v of G' still has at least 5 colors available in L'. Thus, we can list color G' from the list assignment L' by Theorem 2. Now consider each G_i. For every vertex $v \in V(G_i)$, delete the colors from its list used on its colored neighbors. We obtain a new list assignment L''. Note that $|L''(v)| \geq d(v)$ for all $v \in V(G_i)$ and $|L''(v_i)| > d(v_i)$. By Theorem 5, each G_i is L''-list colorable. This completes the L-list coloring of G. $\square$

Note that an analogous proof works if $G[S_6]$ has only one component.

Now we consider $f(v) = \min\{d(v), k\}$ for arbitrary $k \geq 3$. If G is at most 2-connected, then there are graphs that are not f-list colorable. The first examples, with $\kappa(G) = 1$, were given in [3]. Here we give an example with $\kappa(G) = 2$ and minimum degree 3.

Proposition 6 *Let $k \geq 3$ be an integer. There are planar 2-connected non-complete graphs G with $d(S_k) = 2$ and $\delta(G) = 3$ and list assignments L with $|L(v)| = \min\{d(v), k\}$ such that G is not L-list colorable.*

Proof. Let L be the following list assignment for the graph G in Figure 1, with $s = \binom{k}{2}$.

- $L(x) = L(y) = \{1, \ldots, k\}$
- $L(u_i) = L(v_i) = L_i$, where $\{L_1, \ldots, L_s\} = \{\{i, j, 0\} \mid 1 \leq i < j \leq k\}$

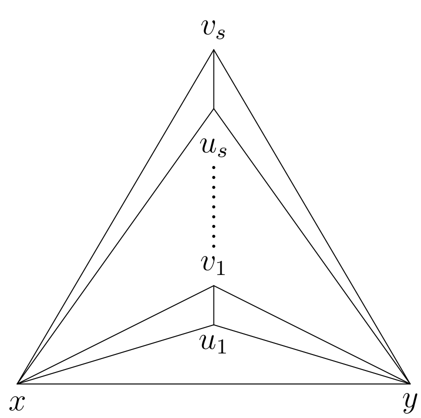

Figure 1: G is planar and 2-connected, with $\delta(G) = 3$.

Since the vertex pair x, y has to be colored by one of the pairs i, j with $1 \leq i < j \leq k$, it follows that one of the vertex pairs u_ℓ, v_ℓ with $1 \leq \ell \leq s$ is not colorable, since the only remaining available color is 0. $\square$

The above proposition shows that in the original Problem 1 we cannot replace the assumption of 3-connectivity by 2-connectivity and/or $\delta(G) = 3$.

Furthermore, Hutchinson [4] pointed out that in Problem 1, even if we keep 3-connectivity, the requirement of planarity cannot be dropped; what is more, it cannot be relaxed to the restriction of being K_5-minor-free.

Proposition 7 *Let $k \geq 3$ be an integer. There are 3-connected, non-complete, K_5-minor-free graphs G with $d(S_k) = 2$ and list assignments L with $|L(v)| = \min\{d(v), k\}$ such that G is not L-list colorable.*

Proof. Let $s = \binom{k}{3}$. Let $G \cong K_{s+3} - E(K_s)$ be the graph with $V(G) = \{v_1, v_2, \ldots, v_s\} \cup \{x, y, z\}$ such that x, y, and z are pairwise adjacent and every v_i is adjacent to each of x, y, and z. Let L be the following list assignment.

- $L(x) = L(y) = L(z) = \{1, \ldots, k\}$
- $L(v_i) = L_i$, where $\{L_1, \ldots, L_s\} = \{\{h, i, j\} \mid 1 \leq h < i < j \leq k\}$

The three vertices x, y, z have to be colored by one of the triples h, i, j with $1 \leq h < i < j \leq k$. It follows that one of the vertices v_ℓ with $1 \leq \ell \leq s$ is not colorable, since the three colors in its list are used on x, y, and z. □

3 Small connectivity

Theorem 6 *If $5 \leq k \leq 7$ and $f(v) = \min\{d(v), k\}$, then there are planar non-complete graphs G with $d(S_k) = 3$ that are not f-list colorable.*

Proof. Let G' be a minimal non-4-list colorable planar graph and L' a 4-assignment such that G' is not L'-list colorable. Thus $\delta(G) \geq 4$. Let $V(G') = \{v_1, \ldots, v_n\}$. Take n copies of K_3 where the vertices of the i-th copy are denoted by x_i, y_i, z_i. Build G by joining each v_i with x_i, y_i and z_i. Thus $d_G(v_i) \geq 7$.

Let $1, 2, 3$ be colors that are not contained in the lists $L'(v_i)$ for $i = 1, \ldots, n$. Let $L(v_i) = L'(v_i) \cup \{1, 2, 3\}$ and $L(x_i) = L(y_i) = L(z_i) = \{1, 2, 3\}$. Thus $|L(v_i)| = 7$ and $|L(x_i)| = |L(y_i)| = |L(z_i)| = d(x_i) = d(y_i) = d(z_i) = 3$.

It is easy to see that G is not L-list colorable. Obviously the same construction with shorter lists works if $k = 5$ or $k = 6$. □

Note that our example is a graph G with $\kappa(G) = 1$. We do not know an analogous example for $\kappa(G) = 2$. However, the example shows that, at least for small k, the additional assumption $d(S_k) \geq 3$ is not sufficient if we do not have an assumption on the degree of connectivity. In contrast, we show next that if $k \geq 8$, then a list coloring is always possible.

Theorem 7 *Let $k \geq 8$ be an integer and let $f(v) = \min\{d(v), k\}$ for all $v \in V$. If G is a K_5-minor-free graph with $d(S_k) \geq 3$ such that no component of G is a Gallai tree, then G is f-list colorable.*

Proof. We prove the theorem for $k = 8$. Assume the theorem is false. Let G be a smallest counterexample and let L be a list assignment such that G is not L-list colorable. Since G is a smallest counterexample it must be connected.

If all vertices of G have degree at most 8, then $d(v) = |L(v)|$ for all $v \in V$. Theorem 5 shows that such a G is L-list colorable if it is not a Gallai tree. This is a hypothesis of the present theorem, so we are done. If all vertices of G have degree at least 5, then we are also done, since every K_5-minor-free graph is 5-list colorable [7]. Thus, we may assume that both S_8 and B_8 are nonempty.

Now we would like to apply the following strategy. Let H be a component of $G[S_8]$. If H is not a Gallai tree, then we remove it from the graph. If H is a Gallai tree, then we color some vertices of H and delete the used colors from the lists of all corresponding neighbors in G (see the case distinction below). If such a neighbor w belongs to B_8, then we shall delete at most three colors from its list. After that, we remove H. If all components of $G[S_8]$ are removed, then the remaining subgraph is K_5-minor-free and all its lists have cardinality at least $8 - 3 = 5$, since $d(S_8) \geq 3$ in G. By Theorem 2 the remaining graph $G[B_8]$ is list colorable from its reduced lists.

Now we re-insert all uncolored vertices and we delete from their lists all colors used for corresponding neighbors. Let H' denote the subgraph of H induced by the set of all uncolored vertices of H. We shall show that each H' is list colorable from the reduced lists L'. For all uncolored vertices, we know that $|L'(v)| \geq d_{H'}(v)$. Of course, it suffices to show that either G is not a smallest counterexample or $V(H') = \emptyset$. By Theorem 5, it also suffices to show in each case that H' is connected and that it satisfies one of the following three conditions:

(i) H' is not a Gallai tree or

(ii) there is a v in $V(H')$ such that $d_{H'}(v) < |L'(v)|$ or

(iii) there are adjacent vertices v and w in H' that are not cut vertices and that satisfy $L'(v) \neq L'(w)$.

If this strategy works, then the graph G is L-list colorable, which contradicts the assumption that it is a counterexample to the theorem.

If H is not a Gallai tree, then we let $H' = H$, and we are done by (i). If H is a Gallai tree, then its blocks are odd cycles and/or complete graphs on at most four vertices, since G is K_5-minor-free. We distinguish the following seven cases.

(1) H is a complete graph K_l, $l \leq 4$.

The absence of a K_5-minor implies that each vertex in $V(G)$ has at most 3 neighbors in H. Since G is connected, at least one vertex v of H satisfies the inequality $|L(v)| > d_H(v)$. Thus we can color H properly from its lists. Note that we use at most 3 colors from the list of each neighbor in G. Now $V(H') = \emptyset$, so we are done.

(2) H is an odd cycle C_{2l+1}, $l \geq 2$.

Denote the vertices of the cycle by $v_1, \ldots, v_{2l+1}$. If there are adjacent vertices v_i, v_j on the cycle such that there exists a color $c \in L(v_i) \setminus L(v_j)$, then we use color c on v_i. Now H' is a path, where v_j is an end vertex, and $|L'(v_j)| > d_{H'}(v_j)$, so we are done by (ii).

Otherwise we have $L(v_1) = \ldots = L(v_{2l+1})$. Since G is connected, $|L(v_i)| \geq 3$ for all i. We color the cycle with 3 colors from the lists. Thus $V(H') = \emptyset$.

(3) H has an end block with a vertex v that is a cut vertex of G.

Let H_1 be the end block. Since v is a cut vertex of G and $|L(u)| = d_H(u)$ for all $u \in (V(H_1) \setminus \{v\})$, we can color all the vertices of $H_1 \setminus \{v\}$ before we color $G[B_8]$. Since G is a smallest counterexample, we can color $G \setminus (H_1 \setminus \{v\})$ from its lists.

(4) H has an end block that is an odd cycle C_{2l+1}, $l \geq 2$.

Denote the vertices of the cycle by $v_1, \ldots, v_{2l+1}$, and let v_1 be the cut vertex. Since we are not in case (3) or (iii), we know that for all $i, j \in \{2, 3, \ldots, 2l+1\}$ we have $|L(v_i)| \geq 3$ and $L(v_i) = L(v_j)$. Thus, we can color v_2 through v_{2l+1} with 3 colors, so that v_2 and v_{2l+1} get the same color. Now $d_{H'}(v_1) < |L'(v_1)|$ and H' is connected, so we are done by (ii).

(5) H has an end block that is a K_2.

Denote the vertices of the end block by v_1 and v_2, where v_1 is the cut vertex. Since we are not in case (3), we have $d_G(v_2) \geq 2$. Choose two colors, say a and b, from $L(v_2)$ and delete them from the lists of all neighbors of v_2 belonging to $G[B_8]$. Remove H from G. After the coloring of $G[B_8]$, add $H' = H$ and assign $L'(v_2) = \{a, b\}$. Since $d_{H'}(v_2) = 1$, we are done by (ii).

(6) H has an end block that is a K_3.

Denote the vertices of the end block by v_1, v_2 and v_3, where v_1 is the cut vertex. Since we are not in case (3), at least one of v_2 and v_3 has a list of cardinality at least 3, say $|L(v_2)| \geq 3$. Choose 3 colors from this list for v_2 and continue as in the previous case.

Note that the same approach would work for endblocks K_4 if we wanted to prove the theorem only for $k \geq 9$, rather than $k \geq 8$. However, to prove the theorem for $k = 8$, we must consider this final case.

(7) All end blocks of H are K_4.

Fix one of the end blocks and denote its vertices by v_1, v_2, v_3 and v_4, where v_1 is the cut vertex. If there is a color $a \in L(v_i) \setminus L(v_j)$ where $\{i, j\} \subseteq \{2, 3, 4\}$ then color v_i by a and continue as prescribed in the strategy. It follows that H' is connected and $d_{H'}(v_j) < |L'(v_j)|$, so we are done by (ii).

Thus $L(v_2) = L(v_3) = L(v_4)$. If $|L(v_i)| = 3$ for $i = 2, 3, 4$ then we are done by case (3). So we know that $|L(v_i)| \geq 4$ for all i, and each of v_2, v_3 and v_4 has at least one neighbor belonging to B_8.

(7a) There are $\{i, j\} \in \{2, 3, 4\}$ and a vertex $w \in B_8$ such that w is a neighbor of v_i, but not a neighbor of v_j.

Remove H from G and join w with all neighbors of v_j belonging to B_8. Note that the new graph is still K_5-minor-free. After removing all components of $G[S_8]$, color the remaining graph from the corresponding lists. This is possible since the graph is K_5-minor-free and all lists have cardinality at least 5. Remove the additional edges, add H and delete the colors of the neighbors of the vertices of H from their lists. Let a be the color used on w. If $a \notin L(v_i)$, then we have $d_{H'}(v_i) < |L'(v_i)|$ and we are done by (ii). Otherwise $a \in L(v_i) = L(v_j)$. But no neighbor of v_j is colored by a since all of them were joined to w by the additional edges. Since $a \notin L'(v_i)$ and $a \in L'(v_j)$, we have $L'(v_i) \neq L'(v_j)$, so we are done by (iii).

(7b) Each vertex $z \in B_8$ that is adjacent to at least one of v_2, v_3, v_4 is adjacent to all of them.

Note that v_2, v_3, v_4 must have some neighbors in B_8 since we are not in case (3). Let $z \in B_8$ be such a neighbor. Then there is no path from v_1 to z that does not use v_2, v_3 or v_4, since otherwise we have a subdivision of K_5. This statement holds also for all further neighbors of v_2, v_3, v_4 in B_8. So by removing the edges v_1v_2, v_1v_3 and v_1v_4, we get two components G_1 containing v_1 and G_2 containing v_2, v_3, v_4 and z.

If G_1 is a Gallai tree, then we can color it since $|L(v_1)| > d_{G_1}(v_1)$. Otherwise we can color it, since it satisfies the hypothesis of the theorem and G is a smallest counterexample to it. Delete the color of v_1 from the lists of v_2, v_3 and v_4 and consider G_2 with the reduced list assignment L'. We need to prove the following claim to argue that G_2 satisfies the hypothesis of the theorem.

Claim: *G_2 is not a Gallai tree.*

Assume to the contrary that G_2 is a Gallai tree. Note that G_2 has at least one vertex of degree at least 8, namely z. Let u be a vertex of degree at least 8 that has the largest distance from v_2 in G_2. This u belongs to at least 3 blocks of G_2, since it has at most 3 neighbors in each block (as G is K_5-minor-free). Moreover, u can have neighbors of degree less than 8 in at most one of these blocks, since $d(S_8) \geq 3$. Thus there are vertices of degree at least 8 that have a greater distance from v_2 than u, contradicting the assumption for u. Hence the claim is proved.

Thus G_2 satisfies the hypothesis of the present theorem and we can color it from the reduced lists, since G is a smallest counterexample. By combining the colorings of G_1 and G_2, we obtain an L-list coloring of G, which is a contradiction.

This completes the proof of the theorem. □

4 G is 3-connected

Theorem 8 *Let $k \geq 7$ be an integer and let G be a K_5-minor-free, non-complete, 3-connected graph. If $d(S_k) \geq 3$, then G is f-list colorable when $f(v) = \min\{d(v), k\}$.*

Proof. We use a strategy similar to the proof of Theorem 7. Let G be a smallest counterexample to the theorem. Since G is 3-connected and non-complete, G is not a Gallai tree. So, if all vertices of G have degree at most 7, then we are done by Theorem 7. Similarly, if all vertices of G have degree at least 5 then we are done by Theorem 2.

For each component H of $G[S_7]$, we would like to color one or more vertices of H such that for each adjacent vertex in B_7 we have to delete at most two colors from its list. If we do this for all components of $G[S_7]$, then we can color $G[B_7]$ from the reduced lists. This is possible because $d(S_7) \geq 3$, which ensures that all reduced lists have cardinality at least 5. Note that we do not need a connectivity assumption for this argument. For a component H of $G[S_7]$, let H' be the subgraph induced by the uncolored vertices of H. For every vertex $v \in V(H')$, we obtain a reduced list $L'(v)$ by deleting all colors from $L(v)$ that were used for the already colored neighbors of v. Finally, we try to color every H' from the list assignment L'. It will suffice to show, for each H', that H' is connected and satisfies one of conditions (i), (ii) and (iii), from Theorem 7.

If H is not a Gallai tree, then let $H' = H$, and we are done by (i). Thus we can assume that H is a Gallai tree.

(1) H is a single vertex v.

Color v. Since $V(H') = \emptyset$, we are done.

(2) H is K_2 or H has an end block K_3.

If H is K_2, then denote the vertices of H by v_1 and v_2. If H has an end block K_3, then denote the vertices of the end block by v_1, v_2 and v_3, where v_3 is the cut vertex. If there is an $a \in L(v_i) \setminus L(v_j)$ ($\{i, j\} = \{1, 2\}$), then color v_i by a. In this case, $|L'(v_j)| > d_{H'}(v_j)$, so we are done by (ii). So instead, we may assume $L(v_1) = L(v_2)$.

(2a) There is a vertex $z \in B_7$ that is adjacent to v_i and not adjacent to v_j ($\{i, j\} = \{1, 2\}$).

Remove H and join z with all neighbors of v_j. The new graph is still K_5-minor-free. Let $H' = H$, with the lists reduced by the coloring of $G[B_7]$ with the additional edges. Let a be the color of z in this coloring. If $a \notin L(v_i)$, then we have $|L'(v_i)| > d_H(v_i)$, so we are done by (ii). Otherwise we have $a \notin L'(v_i)$ and $a \in L'(v_j)$, so we are done by (iii).

(2b) Every $z \in B_7$ that is a neighbor of v_1 or v_2 is a neighbor of both.

Since G is 3-connected, there have to be at least 3 such neighbors (including v_3 if H is an end block K_3). Let $Z = \{z_1, z_2, z_3, \ldots\}$ be the set of all these neighbors.

Consider $G' = G[V \setminus \{v_1, v_2\}]$. This subgraph is connected since G is 3-connected. Thus any vertex pair z_i, z_j is joined by a path in G'. Consider an auxiliary graph T with $V(T) = Z$, where two vertices z_i and z_j are joined by an edge if and only if there is path from z_i to z_j in G' not containing another vertex $z_\ell \in Z$. Note that T has to be a tree since otherwise there is a subdivision of K_5 in G.

Let z_1 be a leaf of T and z_2 its neighbor in T. Since $z_1 \in B_7$ or z_1 is a cut vertex of H, it must have a neighbor $w \notin Z \cup \{v_1, v_2\}$.

Now consider a path P from w to z_3 in G. Since z_1 is a leaf in T, every path from w to z_3 must contain a vertex $u \in Z \cup \{v_1, v_2\}$ as an internal vertex. If P contains $x \in \{v_1, v_2\}$, then, in addition to z_3, P must contain another neighbor of x that is in Z; call it z_i. Thus every path from w to z_3 contains some vertex $z_i \neq z_3$ as an internal vertex. Since there have to be at least three internally disjoint paths from w to z_3, we have a path from z_1 to some $z_j \neq z_2$ that has no internal vertices in $Z \cup \{v_1, v_2\}$. However, now T is not a tree. This again gives us a subdivision of K_5, which contradicts the K_5-minor-freeness of G.

(3) H is an odd cycle C_{2l+1}, $l \geq 2$, or H has an end block that is an odd cycle C_{2l+1}, $l \geq 2$.

Denote the vertices of C_{2l+1} by $v_1, v_2, \ldots, v_{2l+1}$, where the cut vertex (if it exists) is v_{2l+1}. If there is an $a \in L(v_i) \setminus L(v_j)$ ($\{i, j\} \subset \{1, 2, \ldots, 2l\}$), then color v_i by a. In this case, $|L'(v_j)| > d_{H'}(v_j)$, and we are done by (ii). So instead, we have $L(v_i) = L(v_j)$ for all $\{i, j\} \subset \{1, 2, \ldots, 2l\}$.

If there is a vertex $w \in B_7$ that is adjacent to v_i and not adjacent to v_j ($\{i, j\} \subset \{1, 2, \ldots 2l\}$), then there exist $w' \in B_7$ and $v_{i'}$ and $v_{i'+1}$, which are adjacent on the odd cycle, such that w' is adjacent to $v_{i'}$ and not adjacent to $v_{i'+1}$. Remove H and join w' with all neighbors of $v_{i'+1}$. The new graph is still K_5-minor free. Let $H' = H$, with the lists reduced by the coloring of $G[B_7]$ with the additional edges. Let a be the color of w' in this coloring. If $a \notin L(v_{i'})$, then $|L'(v_{i'})| > d_H(v_{i'})$, so we are done by (ii). Otherwise $a \notin L'(v_{i'})$ and $a \in L'(v_{i'+1})$, so we are done by (iii).

Thus we may assume that every vertex $w \in B_7$ that is adjacent to at least one v_i, $i \in \{1, \ldots, 2l\}$, is adjacent to all of them. Since G is 3-connected, there must be at least two such neighbors that lie in the same component of $G[B_7]$. To build a K_5-minor, we take these two neighbors together with v_1, v_2 and v_3.

(4) H is K_3 or K_4 or has an end block that is K_4.

If H is K_3, then denote the vertices of H by v_1, v_2 and v_3. Otherwise, denote the vertices by v_1, v_2, v_3 and v_4, where the cut vertex (if it exists) is v_4. If there is an $a \in L(v_i) \setminus L(v_j)$ ($\{i,j\} \subset \{1,2,3\}$), then color v_i by a. In this case, $|L'(v_j)| > d_{H'}(v_j)$, and we are done by (ii). So instead, we have $L(v_1) = L(v_2) = L(v_3)$.

If there is a vertex $w \in B_7$ that is adjacent to v_i and not adjacent to v_j ($\{i,j\} = \{1,2,3\}$), then remove H and join w with all neighbors of v_j. The new graph is still K_5-minor free. Let $H' = H$, with the lists reduced by the coloring of $G[B_7]$ with the additional edges. Let a be the color of w in this coloring. If $a \notin L(v_i)$, then $|L'(v_i)| > d_H(v_i)$, so we are done by (ii). Otherwise $a \notin L'(v_i)$ and $a \in L'(v_j)$, so we are done by (iii).

Thus we may assume that every vertex $w \in B_7$ that is adjacent to v_1, v_2 or v_3 is adjacent to all of them. Since G is 3-connected, there must be at least 3 such neighbors that all lie in the same component of $G \setminus \{v_1, v_2, v_3\}$. To build a K_5-minor, we take two of these neighbors and v_1, v_2 and v_3.

(5) All end blocks of H are K_2.

Denote the vertices of an end block by v_1 and v_2, where v_1 is the cut vertex. Since G is 3-connected, $|L(v_2)| \geq 3$. Choose two colors a and b from $L(v_2)$. Delete a and b from the lists of all neighbors of v_2 in B_7 and let $L'(v_2) = \{a, b\}$. After the coloring of $G[B_7]$ we have $2 = |L'(v_2)| > d_H(v_2) = 1$, so we are done by (ii).

This completes the proof of the theorem. □

Remark: The above proof also works for $k = 6$, except in the final case.

5 The case $k = 5$

Theorem 9 *Let $k = 5$. There is a planar, 3-connected, non-complete graph G with $d(S_5) = 4$ and a list assignment L with $|L(v)| = \min\{d(v), 5\}$ for all $v \in V$ such that G is not L-list colorable.*

Proof. Consider the subgraph H in Figure 2a with list assignment $L'(x) = \{a\}$, $L'(y) = \{b\}$, and $L'(u_i) = \{a, b, 1, 2, 3\}$, for $i = 1, \ldots, 4$. We claim that three colors can be assigned to each of the triangles $D_1, \ldots, D_6$ such that in every coloring of H at least one of the triangles is colored by its assigned colors; note that the order of the colors in each assignment matters.

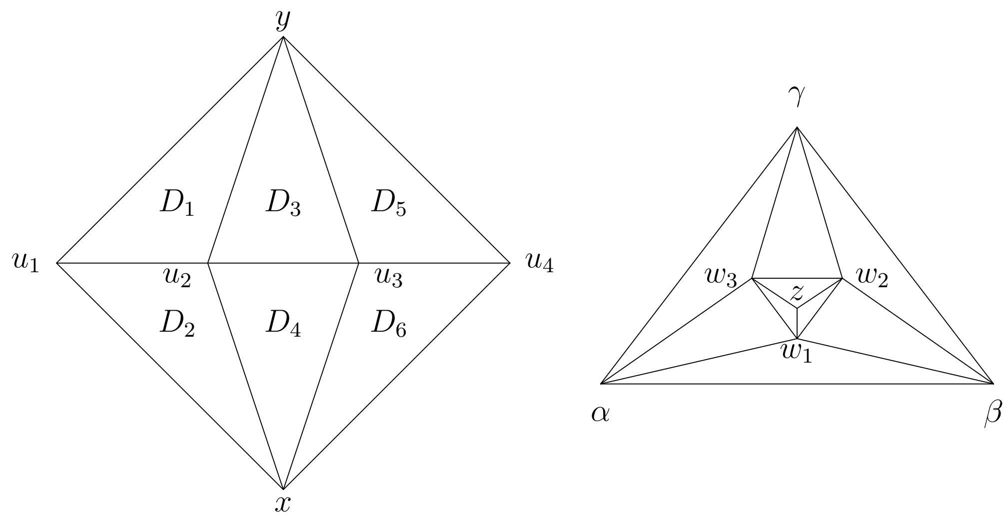

Figure 2: (a) Subgraph H. (b) The structure inside each D_i.

The assignments are

$$D_1 : 2, 1, a; \quad D_2 : 3, 1, b; \quad D_3 : 2, 3, a; \quad D_4 : 3, 2, b; \quad D_5 : 1, 2, a; \quad D_6 : 1, 3, b.$$

Note that the prescribed coloring of D_1 or D_2 occurs if u_2 is colored by 1. Analogously, the prescribed coloring of D_5 or D_6 occurs if u_3 is colored by 1. Thus we may assume that u_2 and u_3 are not colored by 1. But now the prescribed coloring of D_3 or D_4 occurs.

Now assume that inside of each of $D_1, \ldots, D_6$ we have the structure in Figure 2b. For each triangle D_i, let α, β, γ be the colors assigned to D_i. Let $L'(w_1) = \{\alpha, \beta, 4, 5, 6\}, L'(w_2) = \{\beta, \gamma, 4, 5, 6\}$ and $L'(w_3) = \{\alpha, \gamma, 4, 5, 6\}$ and $L'(z) = \{4, 5, 6\}$. Thus H is not L'-list colorable for the given list assignment.

Now build a graph G by taking 25 copies of H and identifying all 25 x-vertices to a vertex x^* and all 25 y-vertices to vertex y^*. Join vertex u_4 in the i-th copy with vertex u_1 in the $(i + 1)$-th copy, for $i = 1, \ldots, 24$. Define a list assignment L, as follows. Let $L(x^*) = \{7, 8, 9, 10, 11\}$ and $L(y^*) = \{12, 13, 14, 15, 16\}$. For all other vertices let $L(v) = L'(v)$, where each pair of $L(x^*) \times L(y^*)$ corresponds with the color pair (a, b) of one copy of H.

Finally, observe that G is a planar, 3-connected non-complete graph with a list assignment L such that $|L(v)| = \min\{d(v), 5\}$ for all $v \in V$, but G is not L-list colorable. $\square$

Note that $d(S_5) = 4$ for the example in the proof above. In fact, it is almost the same example as in [10].

6 Open problems

Despite our progress in this paper, two essential problems remain open.

Problem 3 *Let $k \geq 6$ be an integer and let G be a non-complete planar graph with $\kappa(G) \in \{3, 4\}$ and $d(S_k) = 2$. Is G f-list colorable when $f(v) = \min\{d(v), k\}$ for all $v \in V$?*

Problem 4 *Let $k = 5$ be an integer and let G be a connected planar graph that is not a Gallai tree and that has $d(S_k) \geq 5$. Is G f-list colorable when $f(v) = \min\{d(v), k\}$ for all $v \in V$?*

One can raise the analogous questions for K_5-minor-free graphs, too. We know from Proposition 7 that the answer to the first one with $\kappa = 3$ is negative.

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