List-coloring the Squares of Planar Graphs without 4-Cycles and 5-Cycles

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Abstract

Let G be a planar graph without 4-cycles and 5-cycles and with maximum degree $\Delta \geq 32$. We prove that $\chi_\ell(G^2) \leq \Delta + 3$. For arbitrarily large maximum degree Δ, there exist planar graphs G_Δ of girth 6 with $\chi(G_\Delta^2) = \Delta + 2$. Thus, our bound is within 1 of being optimal. Further, our bound comes from coloring greedily in a good order, so the bound immediately extends to *online* list-coloring. In addition, we prove bounds for $L(p, q)$-labeling. Specifically, $\lambda_{2,1}(G) \leq \Delta + 8$ and, more generally, $\lambda_{p,q}(G) \leq (2q - 1)\Delta + 6p - 2q - 2$, for positive integers p and q with $p \geq q$. Again, these bounds come from a greedy coloring, so they immediately extend to the list-coloring and online list-coloring variants of this problem.

1 Introduction

The *square* G^2 of a graph G is formed from G by adding an edge between each pair of vertices at distance two in G. In 1977, Wegner [12] posed the following conjecture, which has attracted great interest, and led to a remarkable number of results. (Most of our terminology and notation is standard. When it is not, we define terms where they are first used. For reference, we also collect some key definitions in the Appendix.)

Conjecture 1.1 (Wegner [12]). *If G is a planar graph with maximum degree Δ, then*

$$\chi(G^2) \leq \begin{cases} 7 & \text{if } \Delta = 3; \\ \Delta + 5 & \text{if } 4 \leq \Delta \leq 7; \\ \lfloor \frac{3\Delta}{2} \rfloor + 1 & \text{if } \Delta \geq 8. \end{cases}$$

Wegner also gave constructions showing that this conjecture is sharp if true. In particular, his sharpness example for $\Delta \geq 8$ is shown in Figure 1. Although the conjecture remains open in general, Havet et al. [9] showed that the conjectured upper bound holds asymptotically, i.e., $\chi(G^2) \leq \frac{3}{2}\Delta + o(\Delta)$. A more thorough history of Wegner's conjecture appears in the introductions of [8] and [9].

For every graph G, we have the lower bound $\chi(G^2) \geq \Delta + 1$. If we seek to prove an upper bound closer to this trivial lower bound, we clearly must forbid the configuration of Figure 1. Forbidding 3-cycles alone does not really help, since now subdiving the edge vw yields a graph G with no

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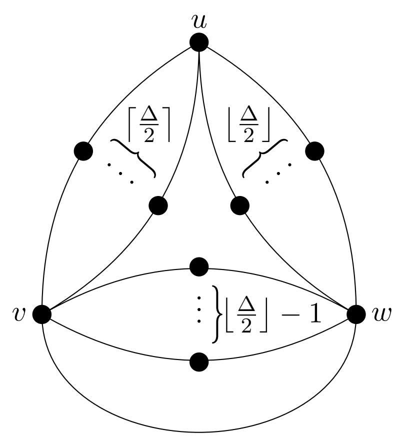

Figure 1: Wegner's construction for $\Delta \geq 8$.

3-cycles and such that G^2 still has clique number $\lfloor \frac{3}{2}\Delta \rfloor$. So we make the obvious choice and forbid 4-cycles, as well as perhaps cycles of other lengths. This line of inquiry has an intriguing history, much of which was motivated by the following conjecture of Wang and Lih [11].

Conjecture 1 (Wang and Lih [11])**.** *For every integer g at least 5, there exists some integer Δ_g such that every planar graph G with girth at least g and maximum degree at least Δ_g satisfies $\chi(G^2) = \Delta + 1$.*

The conjecture was proved by Borodin et al. [4] for $g \geq 7$ and disproved for $g \in \{5, 6\}$ in the same paper. However, Dvořák et al. [8] complemented these results with the following theorem.

Theorem 1.2 ([8])**.** *If G is a planar graph with girth at least 6 and $\Delta \geq 8821$, then $\chi(G^2) \leq \Delta + 2$.*

(Soon after, Borodin et al. [5] weakened the hypothesis to $\Delta \geq 18$.) In the same paper, Dvorak et al. posed the following conjecture.

Conjecture 2. *There exists some constant M such that every planar graph G with girth 5 and maximum degree at least M satisfies $\Delta(G^2) \leq \Delta + 2$.*

If true, Conjecture 2 would be a very nice result. Zhu et al. [13] went in a slightly different direction. They considered planar graphs with no 4-cycles and no 5-cycles (although 3-cycles are allowed). Among other results, they showed that if $\Delta \geq 9$, then $\chi(G^2) \leq \Delta + 5$. In fact, this bound follows from a more general result on $L(p, q)$-labeling, which we will discuss soon.

Our main result is the following theorem.

Main Theorem. *Let G be a planar graph with maximum degree Δ that contains no 4-cycles and no 5-cycles. If $\Delta \geq 32$, then there exists an ordering $v_1, \ldots, v_n$ of $V(G)$ such that each v_i has at most 3 neighbors in G that appear earlier in the ordering and at most $\Delta + 2$ neighbors in G^2 that appear earlier in the ordering.*

This theorem is optimal in the following sense. We cannot reduce the bound of "at most 3 neighbors in G" to "at most 2". To see this, it suffices to construct planar graphs with arbitrarily large maximum degree, no 4-cycles and no 5-cycles, and minimum degree 3. We do so as follows.

Form gadget H from a 6-cycle $v_1 \ldots v_6$ by adding vertices u_1, u_2, u_3 with u_1 adjacent to v_1 and v_2; u_2 adjacent to v_3 and v_4; and u_3 adjacent to v_5 and v_6. Finally, add a pendant edge incident to each u_i. To form graph G_k, begin with a cycle C_k and add a dominating vertex. Now replace,

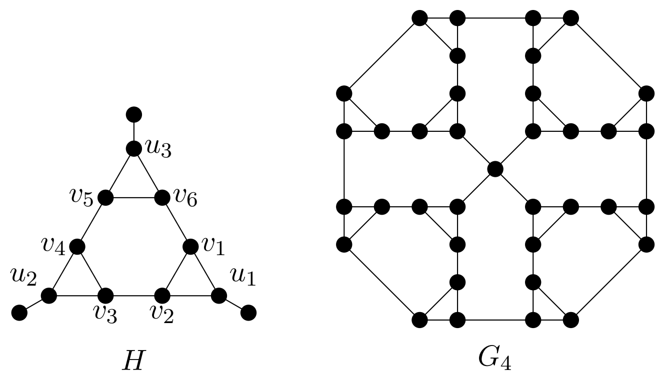

Figure 2: The gadget H (on the left) and G_4.

successively, each 3-vertex x of the resulting graph with a copy of H, joining each neighbor of x to H using its three pendant edges. Clearly the resulting graph has minimum degree 3. Each cycle within a copy of H has length 3 or at least 6, and each cycle through more than one copy of H has length at least 9. Thus, for any ordering σ of the vertices of G_k, the final vertex will have at least 3 neighbors earlier in σ.

To put this theorem in context, we note that this approach of coloring greedily in a good ordering was used implicitly by van den Heuvel and McGuinness [10] in their proof that every planar graph with Δ large enough satisfies $\chi(G^2) \leq 2\Delta + 25$. The method was made explicit by Agnarsson and Halldórsson [1] and Borodin et al. [2, 3] who (independently) improved this result to $\chi(G^2) \leq \lceil \frac{9}{5}\Delta \rceil + 1$ for Δ sufficiently large. Both groups showed that this bound is the best possible with this technique, by constructing planar graphs G_k of arbitrarily high maximum degree k such that G_k^2 has minimum degree $\lceil \frac{9}{5}\Delta \rceil$. This approach has also been used in some results on $L(p,q)$-labeling.

Our interest in our Main Theorem is due primarily to the following two corollaries.

Corollary 1. *If G is a planar graph with $\Delta \geq 32$ and neither 4-cycles nor 5-cycles, then $\chi_\ell(G^2) \leq \Delta + 3$. In fact, this bound holds also for paintability: $\chi_p(G^2) \leq \Delta + 3$.*

The bound on $\chi_\ell(G^2)$ comes directly from the Main Theorem, by coloring greedily in the prescribed ordering. Since each vertex v has at most $\Delta + 2$ earlier neighbors, some color remains for use on v. For paintability, the same argument works: on each round, Painter greedily forms a maximal stable set, by adding vertices in the prescribed order. As we noted above, there exist graphs G_Δ with arbitrarily large maximum degree Δ for which $\chi(G_\Delta^2) = \Delta + 2$. (For completeness, we include in the appendix a construction proving this, due to Dvořák et al. [8].) Hence, these bounds are within 1 of being best possible.

An $L(p,q)$-labeling is an assignment f of nonnegative integers to the vertices such that all adjacent vertices u and v satisfy $|f(u) - f(v)| \geq p$ and vertices u and v at distance two satisfy $|f(u) - f(v)| \geq q$. The $L(p,q)$*-labeling number* $\lambda_{p,q}(G)$ is the minimum value of the largest label k taken over all $L(p,q)$-labelings. For planar graphs with no 4-cyclces, no 5-cyclces, and Δ sufficiently large, Zhu et al.[13] proved that $\lambda_{p,q} \leq (2q-1)\Delta + 6p + 2q - 4$. In particular, for $\Delta \geq 11$, they proved $\lambda_{2,1} \leq \Delta + 10$. In the following corollary, we improve this bound for $\Delta \geq 32$.

Corollary 2. *If G is a planar graph with $\Delta \geq 32$ and neither 4-cycles nor 5-cycles, then $\lambda_{p,q}(G) \leq (2q-1)\Delta + 6p - 2q - 2$. In particular, $\lambda_{2,1}(G) \leq \Delta + 8$.*

As above, these bounds come from coloring greedily in the prescribed order. Consider a vertex v_i. Each of its at most 3 earlier neighbors forbid at most $(2p-1)$ labels; each of its other at most

$(\Delta+2-3)$ earlier neighbors in G^2 forbid at most $(2q-1)$ labels. Since the smallest allowable label is 0, we get $\lambda_{p,q}(G) \leq (2q-1)\Delta + 6p - 2q - 2$. Note that, also by greedily coloring, the bounds generalize immediately to *online list* $L(p,q)$-labeling.

1.1 Reducibility

To avoid some technical difficulties (caused by deleting a vertex and reducing the maximum degree of G) we prove the following theorem, which immediately implies our Main Theorem.

Theorem 1.3. *If G is a planar graph with maximum degree Δ that contains no 4-cycles and no 5-cycles, then there exists an ordering $v_1, \ldots, v_n$ of $V(G)$ such that each v_i has at most 3 neighbors in G that appear earlier in the ordering and at most* $\max(\Delta, 32) + 2$ *neighbors in G^2 that appear earlier in the ordering.*

In what follows, we prove some structural properties of a minimal counterexample to our theorem. Henceforth, let G denote such a minimal counterexample. More precisely, let G be a planar graph with no 4-cycles and no 5-cycles and such that no ordering $v_1, \ldots, v_n$ of $V(G)$ has every vertex v_i with both at most 3 neighbors in G earlier in the ordering and at most $\max(\Delta, 32) + 2$ neighbors in G^2 earlier in the ordering. Moreover, every proper subgraph of G has such an ordering. Let $N^2(u)$ denote the set of neighbors of u in G^2. Let $D = \max(32, \Delta)$. We call the ordering guaranteed by the Main Theorem a *good ordering* for G.

Basic Reducibility Lemma. *A minimal G has no vertex u such that $d(u) \leq 3$ and $|N^2(u)| \leq D+2$ and $(G-u)^2 = G^2 - u$. In particular, (i) $\delta(G) \geq 2$ and (ii) for every 2-vertex u on a 3-cycle uv_1v_2 we have $d(v_1) + d(v_2) \geq D + 5$.*

Proof. If u is such a vertex, then a good ordering for $G - u$ extends to a good ordering for G by appending u to the order. Further, we have $d(u) \leq 2$ and $|N^2(u)| \leq D + 2$ if either u is (i) a 1-vertex or (ii) u is a 2-vertex on a 3-cycle uv_1v_2 with $d(v_1) + d(v_2) \leq D + 4$. □

This lemma is illustrated in Figure 3. Note that here and throughout the paper, a vertex that is drawn as a filled circle has all of its incident edges drawn, while a vertex that is drawn as an empty box may have other incident edges that are not shown.

Figure 3: (i) a 1-vertex is reducible. (ii) a 2-vertex on a 3-cycle uv_1v_2 is reducible if $d(v_1) + d(v_2) \leq D + 4$.

We can extend the idea behind the Basic Reducibility Lemma to give another, stronger reducibility lemma.

Main Reducibility Lemma. *A minimal G has no sequence $S = \{w_1, \ldots, w_k\}$ of distinct vertices in $V(G)$ such that $E(G[S]) \neq \emptyset$, and also $|N(w_i) \setminus \{w_{i+1}, \ldots, w_k\}| \leq 3$ and $|N^2(w_i) \setminus \{w_{i+1}, \ldots, w_k\}| \leq D + 2$ for every $1 \leq i \leq k$.*

Proof. Suppose, to the contrary, that such a sequence S exists. Choose some $e \in E(G[S])$. Since $G - e$ is a proper subgraph, it has some good ordering σ'. To extend σ' to G, we delete all elements of S and append them in order; call this new ordering σ. Note that all edges of G^2 that are absent from $(G-e)^2$ are incident with some vertex of S. So σ is certainly good for each vertex of $V(G)\setminus S$. By hypothesis, it is also good for each vertex of S. $\square$

Whenever we invoke this lemma, we will list the sequence S in the appropriate order. While this result holds in general, we will typically use it when $k=2$ or $k=3$. The case $k=2$ gives the following useful intuition for the proof: For each edge u_1u_2 in G, at least one u_i is either a 5^+-vertex or has $|N^2(u_i)| \geq D+3$ (or possibly they are both 4-vertices). Thus, when we do our discharging analysis later, each edge with an endpoint that needs charge (this will be a vertex of low degree) has some charge "nearby", since it has a nearby vertex of large degree. The work of the proof is formalizing this intuition.

To conclude this section, we prove a Concavity Lemma. Essentially, this lemma implies that if $|N^2(u)|$ is fixed, then vertex u receives the least charge when it has one high degree neighbor and all other neighbors have degree as small as possible (subject to the constraint on $|N^2(u)|$).

Concavity Lemma. *Let $f(x) = 1 - \frac{4}{x}$, considered on some interval $[a,\infty)$ where $a > 0$. If $x_1,\ldots,x_n$ are to be chosen in $[a,\infty)$ such that $\sum_{i=1}^n x_i = C$ for some constant C, then the minimum value of $\sum_{i=1}^n f(x_i)$ is achieved when $x_1 = \ldots = x_{n-1} = a$ and $x_n = C - a(n-1)$.*

Proof. It suffices to show that $f(x_1) + f(x_2) \geq f(a) + f(x_1 + x_2 - a)$ for all $x_1, x_2 \in [a,\infty)$, since we can then proceed by induction on the number of x_i that are not equal to a.

Assume without loss of generality that $x_1 \leq x_2$, and let $t = x_1 - a$. Since f is concave, its derivative is decreasing, and can be bounded at a point by left and right secants there, giving:

$$\frac{f(x_2+t) - f(x_2)}{t} \leq f'(x_2) \leq f'(x_1) \leq \frac{f(x_1) - f(x_1-t)}{t}.$$

Clearing denominators and rearranging terms gives $f(x_2+t) + f(x_1-t) \leq f(x_1) + f(x_2)$. But this is equivalent to $f(x_1+x_2-a) + f(a) \leq f(x_1) + f(x_2)$, as was desired. $\square$

2 Proof of the Main Theorem via Discharging

Our proof of the Main Theorem is by the discharging method, which is most well-known for its central role in the proof of the 4 Color Theorem. (For an introduction to this technique, and a survey of results proved by it, see *A Guide to the Discharging Method* [7], by the first author and West.) We assume the theorem is false, and let G be a counterexample with fewest edges. We assign to each vertex v a charge $d(v)-4$ and to each face f a charge $\ell(f)-4$, where $d(v)$ and $\ell(f)$ denote the degree of v and the length of f. We denote these charges as $\mathrm{ch}(v)$ and $\mathrm{ch}(f)$. By Euler's formula, the sum of these initial charges (over all vertices and faces) is -8, since

$$\sum_{x\in V\cup F} \mathrm{ch}(x) = \sum_{v\in V} d(v) - 4 + \sum_{f\in F} \ell(f) - 4 = 2|E| - 4|V| + 2|E| - 4|F| = -4(2).$$

Now we redistribute charge via the four discharging rules outlined below, giving a final charge function ch^*. Since G is a minimal counterexample, it must not contain any configurations that are reducible under either the Basic Reducibility Lemma or the Main Reducibility Lemma. We use the absence of such configurations to show that each face and vertex finishes with nonnegative final charge. This gives the following contradiction:

$$-8 = \sum_{x\in V(G)\cup F(G)} \mathrm{ch}(x) = \sum_{x\in V(G)\cup F(G)} \mathrm{ch}^*(x) \geq 0.$$

Hence no such minimal counterexample G can exist, so the Main Theorem is true.

2.1 Discharging Rules

The following four discharging rules are applied to the elements of G successively, i.e., (R1) is applied everywhere that it is applicable, then (R2), then (R3), and finally (R4). Examples of these rules are illustrated in Figure 4. We write k-vertex (resp. k^+, k^-) for a vertex of degree k (resp. at least k, at most k). We define k-faces analogously.

R1: Each 6^+-face gives charge $\frac{1}{3}$ to each incident edge. If such an edge e is incident to a 3-face f, then e gives this charge to f. Otherwise, e splits this charge evenly between any 3^--endpoints it has, or else splits it evenly between both endpoints if both have degree at least 4. [1]

R2: Each 6^+-vertex v splits its initial charge evenly among its neighbors of degree at most $d(v)$. Each 5-vertex with a 16^+-neighbor splits its initial charge evenly among its 4^--neighbors. Each 5-vertex v with no 16^+-neighbor splits is initial charge evenly among its neighbors of the following types: 3-vertices on triangular faces with v and no 12^+-neighbor, 2-vertices on triangular faces with v, and other 2-vertices with no $(D-2)^+$-neighbor.

R3: Let u be a 4^+-vertex on a 3-face uvw and suppose u receives some charge c during R2 from v. If w is a 2-vertex, then u passes charge c on to w. If instead w is a 3-vertex with a 2-neighbor whose other neighbor has degree less than D, then u passes charge $\min\{c, \frac{1}{2}\}$ on to w. [2]

R4: If a 3^+-vertex has positive charge after R1-R3, it splits this charge among its neighbors with negative charge, such that a 3-vertex gives charge at most $\frac{4}{15}$ to another 3-vertex, and otherwise all charge splits evenly.

As stated above, we now show that $\text{ch}^*(x) \geq 0$ for each vertex and face x. It turns out that this is easy for everything except 3-vertices and 2-vertices, which require more detailed analysis.

2.2 Faces and High-Degree Vertices

All faces end with nonnegative final charge. Each 6^+-face f starts with charge $\ell(f) - 4$ and gives away charge $\frac{\ell(f)}{3}$. Thus f ends with $\text{ch}^*(f) = \frac{2\ell(f)}{3} - 4$, which is nonnegative since $\ell(f) \geq 6$. A 3-face cannot be adjacent to another 3-face since 4-cycles are forbidden. Since G has no 4-cycles or 5-cycles, each 3-face f must be adjacent to a 6^+-face on each of its edges. Each such 6^+-face passes charge $\frac{1}{3}$ to f via their common edge, so $\text{ch}^*(f) = 3 - 4 + 3(\frac{1}{3}) = 0$.

Each 4^+-vertex v starts out with nonnegative initial charge, and by the design of the discharging rules never gives away more than its current charge, so $\text{ch}^*(v) \geq 0$. Now we must verify that all 3-vertices and 2-vertices end with nonnegative final charge as well, which will complete the proof.

2.3 3-vertices

First consider a 3-vertex u that is not incident to any 3-faces. The three faces meeting at u must all be 6^+-faces, and thus each gives total charge $\frac{2}{3}$ to two of the edges incident to u. Even when all of u's neighbors are 3^--vertices, u receives at least half of this charge, and hence end with $\text{ch}^*(u) \geq 3 - 4 + 3(\frac{1}{3}) = 0$.

Now consider a 3-vertex u on a 3-face uv_1v_2 whose third neighbor is w, as shown in Figure 5. Note that since v_1 and v_2 are adjacent, $|N^2(u)| \leq d(w) + d(v_1) + d(v_2) - 2$. The two faces incident

[1] Edges only ever act as a charge carrier between faces and other faces or vertices. Outside of this phase, edges always have zero charge. Also, G need not be 2-connected. If a cutedge e lies on a face f, then f gives e charge $\frac{2}{3}$.

[2] This rule rarely applies, and it can be largely ignored when seeking the high-level intuition behind the proof.

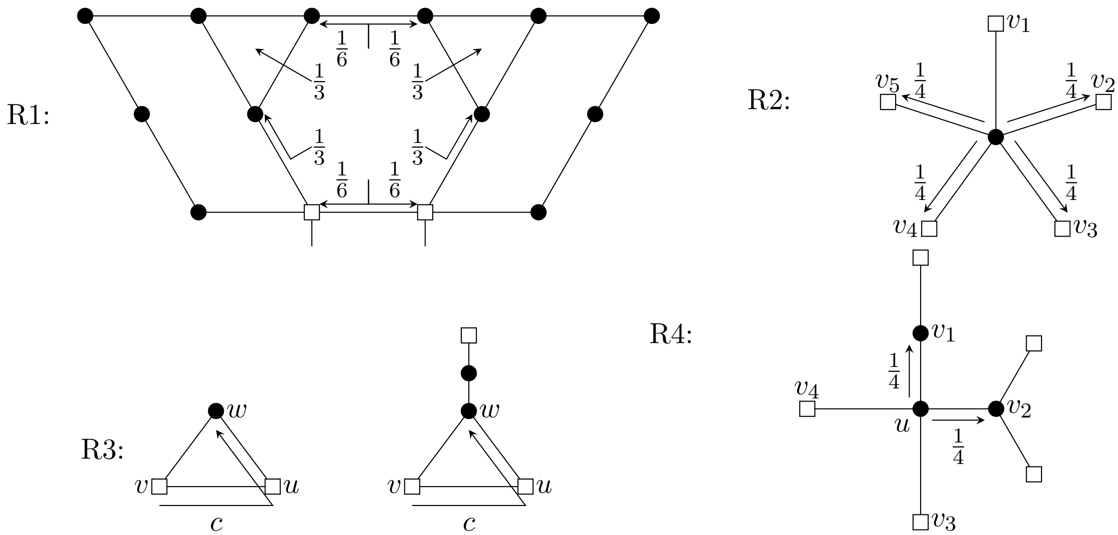

Figure 4: (R1) A 6-face gives charge $\frac{1}{3}$ to each incident edge, and it is passed on to either an incident 3-face or to one or both endpoints of the edge. (R2) If $d(v_1) \geq 5$ and $d(v_i) \leq 4$ for each $i \in \{2,3,4,5\}$, then the 5-vertex splits its charge equally among v_2, v_3, v_4, and v_5. (R3) Here u passes some or all of the charge it receives from v on to w. (R4) If u has charge $\frac{1}{2}$ after R1–R3, then it splits this charge between its two neighbors needing charge, v_1 and v_2.

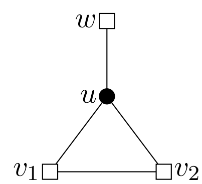

Figure 5: The 3-vertex u on a 3-face under consideration.

to u other than the 3-face must be 6^+-faces, and hence give total charge $\frac{2}{3}$ to the edge uw. If $d(w) \geq 4$, then all of this charge passes to u, while if $d(w) \leq 3$, then u receives charge $\frac{1}{3}$ from this edge.

If $d(v_i) = 2$ for any i, then v_i is reducible under the Basic Reducibility Lemma. Alternately, if $d(v_i) \geq 12$ for some i, then u receives charge at least $\frac{12-4}{12} = \frac{2}{3}$ from v_i via R2, and since uw sends u charge at least $\frac{1}{3}$ via R1, $\mathrm{ch}^*(u) \geq 3 - 4 + \frac{2}{3} + \frac{1}{3} = 0$. Hence we can assume $3 \leq d(v_i) \leq 11$ for $i \in \{1,2\}$. Also, if $d(v_1) + d(v_2) \geq 17$, then the Concavity Lemma (with $a = 5$) implies that u receives at least as much charge as when one v_i is a 12^+-vertex, which, as just shown, ensures that $\mathrm{ch}^*(u) \geq 0$. Thus we assume $d(v_1) + d(v_2) \leq 16$. Now we consider what happens to u based on the degree of w.

Case $d(w) \geq 6$: Here u receives charge at least $\frac{6-4}{6} = \frac{1}{3}$ from w, as well as charge $\frac{2}{3}$ from uw, thus ends with $\mathrm{ch}^*(u) \geq 3 - 4 + \frac{2}{3} + \frac{1}{3} = 0$.

Case $d(w) = 2$: Since $|N^2(u)| \leq d(v_1) + d(v_2) \leq 16$ and $|N^2(w)| \leq D + 3$ and $D \geq 14$, this configuration is reducible under the Main Reducibility Lemma.

Case $d(w) \in \{3,4,5\}$: We will show that u receives charge at least $\frac{1}{2}$ total from w and the edge uw, and at least $\frac{1}{4}$ from each of v_1 and v_2. This ensures that $\mathrm{ch}^*(u) \geq 3 - 4 + \frac{1}{2} + 2(\frac{1}{4}) = 0$. First consider the charge from w and uw: if $d(w) \geq 4$, then as mentioned above, all $\frac{2}{3}$ of the charge that passes through uw goes to u, and $\frac{2}{3} > \frac{1}{2}$. Otherwise, if $d(w) = 3$, then u receives $\frac{1}{3}$ from uw, and so needs at least $\frac{1}{6}$ more from w for this total to reach $\frac{1}{2}$.

Let x_1 and x_2 denote the neighbors of w other than u. Since $\{u, w\}$ is not reducible, the Main Reducibility Lemma implies that $d(x_1) + d(x_2) \geq D + 1$. Now the Concavity Lemma implies that w has at least as much charge to give to u via R4 as when $d(x_1) = D - 4$ and $d(x_2) = 5$ (and x_2 gives no charge to u). If w does not lie on a 3-face, then it receives charge $3(\frac{1}{3})$ from its three incident edges via R1, making its charge nonnegative. Now the additional charge of $\frac{(D-4)-4}{D-4}$ from x_1 is split at most two ways. Since $D \geq 10$, this ensures that u gets an additional charge of at least $\frac{1}{6}$ from w.

Suppose instead that w does lie on a 3-face. Now we know that $d(x_2) \geq 3$, since a 2-vertex on a 3-face with a 3-neighbor is reducible according to the Basic Reducibility Lemma. Now if $d(x_2) \geq 4$, then x_2 always has nonnegative charge and thus never needs to receive charge. If $d(x_2) = 3$, then x_2 receives charge at least $\frac{1}{3}$ from its incident edge not on the 3-face, and at least $\frac{2}{3}$ from x as long as $d(x) \geq 12$, meaning it does not need any charge from w. Thus, whatever the degree of x_2, vertex w does not need to give it any charge via R4. Since $D \geq 25$, this ensures that w gets charge $\frac{1}{3} + \frac{5}{6}$ via R1 and R2, and thus gives charge $\frac{1}{6}$ to u via R4. Hence we have shown that u always gets charge at least $\frac{1}{2}$ from w and the edge uw.

Now we show that u receives charge at least $\frac{1}{4}$ from v_1 and, by symmetry, also from v_2. If $d(v_1) \geq 6$, then v_1 gives charge at least $\frac{1}{3}$ to u via R2, and $\frac{1}{3} > \frac{1}{4}$. Otherwise assume $d(v_1) \leq 5$.

First suppose that $d(v_1) = 5$. If v_1 splits its charge between four or fewer neighbors, then each receives charge at least $\frac{1}{4}$, so we are done. So assume instead that all five neighbors of v_1 should receive some of its charge via R4. We will show that uv_1v_2 is a reducible configuration. By minimality, we can get a good ordering σ' for $G - uv_2$. Let $S = \{v_2, u\}$. To extend σ' to G, delete S and append v_2, u; call this ordering σ. Clearly σ is good for every vertex of $V(G) \setminus S$. Also, each vertex of S has at most three neighbors in G earlier in σ. Finally, each $x \in S$ has at most $D + 2$ neighbors in G^2 earlier in the ordering: $|N^2(v_2) \setminus \{u\}| \leq d(v_1) + d(u) + (D - 3) - 3 = D + 2$ and $|N^2(u)| \leq 5 + 5 + 3$. So assume $d(v_1) \in \{3, 4\}$.

Recall that $|N^2(u)| \leq d(w) + d(v_1) + d(v_2) - 2 \leq 19$. If $\{u, v_1\}$ is not reducible under the Main Reducibility Lemma, then $|N^2(v_1)| \geq D + 4$, i.e., v_1 has at least one high-degree neighbor z. Now v_1 has no excess charge to give to u via R1, but will be able to give the needed charge via R4. Note that by the same reasoning used above, since $\{u, v_2\}$ is not reducible under the Main Reducibility Lemma, v_2 must also either be a 4^+-vertex or have a high-degree neighbor. This means that v_1 never needs to give charge to v_2 via R4, since v_2 only ever needs to receive charge if it is a 3-vertex, and in such a case, it receives all the charge it needs from its high-degree neighbor and incident edge off of the 3-face.

In the case that $d(v_1) = 3$, the neighbor z of v_1 not on the 3-face must have degree at least $D - 8$. Since $D \geq 18$, this ensures that v_1 gets charge at least $\frac{3}{5} + \frac{2}{3}$ from z and the edge v_1z. Thus v_1 is able to pass charge at least $\frac{4}{15} > \frac{1}{4}$ to u.

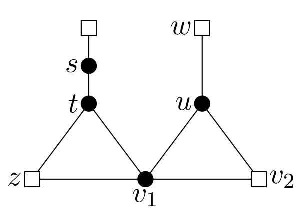

Figure 6: This configuration, where R3 would apply, is reducible by the Main Reducibility Lemma.

If instead $d(v_1) = 4$, then v_1 split any excess charge it receives at most two ways via R4 (since neither z nor v_2 needs charge). Let t be the neighbor of v_1 other than u, v_2, and z, and note that v_1 only sends charge to t via R4 if $d(t) < 4$. By the Concavity Lemma, v_1 receives no less charge than when $d(z) = D - 5$, $d(t) = 3$, and $d(v_2) = 5$ (but it doesn't give charge to v_1). If v_1zt is not a 3-face, then v_1 receives charge at least $\frac{(D-5)-4}{D-5} + \frac{1}{3}$ from z and the edge v_1z. Since $D \geq 10$, vertex v_1 get charge at least $\frac{8}{15}$, so it passes at least $\frac{4}{15} > \frac{1}{4}$ to u via R4.

If instead v_1zt is a 3-face, then we note that t cannot be a 2-vertex, since this would be reducible. Also, t cannot be a 3-vertex with a 2-neighbor s, where the other neighbor of s has degree less than D, because this also would be reducible under the Main Reducibility Lemma (using the vertex sequence $S = \{t, s, u\}$), as shown in Figure 6. Since these are the only times when R3 can apply, we conclude that this rule is not used here. Hence v_1 gets charge at least $\frac{(D-5)-4}{D-5}$ from z, which it can then send at least half of to u. As long as $D \geq 13$, this means v_1 sends at least $\frac{1}{4}$ to u as desired.

2.4 2-vertices

2-vertex on a 3-face: First consider a 2-vertex u on a 3-face uv_1v_2, as depicted in Figure 7. By the Basic Reducibility Lemma, this is reducible unless $d(v_1) + d(v_2) \geq D + 5$. By the Concavity Lemma, we know that u receives at least as much charge as if $d(v_1) = D$ and $d(v_2) = 5$. Now u receives charge at least $\frac{D-4}{D} + \frac{1}{4}$ via R2. However, v_2 also receives charge $\frac{D-4}{D}$ from v_1 via R2, and the conditions are met for R3, so v_2 passes this charge along to u. Hence in total u receives charge at least $2(\frac{D-4}{D}) + \frac{1}{4}$. Since $D \geq 32$, u ends with $\text{ch}^*(u) \geq 2 - 4 + 2(\frac{32-4}{32}) + \frac{1}{4} = 0$.

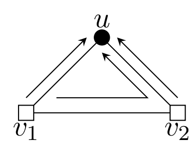

Figure 7: A 2-vertex on a 3-face receives charge via R2 and R3.

2-vertex with one high-degree neighbor: Now we assume that the 2-vertex u, with neighbors v_1 and v_2, does not lie on a 3-face. Note that if $d(v_i) = 2$ for some $i \in \{1, 2\}$, then $\{u, v_i\}$ is reducible under the Main Reducibility Lemma. Hence we assume that $d(v_1) \geq 3$ and $d(v_2) \geq 3$.

Suppose $d(v_1) \geq D - 2$; now u receives charge $\frac{2}{3}$ through the edge uv_1 via R1 and $\frac{(D-2)-4}{D-2}$ from v_1 via R2. If $d(v_2) \geq 4$, then u also gets $\frac{2}{3}$ through the edge uv_2 via R1, and so ends with final charge at least $2 - 4 + 2(\frac{2}{3}) + \frac{(D-2)-4}{D-2}$, which is nonnegative since $D \geq 14$.

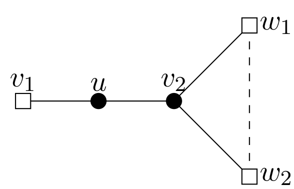

Figure 8: A 2-vertex u with a neighbor v_1 such that $d(v_1) \geq D - 2$.

So assume $d(v_2) = 3$, and denote the other neighbors of v_2 by w_1 and w_2, as pictured in Figure 8. Note that v_2 and u each receive charge $\frac{1}{3}$ from the edge uv_2 via R1. Now $\{u, v_2\}$ is reducible under the Main Reducibility Lemma unless $|N^2(v_2)| \geq D + 3$. First, suppose that v_2 lies on a 3-face, which implies $d(w_1)+d(w_2) \geq D+3$. By the Concavity Lemma, v_2 receives at least as much charge as if $d(w_1) = D-1$ and $d(w_2) = 4$. Hence after R2, v_2 has charge at least $3-4+\frac{1}{3}+\frac{(D-1)-4}{D-1}$. Since $D \geq 26$, this ensures that v_2 has charge at least $-1+\frac{1}{3}+\frac{21}{25} > \frac{1}{6}$ after R2, which it passes to u via R4. (Note that w_2 does not receive charge from v_2 via R4: since $v_2w_1w_2$ is a 3-face, $d(w_2) > 2$. Further, if $d(w_2) = 3$, then w_2 receives enough charge from w_1 and its incident edge off of the 3-face.) Hence $\text{ch}^*(u) \geq 2-4+\frac{2}{3}+\frac{1}{3}+\frac{(26-2)-4}{26-2}+\frac{1}{6} = 0$.

So suppose instead that v_2 does not lie on a 3-face. Now $|N^2(v_2)| \geq D+3$, implying that $d(w_1)+d(w_2) \geq D+1$. Again using the Concavity Lemma, we can assume that $d(w_1) \geq D-4$. Now v_2 gets charge at least $\frac{1}{3}$ from each of the edges uv_2 and v_2w_2, and $\frac{2}{3}$ from the edge v_2w_1 via R1, which already puts its total charge at $3-4+\frac{4}{3} = \frac{1}{3}$. Now v_2 splits this charge at most two ways (giving to u and possibly w_2) via R4. Since v_2 has charge at least $\frac{1}{3}$ after R1, it gives charge at least $\frac{1}{6}$ to u via R4. As shown above, since $D \geq 26$ this ensures that $\text{ch}^*(u) \geq 0$, as desired.

Hereafter we assume that $d(v_1) \leq D-3$ and $d(v_2) \leq D-3$. We show that u must receive total charge at least 1 from edge uv_1 and vertex v_1; by symmetry the same is true of edge uv_2 and vertex v_2. This ensures that u ends with final charge at least $2-4+1+1 = 0$, as desired. If $d(v_1) \geq 6$, then u gets charge $\frac{2}{3}$ from uv_1 via R1 and charge $\frac{d(v_1)-4}{d(v_1)} \geq \frac{6-4}{6} = \frac{1}{3}$ from v_1 via R2. This gives u the charge of 1 from v_1's side as needed, so henceforth we assume $d(v_1) \leq 5$.

2-vertex with a 3-neighbor: Suppose $d(v_1) = 3$, and denote the other neighbors of v_1 by w_1 and w_2, with $d(w_1) \geq d(w_2)$. Now u receives charge $\frac{1}{3}$ from the edge uv_1 via R1, meaning it needs to get $\frac{2}{3}$ from v_1 via R4. First suppose that v_1 does not lie on a 3-face. Since $d(v_2) \leq D-3$, we apply the Main Reducibility Lemma with $S = \{v_1, u\}$, unless $d(w_1)+d(w_2) \geq D+2$. Likewise, if $d(w_2) = 2$, then we simply take $S = \{v_1, w_2, u\}$.

Hence we assume $d(w_2) \geq 3$. If $d(w_2) \geq 4$, then v_1 receives charge $\frac{2}{3}$ from both of the edges v_1w_1 and v_1w_2, along with $\frac{1}{3}$ from the edge uv_1 via R1. This means that after R1 alone, v_1 has charge $3-4+\frac{1}{3}+2(\frac{2}{3}) = \frac{2}{3}$, which it can then send to u via R4 as needed. So instead suppose that $d(w_2) = 3$, which implies $d(w_1) \geq D-1$. Now v_1 gets charge at least $\frac{4}{3}$ via R1 ($\frac{1}{3}$ each from edges uv_1 and v_1w_2, and $\frac{2}{3}$ from edge v_1w_1) and $\frac{(D-1)-4}{D-1}$ from w_1 via R2. Since $D \geq 11$, this ensures that v_1 has charge at least $3-4+\frac{4}{3}+\frac{(11-1)-4}{11-1} = \frac{14}{15}$ after R2. Since v_1 gives no more charge than $\frac{4}{15}$ to w_2 via R4, it can give at least $\frac{10}{15} = \frac{2}{3}$ to u via R4 as needed. So u gets charge at least 1 from v_1 and uv_1.

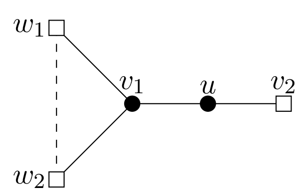

Figure 9: A 2-vertex u with a 3-neighbor v_1.

Now suppose instead that v_1 does lie on a 3-face. If we cannot apply the Main Reducibility Lemma with $S = \{v_1, u\}$, then $d(w_1)+d(w_2) \geq D+4$. By the Concavity Lemma, v_1 receives at least as much charge as if $d(w_1) = D$ and $d(w_2) = 4$. Thus v_1 receives charge $\frac{1}{3}$ from edge uv_1 via R1, and further receives charge at least $\frac{D-4}{D}$ from w_1 via R2. Additionally, w_2 receives at least $\frac{D-4}{D}$ from w_1 via R2, and the criteria are met for R3; since $D \geq 8$, this means w_2 passes charge $\frac{1}{2}$ to v_1. Hence after R3, v_1 has charge at least $3-4+\frac{1}{3}+\frac{1}{2}+\frac{D-4}{D}$. Since $D \geq 24$, this means v_1 has charge at least $-\frac{1}{6}+(\frac{24-4}{24}) = \frac{2}{3}$ that it can pass to u via R4, as needed.

2-vertex with a 4-neighbor: Now suppose $d(v_1) = 4$. In this case, u receives charge $\frac{2}{3}$ from edge uv_1 via R1, and hence only needs to get charge $\frac{1}{3}$ more from v_1 via R4. We can apply the Main Reducibility Lemma with $S = \{v_1, u\}$ unless $|N^2(v_1)| \geq D + 4$, which means the degree sum of the neighbors of v_1 other than u is at least $D + 2$. The least charge that passes from v_1 to u via R4 occurs when v_1 has as many 3^--neighbors as possible, so we assume that v_1 has two 3^--neighbors w_1 and w_2 and one high-degree neighbor z, as shown in Figure 10.

By the Concavity Lemma, v_1 receives at least as much charge via R2 as if $d(z) = D - 8$ and $d(w_1) = d(w_2) = 5$ (but neither w_1 nor w_2 gives charge to v_1). If v_1 and z do not lie on a common 3-face, then v_1 receives charge $\frac{1}{3}$ from edge v_1z via R1. Since $D \geq 20$, v_1 receives charge at least $\frac{(20-8)-4}{20-8} = \frac{2}{3}$ from z via R2, giving v_1 a total charge of at least 1 after R2. Since v_1 splits its charge at most three ways, it passes charge at least $\frac{1}{3}$ to u via R4, as needed.

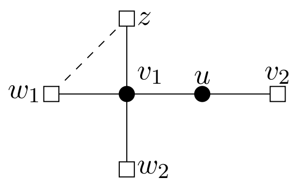

Figure 10: A 2-vertex u with a 4-neighbor v_1, where v_1 has a high-degree neighbor z.

Instead, assume v_1zw_1 is a 3-face. By the Basic Reducibility Lemma, we know w_1 cannot be a 2-vertex, so instead assume $d(w_1) \geq 3$. First, suppose $d(w_1) = 3$, and let x be the third neighbor of w_1 besides v_1 and z. Now w_1 receives charge at least $\frac{1}{3}$ from edge w_1x via R1 and, since $D \geq 20$, receives charge at least $\frac{(20-8)-4}{20-8} = \frac{2}{3}$ from z via R2. Hence w_1 has nonnegative charge after R2, and thus does not need charge from v_1 via R4, meaning v_1 only splits its charge at most two ways. Similarly, if $d(w_1) \geq 4$, then w_1 does not need charge from v_1 via R4. Thus, in every case, v_1 splits its charge after R3 at most two ways.

Now v_1 also receives charge at least $\frac{2}{3}$ from z via R1. If $d(x) = 2$ and the other neighbor of x has degree less than D, then the sequence $S = \{w_1, x, u\}$ is reducible under the Main Reducibility Lemma. If instead $d(x) \geq 3$, or $d(x) = 2$ and the other neighbor of x has degree D, then the conditions for R3 are not met, which means v_1 keeps its charge from z until R4. Splitting at most two ways, v_1 can give charge at least $\frac{1}{3}$ to u via R4, which is all u still needs.

2-vertex with a 5-neighbor: Finally, suppose $d(v_1) = 5$, as shown in Figure 11. Similar to above, u receives charge $\frac{2}{3}$ from edge uv_1 via R1. Now we must consider whether or not v_1 has a 16^+-neighbor. First, suppose that it does.

Since v_1 has a 16^+-neighbor, it splits its initial charge of $5 - 4 = 1$ at most four ways, so it passes charge at least $\frac{1}{4}$ to u via R2. Thus in order for u to receive charge at least 1 from v_1 and the edge uv_1, it only needs to get charge $\frac{1}{12}$ more from v_1 via R4.

Let z denote the highest-degree neighbor of v_1, and denote its other neighbors by w_1, w_2, and w_3. If v_1 and z are not together on a 3-face, then v_1 receives charge $\frac{1}{3}$ from edge v_1z via R1, and does not lose this charge prior to R4. Thus in R4, v_1 has charge at least $\frac{1}{3}$ which it splits at most four ways, meaning it sends charge at least $\frac{1}{12}$ to u, as needed. So instead assume that v_1zw_1 is a 3-face. Now since $|N^2(v_1)| \geq D+4$, we have $d(z)+d(w_1)+d(w_2)+d(w_3) \geq D+4$; by the Concavity Lemma, v_1 receives at least as much charge via R1 and R2 as if $d(z) = D - 10$ and $d(w_1) = 4$ and $d(w_2) = d(w_3) = 5$ (but neither sends charge to v_1 via R2).

Suppose $d(w_1) = 2$. This configuration is not immediately reducible under either the Basic Reducibility Lemma or the Main Reducibility Lemma, but is in fact reducible using a hybrid of the

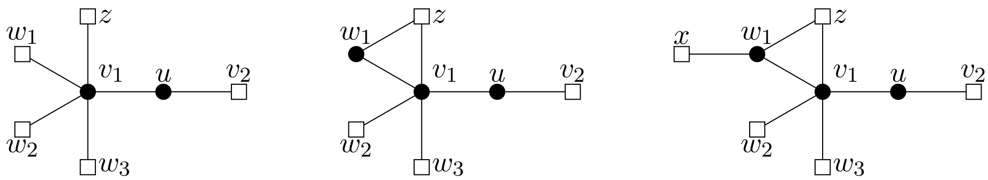

Figure 11: Cases where a 2-vertex u has a 5-neighbor v_1.

two approaches. If we delete vertex w_1 as in the Basic Reducibility Lemma, we get a good ordering σ' for $G - w_1$. To extend this ordering to G, we delete u and append w_1, u. The key point is that now u is not an earlier neighbor of w_1 in G^2, so the number of earlier neighbors for w_1 in G^2 is at most $d(z) + d(v_1) - 2 - 1 \leq D + 5 - 3 = D + 2$. Also, recall that we are assuming $d(v_2) \leq D - 3$, so $|N^2(u)| \leq d(v_1) + d(v_2) \leq 5 + (D - 3) = D + 2$. Hence, this configuration is reducible.

Now assume $d(w_1) \geq 3$. If $d(w_1) \geq 4$ then whatever charge v_1 gets from z via R2 it keeps until R4. Since $d(z) \geq 6$, this means that v_1 receives charge at least $\frac{6-4}{6} = \frac{1}{3}$ in R2, and splits it at most three ways in R4, so it gives u charge at least $\frac{1}{9} > \frac{1}{12}$. Instead suppose $d(w_1) = 3$, and let x be the other neighbor of w_1. If the criteria for R3 are not met (i.e. $d(x) \geq 3$ or $d(x) = 2$ and the other neighbor of x has degree D), then v_1 keeps any charge it receives from z via R2 until R4. Thus, as before, v_1 still gets charge at least $\frac{1}{3}$ since $d(z) \geq 6$, and splitting at most four ways gives charge $\frac{1}{12}$ to u via R4, as needed.

Suppose instead that $d(x) = 2$ and the other neighbor of x has degree at most $D - 1$. Now v_1 passes some charge that it gets from z via R2 to w_1 via R3. Since $d(z) \geq 16$, v_1 receives charge at least $\frac{16-4}{16} = \frac{3}{4}$ from z via R2. Now v_1 gives charge $\frac{1}{2}$ to w_1 via R3, leaving it with charge $\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$. Since w_1 gets charge at least $\frac{1}{3}$ from the edge w_1x via R1, $\frac{3}{4}$ from z via R2, and $\frac{1}{2}$ from v_1 via R3, it has nonnegative charge, and thus needs no charge from v_1 via R4. Hence v_1 splits its remaining $\frac{1}{4}$ charge at most three ways, meaning it gives charge at least $\frac{1}{12}$ to u via R4 as needed.

Now suppose instead that v_1 has no 16^+-neighbor. Since u receives charge $\frac{2}{3}$ from edge uv_1, we must show that in this case u still receives charge at least $\frac{1}{3}$ from vertex v_1. By R2, v_1 splits its charge of 1 among neighbors of the following types: 3-vertices on triangular faces with v_1 and no 12^+-neighbor, 2-vertices on triangular faces with v_1, and other 2-vertices with no $(D-2)^+$-neighbor. If v_1 has at most three neighbors of these types, then clearly v_1 gives charge at least $\frac{1}{3}$ to u, and we are done. So, suppose instead that v_1 has at least four neighbors of these types. In particular, this implies that v_1 has at most one 4^+-neighbor and no 16^+-neighbor. We will show that G contains a reducible configuration.

Note that v_1 can be incident to at most two triangular faces. We will show that v_1 gives charge via R2 to at most neighbor not on a triangular face and at most one neighbor on each of at most two incident triangular faces. Thus, v_1 gives charge to at most 3 neibhbors by R2.

Suppose that v_1 has two 2-neighbors, say u_1 and u_2, such that each u_i has no $(D-2)^+$-neighbor. Form G' from G by deleting u_1 and u_2. By minimality, G' has a good vertex ordering σ'. To reach a good vertex ordering σ for G, delete v_1 from σ', then append v_1, u_1, u_2. Now v_1 has at most three earlier neighbors in σ and at most $15 + (2)3 + (2)1 = 22$ earlier neighbors in G^2. Also, each u_i has at most two earlier neighbors in G and at most $(D - 3) + 5$ earlier neighbors in G^2.

Now we must verify that on each incident triangular face v_1 has at most one neighbor that receives charge. If v_1 has two such neighbors on a common 3-face and one is a 2-neighbor, say u_2, then the configuration is reducible by the Basic Reducibility Lemma, since $|N^2(u_2)| \leq 5 + 3$. So suppose that v_1 has two 3-neighbors, u_2 and u_3, on a common 3-face and they both receive charge from v_1. Form G' from G by deleting edge u_2u_3. By minimality, G' has a good vertex ordering σ'.

To get a good vertex ordering σ for G, delete u_2 and u_3 from σ', then append u_2 and u_3. Clearly, each u_i has at most 3 earlier neighbors in the ordering. Also, v_1 gives charge to u_2 only when u_2 has no 12^+-neighbor. Thus, $|N^2(u_2)| \leq 5 + 3 + 11$; similarly for u_3. Thus, the resulting vertex ordering σ is good for G. □

To conclude the paper, we remark that this vertex ordering guaranteed by the Main Theoerm can be constructed recursively in linear time. The basic idea is to find some reducible configuration in amortized constant time. We assume a data structure that stores for each vertex: its degree, a doubly-linked adjacency list in clockwise order, and for each neighbor a pointer to that neighbor. Note that to handle each reducible configuration, we either delete a vertex of low degree or we delete an edge with both endpoints of low degree. Thus, we can preprocess G in linear time to find all such reducible configurations, storing them in some generic "bag" (for example a stack or a queue). Now at each step, we remove some reducible configuration from the bag, recurse on the appropriate smaller graph, and add to the bag any newly created reducible configurations. (The proof of the Main Theorem guarantees that the bag will never be empty.) The first author and Kim give a lengthier explanation of these ideas in Section 6 of [6].

Acknowledgment

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Appendix

In this section, we first collect a few standard graph theory definitions. We conclude with a construction of Dvořák et al. [8] of planar graphs G of girth 6 and maximum degree Δ such that $\chi(G^2) \geq \Delta + 2$ (for each $\Delta \geq 2$).

The *girth* of a graph is the length of its shortest cycle. The degree $d(v)$ of a vertex v is its number of incident edges. The maximum degree in G is denoted Δ. The set of vertices within distance 2 of a vertex v is denoted $N^2(v)$. We write k-vertex (resp. k^+, k^-) for a vertex of degree k (resp. at least k, at most k). We define k-faces analogously.

A *coloring* of a graph G assigns to each vertex a color (typically denoted by a positive integer). A coloring f is *proper* if the endpoints u and v of each edge uv get distinct colors, i.e., $f(u) \neq f(v)$. A graph is *k-colorable* if it has a proper coloring with at most k colors. The *chromatic number* $\chi(G)$ of a graph G is the least k such that G is k-colorable. A *list assignment* L assigns to each vertex v a set of allowable colors $L(v)$. An *L-coloring* is a proper coloring f such that $f(v) \in L(v)$ for every vertex v. A graph G is *k-choosable* if it is L-colorable whenever $|L(v)| = k$ for every $v \in V(G)$. The *list chromatic number $\chi_\ell(G)$ of G* (or *choice number* of G) is the least k such that G is k-choosable.

The game of k-paintability (or online list k-coloring) is played by two players, *Lister* and *Painter*. In each round i, Lister presents to Painter some nonempty list (set) of uncolored vertices. Painter chooses (paints) some subset of them to receive color i. If Lister lists some particular vertex k times and Painter never paints it, then Lister wins. Otherwise Painter wins. The *paint number* $\chi_p(G)$ is the least k such that G is k-paintable.

Now we present a construction of planar graphs G_Δ with maximum degree Δ and girth 6 such that $\chi(G_\Delta^2) \geq \Delta + 2$. The first such construction appeared in Borodin et al. [4]. The construction we present is due to Dvořák et al [8]. We like it because we find it simpler, and the graphs it produces have fewer vertices.

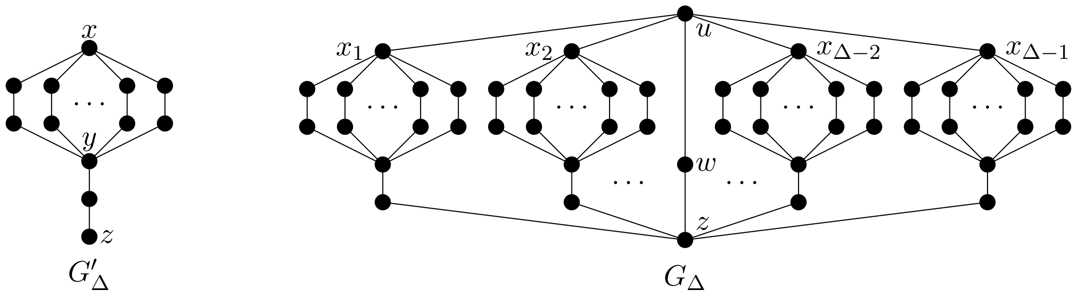

Figure 12: In any $(\Delta+1)$-coloring of the square of G'_Δ, the $(\Delta-1)$-vertex x and the 1-vertex z cannot receive the same color. Because of this, no $(\Delta+1)$-coloring of the square of G_Δ is possible, hence $\chi(G_\Delta^2) \geq \Delta+2$.

The key to the construction is a gadget G'_Δ, show on the left in Figure 12. It consists of two vertices x and y joined by $\Delta-1$ paths of length 3, as well as another path of length 2 incident to vertex y; call the other endpoint of this 2-path z. The key observation is that in any coloring of $(G'_\Delta)^2$ with $\Delta+1$ colors, vertices x and z must receive distinct colors. The reason is that y and all of its neighbors must receive the $\Delta+1$ distinct colors. So z must receive the same color as some neighbor t of y other than its common neighbor with z. This neighbor t will be distance 2 from x, so it cannot recieve the same color as x. To form G_Δ, we take $\Delta-1$ copies of the gadget, identifying vertex z in all of them. Further, we add a new vertex u adjacent to x in each gadget, and we add a new vertex w adjacent to u and z. Now the vertex set $\{u, w, z, x_1, \ldots, x_{\Delta-1}\}$ has size $\Delta+2$ and in a coloring of G^2 each pair of its vertices must receive distinct colors. Thus, $\chi(G_\Delta^2) \geq \Delta+2$.

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