Injective colorings of sparse graphs

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Abstract

Let $\text{Mad}(G)$ denote the maximum average degree (over all subgraphs) of G and let $\chi_i(G)$ denote the injective chromatic number of G. We prove that if $\text{Mad}(G) \leq \frac{5}{2}$, then $\chi_i(G) \leq \Delta + 1$; similarly, if $\text{Mad}(G) < \frac{42}{19}$, then $\chi_i(G) \leq \Delta$. Suppose that G is a planar graph with girth $g(G)$ and $\Delta \geq 4$. We prove that if $g(G) \geq 9$, then $\chi_i(G) \leq \Delta + 1$; similarly, if $g(G) \geq 13$, then $\chi_i(G) = \Delta$.

Keywords: injective coloring, maximum average degree, planar graph, discharging

MSC: 05C15

1 Introduction

An *injective coloring* of a graph G is an assignment of colors to the vertices of G so that any two vertices with a common neighbor receive distinct colors. The *injective chromatic number*, $\chi_i(G)$, is the minimum number of colors needed for an injective coloring. Injective colorings have their origin in complexity theory [6], and can be used in coding theory. Note that $\chi_i(G) = \chi(G^{(2)})$, where the *neighboring graph* $G^{(2)}$ is defined by $V(G^{(2)}) = V(G)$ and $E(G^{(2)}) = \{uv : u \text{ and } v \text{ have a common neighbor in } G\}$.

Injective colorings were introduced by Hahn et al. [6], who showed that $\Delta \leq \chi_i(G) \leq \Delta^2 - \Delta + 1$, where Δ is the maximum degree of graph G. They also determined when the upper bound is attained and when the lower bound is attained by regular graphs. Injective colorings were further examined in [4, 7, 8]. Luzar et al. studied injective colorings of planar graphs with high girth.

Theorem 1 (Lužar, Škrekovski, and Tancer [8]). *Let G be a planar graph with girth $g(G)$ and maximum degree Δ.*

(a) If $g(G) \geq 5$, then $\chi_i(G) \leq \Delta + 4$.

(b) If $g(G) \geq 10$, then $\chi_i(G) \leq \Delta + 1$.

(c) If $g(G) \geq 19$, then $\chi_i(G) = \Delta$.

Doyon, Hahn, and Raspaud studied injective colorings of graphs G with bounded *maximum average degree*, $\text{Mad}(G)$, where the average is taken over all subgraphs of G.

Theorem 2 (Doyon, Hahn, and Raspaud [4]). *Let G be a graph with maximum degree Δ.*

(a) If $\text{Mad}(G) < \frac{10}{3}$, then $\chi_i(G) \leq \Delta + 8$.

(b) If $\text{Mad}(G) < 3$, then $\chi_i(G) \leq \Delta + 4$.

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(c) *If* $Mad(G) < \frac{14}{5}$*, then* $\chi_i(G) \leq \Delta + 3$.

Note that every planar graph G with girth at least g satisfies $\text{Mad}(G) < \frac{2g}{g-2}$.

In the current paper, we extend the work of Doyon, Hahn, and Raspaud by considering bounds on $\text{Mad}(G)$ that imply $\chi_i(G) \leq \Delta + 1$ and $\chi_i(G) \leq \Delta$. We also generalize Theorem 1(b), by proving a result in terms of $\text{Mad}(G)$ that implies Theorem 1(b). Finally, for the case $\Delta \geq 4$, we improve Theorems 1(a) and 1(b). In an upcoming paper [3], we improve Theorem 2(c) for $\Delta \geq 4$ and show that the result is best possible for $\Delta = 3$.

Theorem 3. *Let* G *be a graph with maximum degree* Δ.

(a) *If* $Mad(G) \leq \frac{5}{2}$*, then* $\chi_i(G) \leq \Delta + 1$.

(b) *If* $Mad(G) < \frac{42}{19}$*, then* $\chi_i(G) \leq \Delta$.

Theorem 4. *Let* G *be a planar graph with girth* $g(G)$ *and maximum degree* $\Delta \geq 4$.

(a) *If* $g(G) \geq 9$*, then* $\chi_i(G) \leq \Delta + 1$.

(b) *If* $g(G) \geq 13$*, then* $\chi_i(G) = \Delta$.

In Section 2, we introduce definitions, as well as a lemma of Vizing and a theorem of Erdos, Rubin, and Taylor; we use this lemma and theorem in Sections 3 and 4. In Section 3, we prove Theorems 3(a) and 4(a). In Section 4, we prove Theorem 3(b). Finally, in Section 5, we prove Theorem 4(b).

2 Preliminaries

We write $\Delta(G)$ to denote the maximum degree of G; when the context is clear, we simply write Δ. A k-vertex is a vertex of degree k; a k^+- and a k^--vertex have degree at least and at most k, respectively. A *thread* is a path with 2-vertices in its interior and 3^+-vertices as its endpoints. A k-thread has k interior 2-vertices. If u and v are the endpoints of a thread, then we say that u and v are *pseudo-adjacent*. If a 3^+-vertex u is the endpoint of a thread containing a 2-vertex v, then we say that v is a *nearby vertex* of u and vice versa. We will need one or both of the following two results in many of our proofs.

Lemma A (Vizing)**.** *For a connected graph* G*, let* L *be a list assignment such that* $|L(v)| \geq d(v)$ *for all* v.

a) *If* $|L(y)| > d(y)$ *for some vertex* y*, then* G *is* L*-colorable.*

b) *If* G *is 2-connected and the lists are not all identical, then* G *is* L*-colorable.*

We say that a graph is *degree-choosable* if it can be colored from lists when each vertex is given a list of size equal to its degree.

Theorem B (Erdős-Rubin-Taylor)**.** *A graph* G *fails to be degree-choosable if and only if every block is a complete graph or an odd cycle.*

All of our proofs rely on counting arguments, to show that a minimal counterexample must have higher maximum average degree than is allowed by the hypothesis. Often, these proofs use the techniques of reducibility and discharging. In the *reducibility phase*, we assume that G is a minimal counterexample to the theorem we are proving, and we show that G cannot contain a certain subgraph, with specified degrees in G; we call this subgraph a *reducible configuration*. To prove that configuration H is reducible, we assume that $G - H$ has an injective coloring (with the desired number of colors), and we show how to extend the coloring to G. Most commonly, when we color the vertices of H, we do so in order of increasing number of remaining colors available; we call this *coloring greedily*. When we color H greedily, we do not specify the order in which we color the vertices; otherwise, we specify the order. In the *discharging phase*, we use a counting argument to show that every supposed minimal counterexample must contain a reducible configuration; this yields a contradiction.

The standard method for proving upper bounds on the injective chromatic number is to use reducible configurations and discharging, as we do in this paper. Our main new idea is to allow reducible configurations of arbitrary size, similar to the 2-alternating cycles introduced by Borodin [1] and generalized by Borodin, Kostochka, and Woodall [2]. As far as we know, this approach has not previously been applied to injective chromatic number.

3 Conditions that imply $\chi_i(G) \leq \Delta + 1$

We split the proof of Theorem 3(a) into Lemmas 5 and 6.

Lemma 5. *Let G be a graph with $\Delta = 3$. If Mad(G) $\leq \frac{5}{2}$, then $\chi_i(G) \leq 4$.*

Proof. We prove the more general statement: If $\Delta = 3$ and $\text{Mad}(G) \leq \frac{5}{2}$, then G can be injectively colored from *lists* of size 4. Let G be a minimal counterexample. It is easy to see that G has no 1-vertex and G has no 2-thread; in each case, we could delete the subgraph H, injectively color $G - H$, then greedily color H. First, we consider the case $\text{Mad}(G) < \frac{5}{2}$. Let G_{23} denote the subgraph induced by edges with one endpoint of degree 2 and the other of degree 3. If each component of G_{23} contains at most one cycle, then each component of G_{23} contains at least as many 3-vertices as 2-vertices, so $\text{Mad}(G) \geq \frac{5}{2}$; this contradicts our assumption. Hence, some component H of G contains a cycle C with a vertex u on C such that $d_H(u) = 3$. Let $J = V(C) \cup N(u)$. Because G is a minimal counterexample, we can injectively 4-color $G \setminus J$. We will now use Lemma A to extend the coloring of $G \setminus J$ to J.

Since C is an even cycle, $G^{(2)}[J]$ consists of two components; one component contains u and the other does not. We will use part (a) of Lemma A to color the component that contains u and we will use part (b) to color the other component; note that this second component is 2-connected. Let $L(v)$ be the list of colors available for each vertex v before we color $G \setminus J$ and let $L'(v)$ be the list of colors available after we color $G \setminus J$. To apply Lemma A as desired, we need to prove three facts: first, $|L'(v)| \geq d_{J^{(2)}}(v)$ for each $v \in J$; second, $|L'(u)| > d_{J^{(2)}}(u)$; and third, the lists L' for the second component of $G^{(2)}[J]$ are not all identical.

First, observe that for each vertex $v \in J$ we have the inequality $|L(v)| = 4 \geq d_{G^{(2)}}(v)$; because each colored neighbor in $G^{(2)}$ of v forbids only one color from use on v, this inequality implies that $|L'(v)| \geq d_{J^{(2)}}(v)$ for each $v \in J$. Second, note that $|L(u)| = 4 > 3 = d_{G^{(2)}}(u)$; this inequality implies that $|L'(u)| = 3 > 2 = d_{J^{(2)}}(u)$. Third, let x and y be the two vertice on C that are adjacent to u in G; note that $|L'(x)| = 3$ and $|L'(y)| = 3$, while $|L'(v)| = 2$ for every other vertex v in the second component of $J^{(2)}$. Since vertices x and y have lists of size 3, while the other vertices have lists of size 2, it is clear that not all lists are identical. Hence, we can color the first component of $J^{(2)}$ by Lemma A part (a), and we can color the second component of $J^{(2)}$ by Lemma A part (b).

Now consider the case $\text{Mad}(G) = \frac{5}{2}$. If G_{23} does not contain any cycle C with a vertex u such that $d_{G_{23}}(u) = 3$, then each component of G_{23} is an even cycle. This means that $d_{G^{(2)}}(v) = 4$ for every vertex v. By applying Theorem B to $G^{(2)}$, we see that $\chi_i(G) \leq 4$; it is straightforward to verify that in each component of $G^{(2)}$, some block is neither an odd cycle nor a clique. □

Lemma 6. *Let G be a graph with $\Delta \geq 4$. If Mad(G) $\leq \frac{5}{2}$, then $\chi_i(G) \leq \Delta + 1$.*

Proof. Let G be a minimal counterexample. It is easy to see that G has no 1-vertex and G has no 2-thread. Observe that G also must not contain the following subgraph: a 3-vertex v adjacent to three 2-vertices such that one of these 2-vertices u is adjacent to a second 3-vertex. If G contains such a subgraph, let H be the set of v and its three neighbors. By minimality, we can injectively color $G - H$ with $\Delta + 1$ colors; now we greedily color H, making sure to color u last.

We now use a discharging argument, with initial charge $\mu(v) = d(v)$. We have two discharging rules:

R1) Each 3-vertex divides a charge of $\frac{1}{2}$ equally among its adjacent 2-vertices.

R2) Each 4^+-vertex sends a charge of $\frac{1}{3}$ to each adjacent 2-vertex.

Now we show that $\mu^*(v) \geq \frac{5}{2}$ for each vertex v.

$d(v) = 3$: $\mu^*(v) = 3 - \frac{1}{2} = \frac{5}{2}$

$d(v) \geq 4$: $\mu^*(v) \geq d(v) - \frac{d(v)}{3} = \frac{2d(v)}{3} \geq \frac{8}{3} > \frac{5}{2}$

$d(v) = 2$: If v is adjacent to a 4^+-vertex, then $\mu^*(v) \geq 2 + \frac{1}{3} + \frac{1}{6} = \frac{5}{2}$. If v has two neighboring 3-vertices and each neighbor gives v a charge of at least $\frac{1}{4}$, then $\mu^*(v) \geq 2+2(\frac{1}{4}) = \frac{5}{2}$. However, if v has two neighboring 3-vertices and at least one of them gives v a charge of only $\frac{1}{6}$, then v is in a copy of the forbidden subgraph described above; so $\mu^*(v) \geq \frac{5}{2}$.

Hence, each vertex v has charge $\mu^*(v) \geq \frac{5}{2}$. Furthermore, each 4^+-vertex has charge at least $\frac{8}{3}$, so $\text{Mad}(G) > \frac{5}{2}$. □

This completes the proof of Theorem 3(a). Next, we prove Theorem 4(a), which improves Theorem 1(b) when $\Delta \geq 4$. For convenience, we restate the theorem.

Theorem 4(a). *If G is planar, $\Delta(G) \geq 4$, and girth$(G) \geq 9$, then $\chi_i(G) \leq \Delta + 1$.*

Proof. It is easy to see that G has no 1-vertex and G has no 2-thread. We also need one more reducible configuration. Let u_1, u_2, u_3, u_4, u_5 be five consecutive vertices along a face f. Suppose that $d(u_1) = d(u_3) = d(u_5) = 2$ and $d(u_2) = d(u_4) = 3$ and the neighbor of u_2 not on f has degree at most 3. We call this subgraph H and we show that H is a reducible configuration, as follows. By assumption, $G - u_3$ has an injective coloring with $\Delta + 1$ colors; we now modify this coloring to get an injective coloring of G. First uncolor vertices u_1, u_2, u_4, and u_5. Now color the uncolored vertices in the order: u_2, u_4, u_1, u_5, u_3.

We use a discharging argument with intial charge $\mu(v) = d(v) - 4$ and $\mu(f) = d(f) - 4$. We use the following discharging rules.

R1) Each face gives charge 1 to each 2-vertex and gives charge 1/3 to each 3-vertex.

R2) If face f contains the degree sequence $(4^+, 3, 2, 3, 4^+)$, then f gives charge 1/3 to the face adjacent across the 2-vertex.

Now we show that $\mu^*(x) \geq 0$ for each $x \in V(G) \cup F(G)$.

$d(v) = 2$: $\mu^*(v) = -2 + 2(1) = 0$.

$d(v) = 3$: $\mu^*(v) = -1 + 3(1/3) = 0$.

$d(v) \geq 4$: $\mu^*(v) = \mu(v) \geq 0$.

To argue intuitively without handling R2 separately, observe that wherever $(4, 3, 2, 3, 4)$ appears we could replace it with $(4, 2, 4, 2, 4)$ without creating a 2-thread; after this replacement, face f gives away 1/3 more charge (to account for R2), so we only consider R1).

For each face f, let t_2, t_3, t_4 denote the number of 2-vertices, 3-vertices, and 4^+-vertices on f, respectively. The charge of a face f is $\mu^*(f) = d(f) - 4 - t_2 - 1/3t_3 = t_4 + 2/3t_3 - 4$. If a face has negative charge, then $t_4 + 2/3t_3 < 4$. This implies $3/2t_4 + t_3 < 6$, and hence $t_4 + t_3 \leq 5$. Since G contains no 2-threads, $t_2 \leq t_3 + t_4$. So, if a face has negative charge, $d(f) = t_2 + t_3 + t_4 \leq 2(t_3 + t_4) \leq 10$. Hence, we only need to verify that $\mu^*(f) \geq 0$ for faces of length at most 10; since *girth*$(G) \geq 9$, we have only two cases: $d(f) = 10$ and $d(f) = 9$.

Case 1: face f of length 10: If $\mu^*(f) < 0$, then $t_2 = 5$, $t_3 \geq 4$, and $t_4 \leq 1$. So our degree sequence around f must look like either (a) $(2, 4^+, 2, 3, 2, 3, 2, 3, 2, 3)$ or (b) $(2, 3, 2, 3, 2, 3, 2, 3, 2, 3)$.

Case 1a) $(2, 4^+, 2, 3, 2, 3, 2, 3, 2, 3)$: Let u_1 and u_2 be the vertices not on f adjacent to the second and third vertices on f of degree 3. If G does not contain reducible configuration H, then $d(u_1) \geq 4$ and $d(u_2) \geq 4$; but then f receives charge 1/3 by R2, so $\mu^*(f) \geq 0$.

Case 1b) $(2, 3, 2, 3, 2, 3, 2, 3, 2, 3)$: If any neighbor not on f of a 3-vertex on f has degree at most 3, then G contains reducible configuration H. If all such neighbors have degree at least 4, then f receives charge 1/3 from each adjacent face, so $\mu^*(f) = t_4 + \frac{2}{3}t_3 - 4 + (\frac{1}{3})5 = 0 + (\frac{2}{3})5 - 4 + (\frac{1}{3})5 = 1 > 0$.

Case 2: face f of length 9: If $\mu^*(f) < 0$, then $t_2 = 4$, $t_3 \geq 4$, and $t_4 \leq 1$.

Our degree sequence around f, beginning and ending with vertices of degree at least 3, must look like one of the four following: (a) $(3,2,3,2,3,2,3,2,3)$, (b) $(4^+,2,3,2,3,2,3,2,3)$, (c) $(3,2,4^+,2,3,2,3,2,3)$, or (d) $(3,2,3,2,4^+,2,3,2,3)$.

Case 2a) $(3,2,3,2,3,2,3,2,3)$: Because f contains the degree sequence $(2,3,2,3,2,3,2)$, either G contains the reducible configuration H or f receives a charge of $\frac{1}{3}$ from at least two faces. Hence $\mu^*(f) \geq \frac{2}{3}(5) - 4 + \frac{1}{3}(2) = 0$.

Cases 2b) $(4^+,2,3,2,3,2,3,2,3)$ and **2c)** $(3,2,4^+,2,3,2,3,2,3)$: Again f contains the degree sequence $(2,3,2,3,2)$. So in each case, if f doesn't contain the reducible configuration H, then f receives a charge of $\frac{1}{3}$ from some adjacent face; hence $\mu^*(f) \geq 1 + \frac{2}{3}(4) - 4 + \frac{1}{3}(1) = 0$.

Case 2d) $(3,2,3,2,4^+,2,3,2,3)$: Let $v_1, w_1, v_2, w_2, v_3, w_3, v_4, w_4, v_5$ denote the vertices on f, in order around the face, beginning and ending with 3-vertices. If both v_1 and v_2 are adjacent to vertices of degree at least 4, then f receives a charge of $\frac{1}{3}$ from an adjacent face, by R2). In this case $\mu^*(f) \geq 1+\frac{2}{3}(4)-4+\frac{1}{3}(1) = 0$. Conversely, we will show that if either v_1 or v_2 is not adjacent to any vertex of degree at least 4, then G contains a reducible configuration.

Let u_1 and u_2 denote the neighbors of v_1 and v_2 not on f. By minimality, we have an injective coloring of $G - \{w_1, v_2, w_2\}$ with $\Delta + 1$ colors. If $d(u_1) < 4$, then we finish as follows: uncolor v_1 and w_4, now color w_1, w_2, v_2, v_1, w_4. If instead $d(u_2) < 4$, then we finish by coloring w_1, w_2, v_2. □

4 $\mathrm{Mad}(G) < \frac{42}{19}$ implies $\chi_i(G) = \Delta$

We split the proof of Theorem 3(b) into Lemmas 7 and 8. In Lemma 7, we prove a stronger result than we need for Theorem 3(b), since our hypothesis here is $\mathrm{Mad}(G) \leq \frac{9}{4}$, rather than $\mathrm{Mad}(G) \leq \frac{42}{19}$.

Lemma 7. *Let G be a graph with $\Delta \geq 4$. If $Mad(G) \leq \frac{9}{4}$, then $\chi_i(G) = \Delta$.*

Proof. We always have $\chi_i(G) \geq \Delta$, so we only need to prove $\chi_i(G) \leq \Delta$. Let G be a minimal counterexample. Clearly, G has no 1-vertex and G has no 4-thread. Note that G also has no 3-thread with a 3-vertex at one of its ends. We use a discharging argument, with initial charge: $\mu(v) = d(v)$. We have one discharging rule:

R1) Each 3^+-vertex gives a charge of $\frac{1}{8}$ to each nearby 2-vertex.

Now we show that $\mu^*(v) \geq \frac{9}{4}$ for each vertex v.

2-vertex: $\mu^*(v) \geq 2 + 2(\frac{1}{8}) = \frac{9}{4}$.

3-vertex: $\mu^*(v) \geq 3 - 6(\frac{1}{8}) = \frac{9}{4}$.

4^+-vertex: $\mu^*(v) \geq d(v) - 3d(v)(\frac{1}{8}) = \frac{5}{8}d(v) \geq \frac{5}{2} > \frac{9}{4}$.

Hence, each vertex v has charge $\mu^*(v) \geq \frac{9}{4}$. Since each 4^+-vertex v has charge $\mu^*(v) \geq \frac{5}{2}$, we have $\mathrm{Mad}(G) > \frac{9}{4}$. □

Lemma 8. *Let G be a graph with $\Delta = 3$. If $Mad(G) < \frac{42}{19}$, then $\chi_i(G) = 3$.*

Proof. Let G be a minimal counterexample. As above, G has no 1-vertex and G has no 4-thread. We form an auxiliary graph H as follows. Let $V(H)$ be the 3-vertices of G. If u and v are ends of a 3-thread in G, then add the edge uv to H. Suppose instead that u and v are ends of a 2-thread in G. If one of the other threads incident to u is a 3-thread and the third thread incident to u is either a 2-thread or 3-thread, then add edge uv to H. Our plan is similar to the proof of Lemma 5. First, we show that H must contain a cycle with a vertex of degree 3 in H. Second, we show that such a cycle is a reducible configuration.

Claim 1. *If $Mad(G) < \frac{42}{19}$, then H contains a cycle with a vertex v such that $d_H(v) = 3$.*

Proof. To prove Claim 1, it is sufficient to show that H (or some subgraph of H) has average degree greater than 2; so this is our goal. Form subgraph $\widehat{H}$ from H by deleting all of the isolated vertices in H. For $i \in \{0, 1, \ldots, 9\}$, let a_i denote the number of 3-vertices of G that have exactly i nearby 2-vertices and let $\widehat{a}_i$ denote the number of 3-vertices of G that have exactly i nearby 2-vertices and also have a corresponding vertex in $\widehat{H}$. Let n and $\widehat{n}$ denote the number of vertices in H and $\widehat{H}$, respectively. Note that $\sum_{i=0}^{9} a_i = n$ and $\sum_{i=0}^{9} \widehat{a}_i = \widehat{n}$. We now consider two weighted averages of the integers $0, 1, \ldots, 9$; the first average uses the weights a_i and the second average uses the weights $\widehat{a}_i$.

Let V_2 and V_3 denote the number of 2-vertices and 3-vertices in G. Since $\text{Mad}(G) < \frac{42}{19}$, by simple algebra we deduce that $2V_2/V_3 > \frac{15}{2}$. By rewriting this inequality in terms of the a_is, we get: $\frac{1}{n}\sum_{i=0}^{9} a_i i > \frac{15}{2}$. Note that 8 and 9 are the only numbers in this weighted average that are larger than the average. Thus, since $\widehat{a}_i = a_i$ for $i \in \{8, 9\}$ and $\widehat{a}_i \leq a_i$ for $0 \leq i \leq 7$, we also have $\frac{1}{\widehat{n}}\sum_{i=0}^{9} \widehat{a}_i i \geq \frac{1}{n}\sum_{i=0}^{9} a_i i > \frac{15}{2}$. We need one more inequality, which we prove in the next paragraph.

The table below lists the values of three quantities: i, the minimum degree of a vertex in $\widehat{H}$ that has i nearby 2-vertices in G, and the expression $2i/3 - 3$. Note that for all values of i, we have $d_{\widehat{H}}(v) \geq 2i/3 - 3$. We end the table at $i = 4$, since thereafter $d_{\widehat{H}}(v) = 1$ and $2i/3 - 3$ is negative.

i	9	8	7	6	5	4
$d_{\widehat{H}}(v)$	3	3	2	1	1	1
$2i/3 - 3$	3	7/3	5/3	1	1/3	-1/3

By taking the average over all vertices in $\widehat{H}$ of the inequality $d_{\widehat{H}}(v) \geq 2i/3 - 3$, we get the inequality

$$\frac{1}{\widehat{n}} \sum_{v \in V(\widehat{H})} d_{\widehat{H}}(v) \geq \frac{1}{\widehat{n}} \sum_{i=0}^{9} \widehat{a}_i \left(\frac{2}{3}i - 3\right).$$

By expanding this second sum into a difference of two sums, then substituting the values given above for these two sums, we conclude that the average degree of $\widehat{H}$ is greater than 2. This finishes the proof of Claim 1. □

Now we use Claim 1 to show that $\chi_i(G) = 3$. Let C be a cycle in H that contains a vertex v with $d_H(v) = 3$. Let C' be the cycle in G that corresponds to C; i.e. C' in G passes through all the vertices corresponding to vertices of C in the same order that they appear on C; furthermore, between each pair of successive 3-vertices on C' there are either two or three 2-vertices. Let J be the subgraph of G consisting of cycle C' in G, together with the neighbor w of v that is not on C'. By assumption, $G^{(2)} - J^{(2)}$ has a proper 3-coloring.

Note that each vertex u in J satisfies $d_{G^{(2)}}(u) \leq 3$ unless u is a 3-vertex and is also adjacent to a 3-vertex not on J. In that case, $d_{G^{(2)}}(u) = 4$; however, then two of the vertices on C' that are adjacent to u in $G^{(2)}$ have degree 2 in $G^{(2)}$, and hence, these two vertices can be colored after all of their neighbors. To simplify notation, we now speak of finding a proper vertex coloring of $G^{(2)}$. Let $K = J^{(2)}$. We now delete from K all vertices with degree 2 or 4 in $G^{(2)}$, since they can be colored last; call the resulting graph $\widehat{K}$. Our goal is to extend the 3-coloring of $G^{(2)} - K$ to $\widehat{K}$; we can then further extend this 3-coloring to K.

Observe that each vertex u in $\widehat{K}$ has list size at least 2. Furthermore, all vertices of $\widehat{K}$ have at most two uncolored neighbors in $\widehat{K}$ except for the two vertices x and y that are adjacent on C' to v; however, $d_{\widehat{K}}(x) = d_{\widehat{K}}(y) = 3$ and x and y both have three available colors. Hence, for every vertex u in $\widehat{K}$, the number of available colors is no smaller than its degree $d_{\widehat{K}}(u)$. We use this fact as follows.

If any component of $\widehat{K}$ is a path, then we can clearly color each vertex of the component from its available colors, by Lemma A part (a). In that case, each component of $\widehat{K}$ contains a vertex u with number of colors greater than $d_{\widehat{K}}(u)$; so by Lemma A part (a), we can color each vertex of $\widehat{K}$ from its available colors. If instead $\widehat{K}$ is a single component, then since the single component is neither a clique nor an odd cycle, we can color all of $\widehat{K}$, by Theorem B. Hence, we only need to consider the case when $\widehat{K}$ has two components: one contains a cycle, where two adjacent vertices have a common neighbor not on the cycle (this is w, the off-cycle neighbor of v); the other component is a cycle containing v.

Because the first component is neither an odd cycle nor a clique, we can color it by Theorem B. Now observe that since $d_H(v) = 3$ (and v is not adjacent to a 3-thread on C'), we know that v is adjacent in $G^{(2)}$

to a vertex z not on C' such that $d_{G^{(2)}}(z) = 2$. By uncoloring z, we make a third color available at v, so we can extend the coloring to the second component of $\widehat{K}$, using Lemma A part (a). Lastly, we recolor z. □

5 Planar graphs with $\Delta \geq 4$ and girth at least 13

In this section, we prove Theorem 4(b); for convenience, we restate it.

Theorem 4(b). *If G is planar, $\Delta(G) \geq 4$, and girth$(G) \geq 13$, then $\chi_i(G) = \Delta$.*

Proof. Suppose the theorem is false; let G be a minimal counterexample. Below we note six configurations that must not appear in G. In each case, we can delete the configuration H, color $G - H$ (by the minimality of G), and extend the coloring to H greedily.

(RC1) G contains no 1-vertices.
(RC2) G contains no 4-threads.
(RC3) G contains no 3-thread with one end having degree 3.
(RC4) G contains no 2-threads with both ends having degree 3.
(RC5) G contains no 3-vertex that is incident to one 1-thread and two 2-threads.
(RC6) G contains no 3-vertex that is incident to one 2-thread and two 1-threads such that the other end of the one 1-thread has degree 3.

For a specified face f, let t_2, t_3, and t_4 be the number of vertices incident to f of degrees 2, 3, and at least 4. Then by (RC2) and (RC3), for any face f, $t_2 \leq 2t_3 + 3t_4$, equality holds only if every 4^+-vertex is followed by a 3-thread and every 3-vertex is followed by a 2-thread. Thus if $t_3 > 0$, the equality does not hold. So

$$t_2 \leq 2t_3 + 3t_4;\ \text{and if } t_3 > 0, \text{ then } t_2 < 2t_3 + 3t_4. \tag{1}$$

We use a discharging argument. Let the initial charge be $\mu(x) = d(x) - 4$ for $x \in V(G) \cup F(G)$, where $d(x)$ is the degree of vertex x or the length of face x. Then by Euler Formula,

$$\sum_{x \in V \cup F} \mu(x) = -8. \tag{2}$$

We will distribute the charges of the vertices and faces in two phases. In Phase I, we use a simple discharging rule and show that only three types of faces have negative charge. In Phase II, we introduce two more discharging rules and show that the final charge of every face and every vertex is nonnegative. We thus get a contradiction to equation (2).

Discharging Phase I

We use the following discharging rule.

R1) Each face f gives charge $\frac{1}{3}$ to each incident 3-vertex and gives charge 1 to each incident 2-vertex.

Remark: Another way to state this discharging rule is that f gives charge 1 to each vertex, and every 3-vertex returns charge $\frac{2}{3}$ and every 4^+-vertex returns charge 1. Thus each vertex has final charge 0, and the final charge of each face f is

$$\mu^*(f) = \frac{2}{3}t_3 + t_4 - 4. \tag{3}$$

Claim 1. *If $\mu^*(f) < 0$, then f must have one of the following degree sequences:*
(a) $(4^+, 2, 2, 2, 4^+, 2, 2, 2, 4^+, 2, 2, 3, 2, 2)$
(b) $(4^+, 2, 2, 2, 4^+, 2, 2, 2, 4^+, 2, 3, 2, 2)$
(c) $(4^+, 2, 2, 2, 4^+, 2, 2, 4^+, 2, 2, 3, 2, 2)$.

Proof. Consider a face f with negative charge $\mu^*(f)$. Note that $\mu^*(f) \le -1/3$; by equation (3) and inequality (1), we have $-1/3 \ge \mu^*(f) = \frac{2}{3}t_3 + t_4 - 4 \ge \frac{1}{3}t_2 - 4$. We rewrite these inequalities as:

$$t_2 \le 2t_3 + 3t_4 \le 11. \tag{4}$$

By inequality (4), we know that $t_3 + t_4 \le 5$. If $t_3 + t_4 \le 3$, then $t_2 \le 3(t_3 + t_4) \le 9$, and hence $d(f) = t_2 + (t_3 + t_4) \le 9 + 3 = 12$. Since $girth(G) \ge 13$, this is a contradiction. So we must have $4 \le t_3 + t_4 \le 5$. From inequality (4), note that $t_4 < 4$; thus $t_3 > 0$.

If $t_3 + t_4 = 5$, then inequality (4) implies that $t_4 \le 1$, and thus $t_3 \ge 4$. By (RC2), (RC3), and (RC4), f contains at most two 2-threads, and three 1-threads; thus $d(f) \le 2(2)+3(1)+5 = 12$. Again, this contradicts $girth(G) \ge 13$, so we must have $t_3 + t_4 = 4$. If $t_2 = 11$, then $t_2 = 2t_3 + 3t_4$. Now inequality (1) implies that $t_3 = 0$, which is a contradiction. So instead $t_2 \le 10$. Combining this inequality with $t_3 + t_4 = 4$, we have $d(f) = t_2 + (t_3 + t_4) \le 14$. We now consider two cases: $d(f) = 14$ and $d(f) = 13$.

If $d(f) = 14$, then $t_2 = d(f) - (t_3 + t_4) = 10$. Since $t_3 > 0$, inequality (1) yields $2t_3 + 3t_4 > 10$. So $t_3 = 1$ and $t_4 = 3$. By (RC2) and (RC3), the degree sequence of f must be $(4^+, 2, 2, 2, 4^+, 2, 2, 2, 4^+, 2, 2, 3, 2, 2)$.

If $d(f) = 13$, then $t_2 = d(f) - (t_3 + t_4) = 9$. Since $t_3 > 0$, inequality (1) yields $2t_3 + 3t_4 > 9$. Now $(t_3, t_4) \in \{(2, 2), (1, 3)\}$. If $t_3 = t_4 = 2$, then we have one of the following two cases. If the two 3-vertices are pseudo-adjacent, then f has at most one 1-thread, two 2-threads, and one 3-thread; so $d(f) \le 1(1) + 2(2) + 1(3) + 4 = 12$. If the two 3-vertices are not pseudo-adjacent, then f has at most four 2-threads, so $d(f) \le 4(2) + 4 = 12$. Both of these cases contradict $girth(G) \ge 13$, so we must have $t_4 = 3$ and $t_3 = 1$. By (RC2) and (RC3), the degree sequence of f must be $(4^+, 2, 2, 2, 4^+, 2, 2, 2, 4^+, 2, 3, 2, 2)$ or $(4^+, 2, 2, 2, 4^+, 2, 2, 4^+, 2, 2, 3, 2, 2)$. □

Discharging Phase II

Note that each type of bad face listed in Claim 1 ends Phase I with charge $-\frac{1}{3}$. We now introduce two new discharging rules to send an additional charge of $\frac{1}{3}$ to these bad faces.

A *type-1 3-vertex* u is a 3-vertex that is incident with one 2-thread, and two 1-threads, with the other ends of the 1-threads each having degree at least 4. The vertex u is called a *weak vertex* in the face incident with the two 1-threads and is called a *strong vertex* in the other two faces incident to u.

A *type-2 3-vertex* u is a 3-vertex that is incident with one 0-thread, one 2-thread, and one 1^+-thread, with the other ends of the 2-thread and 1^+-thread having degree at least 4. The vertex u is called a *slim vertex* in the faces incident with the 0-thread and is called a *fat vertex* in the other face incident to u.

In our second discharging phase, we use the following two discharging rules.

R2) Each face gives charge $\frac{2}{3}$ to each of its weak vertices and gives charge $\frac{1}{6}$ to each of its slim vertices.

R3) Each face receives charge $\frac{1}{3}$ from each of its strong vertices and receives charge $\frac{1}{3}$ from each of its fat vertices.

Let $\mu^{**}(f)$ be the charge after the second discharging phase. Let $t_3'(f)$ be the number of non-special 3-vertices on f, i.e. 3-vertices on f that are not: slim, fat, strong, or weak. Beginning with equation (3) and applying rules R2) and R3), we write the final charge of a face f as:

$$\mu^{**}(f) = \frac{2}{3}t_3' + t_4 - 4 + 0 \cdot \#(weak) + \frac{1}{2} \cdot \#(slim) + 1 \cdot \#(strong) + 1 \cdot \#(fat).$$

Recall that each vertex had nonnegative charge at the end of Phase I. Since rules R2) and R3) do not change the charge at any vertex, it is clear that every vertex has nonnegative final charge. Now we will show that every face also has nonnegative final charge. This will contradict equation 2.

For a face f with negative charge after Phase I, by Claim 1, it contains a 3-vertex v incident to a 2-thread and a 1^+-thread with the other ends having degree at least 4. By (RC3) and (RC5), the third thread incident to v is either a 0-thread, or a 1-thread. If it is a 0-thread, then v is a fat vertex; if it is a 1-thread, then v is a strong vertex. In each case, either rule R2) or R3) sends an additional charge of $\frac{1}{3}$ to f. Since $\mu^*(f) = -\frac{1}{3}$, we conclude that $\mu^{**}(f) = 0$.

If a face has nonnegative charge after Phase I and contains no weak or slim vertices, then it does not give away charge, and therefore remains nonnegative. So we only consider the faces containing weak and slim vertices. Before proceeding, we have the following claims. Note that even after applying R2) and R3), the net charge given from each 3-vertex to each face is nonnegative; we use this fact implicitly when we prove Claims 2 and 4.

Claim 2. *If a face f contains a path $P = u \dots v$ with $d(u), d(v) \geq 3$, and if $|V(P) - \{u, v\}| \geq 4$, then f receives a total charge of at least 1 from the vertices in $V(P) - \{u, v\}$.*

Proof. Let $P_0 = P - \{u, v\}$ and assume $|V(P_0)| \geq 4$. By (RC2), path P_0 contains at least one 3^+-vertex. If P_0 contains a 4^+-vertex, we are done. Thus we may assume that P_0 contains no 4^+-vertices. Recall that if a 3-vertex v is weak on face f, then each pseudo-neighbor of v that is on f must be a 4^+-vertex. Hence, if P_0 contains more than one 3-vertex, then none of these 3-vertices can be weak, since each one has a pseudo-neighbor on f that is a 3-vertex; thus f gains at least $\frac{2}{3}$ from each 3-vertex, and hence gains more than 1 from P_0. So we assume that P_0 contains exactly one 3-vertex; call it x. By (RC3), vertex x splits P_0 into a 2-thread and a 1^+-thread. By (RC5) the third thread incident to x must be a 1-thread or a 0-thread. Therefore x is either a fat vertex or a strong vertex; hence, f receives a total charge of 1 one from x. □

To prove Claims 2, 3, and 4 we will be interested in the total charge that f receives from a slim vertex and its pseudo-neighbors and the total charge that f receives from a weak vertex and its pseudo-neighbors.

Claim 3. *A face with negative charge after Phase II must satisfy the following properties.*
(C1) It contains at most one weak vertex.
(C2) It contains at most one slim vertex.
(C3) It does not contain both weak and slim vertices.

Proof. Call weak vertices and slim vertices *bad* vertices. Suppose that f has at least 2 bad vertices v_1 and v_2. Note that each bad vertex and its pseudo-neighbors give at least 2 to f, so if the set of v_1 and its pseudo-neighbors is disjoint from the set of v_2 and its pseudo-neighbors, then f receives a total of at least $2(2) = 4$. Hence, we must assume these sets are not disjoint; however, in this case v_1, v_2, and their pseudo-neighbors give f a total of at least three (this case analysis is straightforward). By Claim 2, the contribution from the remaining vertices is at least 1 (we need to verify that these vertices contain a path of at least four vertices, but this is again straightforward). □

Similar to Claim 2, we have the following claim.

Claim 4. *If a face f contains a path $P = u \dots v$ with $d(u), d(v) \geq 3$ and $|V(P) \setminus \{u, v\}| \geq 8$, then either P contains a slim vertex or else f receives a total charge of at least 2 from the vertices of $P \setminus \{u, v\}$.*

Proof. Let $P_0 = P \setminus \{u, v\}$ and assume $|P_0| \geq 8$. If P_0 contains at least two 4^+-vertices, then f receives total charge 1 from each of these 4^+-vertices, and hence receives total charge at least 2 from the vertices of P_0. If P_0 contains a single 4^+-vertex, then call the 4^+-vertex y; note that P_0 contains a path P_1 with at least 5 vertices, such that one endpoint is y and the other endpoint is adjacent to either u or v. Clearly f receives charge 1 from y, and by Claim 2, f receives charge at least 1 from the vertices of $P_1 - y$; hence, f receives a total charge of 2.

Assume instead that P_0 contains no 4^+-vertices. Note that by (RC2), path P_0 must contain at least two 3-vertices. If P_0 contains at least three 3-vertices, then none of them can be weak, since each is pseudo-adjacent to a 3-vertex. If also none is slim, then f receives at least $3(\frac{2}{3})$ from P_0; if P_0 contains a slim vertex, then the claim holds. Finally, if P_0 contains at most two 3-vertices, then by (RC3) and (RC4), it contains at most two 2-threads and one 1-thread; thus $|V(P_0)| \leq 2(2) + 1(1) + 2 = 7$, which is a contradiction. □

Claim 5. *A face with negative charge after phase II must contain no slim vertex and no weak vertex.*

Proof. Assume that a face f contains exactly one vertex that is weak or slim; call this vertex y. Let the pseudo-neighbors of y on f be v_1 and v_2. There is a path P in f from v_1 to v_2 such that $y \notin P$, and $|V(P) \setminus \{v_1, v_2\}| \geq 8$. Note that $d(v_1), d(v_2) \geq 3$. Furthermore, since y is weak or slim, $V(P) \setminus \{v_1, v_2\}$ cannot contain a slim vertex; hence, by Claim 4, f receives total charge at least two from $P \setminus \{v_1, v_2\}$.

If y is weak, then f receives $1+0+1$ from v_1, y, v_2; similarly, if y is slim, then f receives at least $1+\frac{1}{2}+\frac{1}{2}$ from v_1, y, v_2. In each case, we see that $\mu^*(f) \geq -4+2+2=0$; this is a contradiction. □

This completes the proof of Theorem 4(b). □

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