Planar Graphs of Girth at least Five are Square $(\Delta + 2)$-Choosable

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ABSTRACT

We prove a conjecture of Dvořák, Král, Nejedlý, and Škrekovski that planar graphs of girth at least five are square $(\Delta + 2)$-colorable for large enough Δ. In fact, we prove the stronger statement that such graphs are square $(\Delta+2)$-choosable and even square $(\Delta+2)$-paintable.

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1 Introduction

Graph coloring is a central area of research in discrete mathematics. Historically, much work has focused on coloring planar graphs, particularly in an effort to prove the 4 Color Theorem. Since its proof in 1976, research has expanded to numerous related problems. One that has received significant attention is coloring the *square* G^2 of a planar graph G, where $V(G^2) = V(G)$ and $uv \in E(G^2)$ if $\mathrm{dist}_G(u,v) \leq 2$. Wegner [11] conjectured that every planar graph G with maximum degree $\Delta \geq 8$ satisfies $\chi(G^2) \leq \left\lfloor \frac{3\Delta}{2} \right\rfloor + 1$. He also constructed graphs showing that this number of colors may be needed (his construction is a minor variation on that shown in Figure 2, which requires $\left\lfloor \frac{3\Delta}{2} \right\rfloor$ colors). The *girth* of a graph G, denoted $g(G)$, is the length of its shortest cycle. Since Wegner's construction contains many 4-cycles, it is natural to ask about coloring the square of a planar graph G with girth at least 5. First, we need a few more definitions.

A *list assignment* L for a graph G assigns to each vertex $v \in V(G)$ a list of allowable colors $L(v)$. A proper *L-coloring* φ is a proper vertex coloring of G such that $\varphi(v) \in L(v)$ for each $v \in V(G)$. A graph G is *k-choosable* if G has a proper L-coloring from each list assignment L with $|L(v)| = k$ for each $v \in V(G)$. The *list chromatic number* $\chi_\ell(G)$ is the minimum k such that G is k-choosable. Finally, a graph G is *square k-choosable* if G^2 is k-choosable.

Conjecture 1.1 (Wang and Lih [10]) *For every $k \geq 5$ there exists Δ_k such that if G is a planar graph with girth at least k and $\Delta \geq \Delta_k$, then $\chi(G^2) = \Delta + 1$.*

Borodin et al. [4] proved the Wang–Lih Conjecture for $k \geq 7$. Specifically, they showed that $\chi(G^2) = \Delta + 1$ whenever G is a planar graph with girth at least 7 and $\Delta \geq 30$. In contrast, for each integer D at least 2, they constructed a planar graph G_D with girth 6 and $\Delta = D$ such that $\chi(G_D^2) \geq \Delta + 2$.

In 2008, Dvorak et al. [8] showed that for $k = 6$ the Wang–Lih Conjecture fails only by 1. More precisely, let G be a planar graph with girth at least 6. They showed that if $\Delta \geq 8821$, then $\chi(G^2) \leq \Delta + 2$. Borodin and Ivanova strengthened this result: in 2009 they showed [5] that $\Delta \geq 18$ implies $\chi(G^2) \leq \Delta + 2$ (and also [6] that $\Delta \geq 36$ implies $\chi_\ell(G^2) \leq \Delta + 2$). Dvorak et al. conjectured that a similar result holds for girth 5.

Conjecture 1.2 (Dvořák, Král, Nejedlý, and Škrekovski [8]) *There exists Δ_0 such that if G is a planar graph with girth at least 5 and $\Delta \geq \Delta_0$ then $\chi(G^2) \leq \Delta + 2$.*

Our main result verifies Conjecture 1.2, even for list-coloring. Further, in Section 4, we extend the result to paintability, also called online list-coloring.

Theorem 1.3 *There exists Δ_0 such that if G is a planar graph with girth at least five and $\Delta(G) \geq \Delta_0$, then G is square $(\Delta(G)+2)$-choosable. In particular, we can let $\Delta_0 = 1,640^2 + 1 = 2,689,601$.*

The number of colors in Theorem 1.3 is optimal, as shown by the family of graphs introduced in [4] and depicted in Figure 1. The vertex u and its p neighbors together require $p+1$ colors; since v is at distance 2 from each of them, in total we need $p+2$ distinct colors.

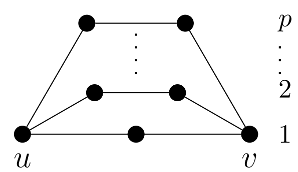

Figure 1: A graph G_p with girth 5, $\Delta(G_p) = p$, and $\chi^2(G_p) = \Delta(G_p) + 2$.

The girth assumption is tight as well, due to a construction directly inspired from Shannon's triangle (see Figure 2). When coloring the square, all $3p$ degree 2 vertices need distinct colors, since each pair has a common neighbor.

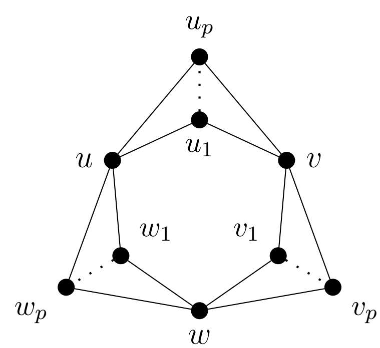

Figure 2: A graph G_p with girth 4, $\Delta(G_p) = 2p$, and $\chi^2(G_p) = 3p$.

Theorem 1.3 is also optimal in another sense. But before we can explain it, we must introduce a more refined measure of a graph's sparsity: its maximum average degree. The average degree of a graph G, denoted $\mathrm{ad}(G)$, is $\frac{\sum_{v \in V} d(v)}{|V|} = \frac{2|E|}{|V|}$. The *maximum average degree* of G, denoted $\mathrm{mad}(G)$, is the maximum of $\mathrm{ad}(H)$ over every subgraph H of G. For planar graphs, Euler's formula links girth and maximum average degree.

Lemma 1.4 (Folklore) *For every planar graph G, $(\mathrm{mad}(G) - 2)(g(G) - 2) < 4$.*

Note that every planar graph G with $g(G) \geq 7$ satisfies $\mathrm{mad}(G) < \frac{14}{5}$. It was proved [3] that Conjecture 1.1 is true not only for planar graphs with $g \geq 7$, but also for all graphs with $\mathrm{mad} < \frac{14}{5}$ (in fact even for all graphs with $\mathrm{mad} < 3 - \epsilon$, for any fixed $\epsilon > 0$). Similarly, every planar graph G with $g(G) \geq 6$ satisfies $\mathrm{mad}(G) < 3$. The theorem of Borodin and Ivanova [5] for planar graphs with $g(G) \geq 6$ was also strengthened in that setting: every graph G with mad$(G) < 3$ and $\Delta(G) \geq 17$ satisfies $\chi^2_\ell(G) \leq \Delta(G) + 2$ [2]. These results suggested that perhaps sparsity was the single decisive characteristic when square list coloring planar graphs of high girth. However, as we show below, Theorem 1.3 cannot be strengthened to require only mad$(G) < \frac{10}{3}$ (rather than planar, with girth 5).

Charpentier [7] generalized the family of graphs presented in Figure 1 to obtain for each $C \in \mathbb{Z}^+$ a family of graphs with maximum average degree less than $\frac{4C+2}{C+1}$, with unbounded maximum degree, and whose squares have chromatic number $\Delta + C + 1$. For $C = 2$ the construction, shown in Figure 3, yields a family of graphs with arbitrarily large maximum degree, maximum average degree less than $\frac{10}{3}$, and whose squares are not $(\Delta + 2)$-colorable. In the square, all $p + 4$ vertices $u, v_1, \ldots, v_p, w, x_1, x_2$ must receive distinct colors, since they are pairwise adjacent. The maximum average degree of G_p is reached on the graph G_p itself. We can argue by induction that mad$(G_p) < \frac{10}{3}$: note that mad$(G_0) = \frac{3}{2} < \frac{10}{3}$ and that G_{p+1} is built from G_p by adding precisely 3 vertices and 5 edges (if $\frac{2(a+5)}{b+3} \geq \frac{10}{3}$ then necessarily $\frac{2a}{b} \geq \frac{10}{3}$).

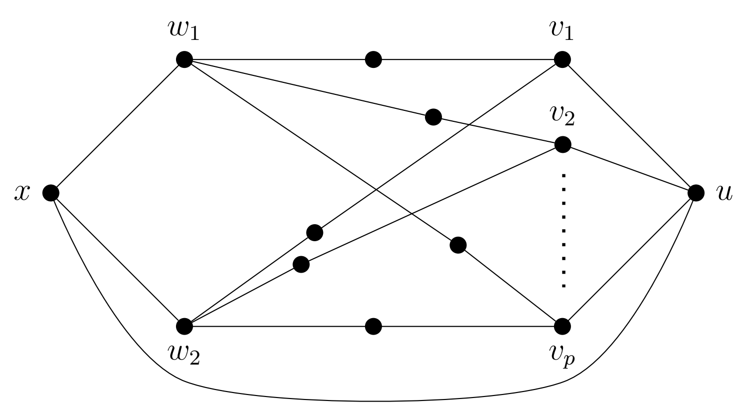

Figure 3: A graph G_p with $\Delta(G_p) = p + 1$, mad$(G_p) = \frac{10p+6}{3p+4} < \frac{10}{3}$ and $\chi^2(G_p) = p + 4$.

2 Definitions and notation

Most of our definitions and notation are standard; for reference, though, we collect them below. Let G be a multigraph with no loops. The *neighborhood* of a vertex v in G, denoted $N(v)$, is the set of neighbors of v, i.e., $N(v) = \{u : uv \in E(G)\}$. The *closed neighborhood*, denoted $N[v]$, is defined by $N[v] = N(v) \cup \{v\}$. For a vertex set S, let $N(S) = \cup_{v \in S} N(v)$ and $N[S] = \cup_{v \in S} N[v]$. For a multigraph G or digraph D and a subset S of its vertices, $G[S]$ or $D[S]$ is the subgraph induced by S. The degree of a vertex v is $d(v)$, and if H is a subgraph of G, then $d_H(v)$ is the number of edge of H incident to v. The maximum and minimum degrees of G are $\Delta(G)$ and $\delta(G)$, respectively; when G is clear from context, we may write simply Δ or δ. The *girth* of G is the length of its shortest cycle.

A multigraph G is *planar* if it can be drawn in the plane with no crossings. A *plane map* is a planar embedding of a planar multigraph such that each face has length at least 3. The *underlying map* G' of a plane embedding of a plane multigraph G is formed from the embedding of G by deleting (the minimum number of) edges to remove all faces of length two. Suppose that $d_G(v) = 2$ and that $N(v) = \{u, w\}$. To *suppress* v, delete v and add an edge uw. The degree of a vertex v in a multigraph G is the number of incident edges. So, in particular, we may have $d_G(v) > |N_G(v)|$. For vertices u and v with $u \in N(v)$, the *multiplicity* of the edge uv is the number of edges with u and v as their two endpoints.

3 Proof of Main Theorem

Let $\Delta_0 = 2,689,601$ and let $k \geq \Delta_0$. To prove Theorem 1.3, it suffices to prove that every plane graph G of girth at least five with $\Delta(G) \leq k$ is square $(k+2)$-choosable. Assume, for a contradiction, that this does not hold, and consider a counterexample $G = (V, E)$ with the fewest possible edges. Let L be a list assignment of $(k+2)$ colors to each vertex of G such that G^2 has no L-coloring. We reach a contradiction, by showing that G must contain some subgraph H such that every L-coloring of $(G \setminus E(H))^2$ can be extended to an L-coloring of G^2; such an H is *reducible*. An unusual feature of our proof is that we don't use the discharging method. Instead, we use only the fact that every planar map has a vertex of degree at most 5.

A vertex v of G is *big* if $\deg(v) \geq \sqrt{k}$, and otherwise v is *small*. The sets of big and small vertices of G are denoted, respectively, by B and S. Further, let $S_i = \{v \in S : |N_G(v) \cap B| = i\}$, i.e., small vertices with exactly i big neighbors. By Lemma 1.4, $\text{mad}(G) < \frac{10}{3}$, so $|E(G)| < 5|V(G)|/3$. Thus, only a tiny fraction of $V(G)$ can be big vertices. Likewise, by planarity $\bigcup_{i\geq 3} S_i$ has size linear in the number of big vertices, again a very small fraction of $|V(G)|$. Hence, the vast majority of $V(G)$ is the subset $\bigcup_{i=0}^{2} S_i$. We show that S_0, the set of small vertices with only small neighbors, induces an independent set. Thus, we can decompose the planar embedding into regions, each defined by a pair of big vertices. We prove that any region with many vertices is reducible. To complete the proof, we show that some big vertex v is adjacent to few regions (here we use that every planar map has a vertex of degree at most 5), so v must be adjacent to a region with many vertices.

We begin with a few simple observations about G.

Lemma 3.1 *Graph G is connected and $\delta(G) \geq 2$.*

Proof. If G is not connected, then one of its components is a smaller counterexample, contradicting the minimality of G.

If G contains a vertex u of degree 1, color $G \setminus \{u\}$ by minimality, and extend the coloring to G as follows. Vertex u has exactly one neighbor v, whose degree is at most k, by assumption on G. So, $|N_{G^2}(u)| = |N(v) \setminus \{u\}| + |\{v\}| \leq k$. Since $|L(u)| \geq k+2$, some color c in $L(u)$ is available to use on u. Coloring u with c gives an L-coloring of G^2, a contradiction. □

A key observation is the next lemma, which shows that in a minimal counterexample at least one endpoint of every edge is either big or adjacent to a big vertex.

Lemma 3.2 *For every edge uv of G, either $u \in N[B]$ or $v \in N[B]$.*

Proof. Assume, for a contradiction, that there is an edge $uv \in E(G)$ such that $u, v \notin N[B]$. In other words, u and v and all their neighbors have degree at most $\sqrt{k}$. By the minimality of G, there exists an L-coloring φ of $(G - uv)^2$. Now we recolor both u and v to obtain an L-coloring of G^2. Since u has at most $\sqrt{k}$ neighbors in G, each of which has degree at most $\sqrt{k}$, u has at most k neighbors in G^2. Thus, at most k colors appear on $N_{G^2}(u)$. Since $|L(u)| = k+2$, at least two colors remain available for u, counting its own color. Similarly, v has at least two available colors. Thus, we may extend φ to u and v to obtain a proper L-coloring of G^2, a contradiction. □

The next lemma extends Lemma 3.2, by showing that if both endpoints of an edge have degree two then both endpoints are adjacent to big vertices.

Lemma 3.3 *If u and v are adjacent vertices of degree two, then $u \in N(B)$ and $v \in N(B)$.*

Proof. Suppose not. Let u and v be adjacent vertices of degree 2 such that $v \notin N(B)$. Let w be the neighbor of v distinct from u. By the minimality of G, there exists an L-coloring φ of $(G \setminus \{u, v\})^2$. Since u has at most $k+1$ neighbors in G^2 that are already colored, we can extend φ to u. Now $|N_{G^2}(v)| = |N(w) \setminus \{v\}| + |\{w, u\}| + |N(u) \setminus \{v\}| \leq d(w) + 2$. Since $w \notin B$, $d(w) < \sqrt{k}$, so we can extend φ to v, which yields an L-coloring of G^2, a contradiction. □

The intuition behind much of the proof is that small vertices with only small neighbors can always be colored last. The next key ingredient in formalizing this intuition is a new plane multigraph. Let G' denote the plane multigraph obtained from G by first suppressing vertices of degree 2 in $S \setminus N(B)$ and then contracting each edge with one endpoint in each of S_1 and B. Note that there is a natural bijection between the faces of G' and those of G. We will use Lemmas 3.2 and 3.3 to prove structural properties of G'.

Since big vertices of G are not identified with each other in the construction of G', we also let B denote the vertices of G' that contain a big vertex of G. Let $S' = V(G') \setminus B$. Note that neither suppressing nor contracting decreases the degree of a vertex in B; thus, we conclude the following.

Observation 3.4 *For every vertex v in B, we have $d_{G'}(v) \geq d_G(v)$.*

Let G'' denote the underlying map of G'. We will next show that there is a big vertex (in G and G') whose degree in G'' is small; in other words, v has many edges in G' to the same neighbor. But first we need the following general lemma about plane maps with certain properties, the hypotheses of which (as we will show) are satisfied by G''.

Lemma 3.5 *Let H be a plane map and A, C, and D be disjoint vertex sets such that $V(H) = A \cup C \cup D$, every $v \in C$ satisfies $|N(v) \cap D| \geq 2$, and for all $v_1, v_2 \in C$ such that $|N(v_1) \cap D| = |N(v_2) \cap D| = 2$, it holds that $N(v_1) \cap D \neq N(v_2) \cap D$. If A is an independent set and $d(v) \geq 3$ for all $v \in A$, then there exists $u \in D$ with $d_{H[D]}(u) \leq 10$ and $d_H(u) \leq 40$.*

Before proving this lemma, we show how we apply it.

Lemma 3.6 *There exists $v \in B$ with $d_{G''[B]}(v) \leq 10$ and $d_{G''}(v) \leq 40$. Further, there exists $u \in V(G')$ (recall that $V(G'') = V(G')$) such that at least $\frac{\sqrt{k}}{40} - 1$ consecutive faces of length two in G' have boundary (u, v).*

Proof. To prove the first statement, apply Lemma 3.5 to G'' with $D = B$, $C = \bigcup_{i \geq 2} S_i$, and $A = \{v \in S_0 : d(v) \geq 3\}$. (Recall that when forming G', we suppressed all $w \in S_0$ with $d(w) = 2$ and we contracted into B all $w \in S_1$.) Note that A is an independent set, by Lemma 3.2. Now suppose there exist $v_1, v_2 \in C$ such that $|N_{G''}(v_1) \cap B| = |N_{G''}(v_2) \cap B| = 2$ and $N_{G''}(v_1) \cap B = N_{G''}(v_2) \cap B$; say $N_{G''}(v_1) \cap B = \{b_1, b_2\}$. When G'' was formed from G, vertices v_1 and v_2 may have gained neighbors in B, but they did not lose neighbors. Since $v_1, v_2 \in \bigcup_{i \geq 2} S_i$, each already had 2 neighbors in B in G; these must be b_1 and b_2. Hence, G contains the 4-cycle $b_1 v_1 b_2 v_2$, contradicting the assumption that G has girth at least 5.

Now consider the second statement. Since $d_G(v) \geq \sqrt{k}$, also $d_{G'}(v) \geq \sqrt{k}$. Since $d_{G''}(v) \leq 40$, by Pigeonhole some edge uv in G' has multiplicity at least $\frac{\sqrt{k}}{40}$; furthermore, these copies of the edge uv are embedded in G' to create at least $\frac{\sqrt{k}}{40} - 1$ consecutive 2-faces. (It is possible that multiple copies of uv, say t copies, are embedded in G', and thus in G'', such that they don't create a 2-face. However, now the t copies of uv contribute t to $d_{G''}(v)$, so they do not impede this Pigeonhole argument.) □

Now we prove Lemma 3.5.

Proof. It is convenient to assume that no vertex of $C \cup D$ is a cut-vertex. So we strengthen the lemma to prove that G has two such vertices u_1 and u_2. We also allow the unbounded face to have length 2. We use induction on $|D|$. Suppose, to the contrary, that some vertex $v \in C \cup D$ is a cut-vertex and let V_1 and V_2 be the vertices of two components of $G - v$. (We can assume $|V_1 \cap D| \geq 2$ and $|V_2 \cap D| \geq 2$, since the case otherwise is easy.) If $v \in C$, then move v to D. By hypothesis, $G[V_1 \cup \{v\}]$ has two such vertices, and one of them, call it u_1, is not v. Similarly, by hypothesis, $G[V_2 \cup \{v\}]$ has two such vertices, and one of them, call it u_2, is not v. Thus, G has the desired vertices, u_1 and u_2. So we can assume that no vertex of $C \cup D$ is a cut-vertex.

Informally, we want to argue that for fixed $|D|$ the worst case H is formed via the following successive steps: (i) triangulating D, (ii) adding a vertex of C inside every triangular face (adjacent to every vertex on that triangle), (iii) replacing every edge uv with both endpoints in D by two parallel edges with endpoints u and v, as well as a vertex in C of degree 2 with neighborhood $\{u, v\}$ between the parallel edges, and (iv) adding in each face f (which must be a triangle) a vertex in A, adjacent to every vertex on f. This immediately yields that for every $u \in D$, if u has p incident edges in the initial triangulation on D, then u has at most $2p$ incident edges with both endpoints in D, at most $2p$ incident edges with an endpoint in C, and at most $4p$ incident edges with an endpoint in A. By Euler's formula (which still holds for plane maps), at least two vertices in D have $p \leq 5$; hence, the desired conclusion holds.

For a given choice of D, take an edge-maximal plane multigraph $H = (V, E, F)$, with $V = A \cup C \cup D$. More precisely, choose H to maximize the number of edges incident to D and, subject to that, to maximize the number of edges incident to $A \cup C$. In particular, adding any edge or increasing $|A \cup C|$ must break one of the hypotheses. Let $\mathcal{M}$ be a given embedding of H. Now for every vertex $u \in V(H)$ and every pair of vertices v_1, v_2 that appear consecutively in the cyclic order of the neighborhood of u, we can assume that v_1 and v_2 are adjacent, and moreover that (u, v_1, v_2) induces a face of length 3. (If some face of length at least 4 contains two vertices of A, then we can add a chord while maintaining that A is independent. Here, we use that no vertex of $C \cup D$ is a cut-vertex, to ensure that this chord is not a loop.) Note that every face f contains a vertex of A; otherwise, we can add a new vertex to A and make it adjacent to every vertex on f.

We claim that every vertex in A is of degree 3. Indeed, assume there is a vertex u in A of degree at least 4, and denote its consecutive neighbors in a cyclic order by $v_1, v_2, \ldots, v_p$ ($p \geq 4$). We add an edge v_1v_3 and replace u by two new vertices u_1 and u_2 in A, where u_1 is adjacent to v_1, v_2, v_3 and u_2 to $v_3, \ldots, v_p, v_1$. (Note also that every face is a triangle and contains a vertex in A.) Similarly, every vertex in C has degree at most 3 in D. We also claim that no vertex $v \in A$ has all three neighbors in D. Indeed, in that case we could move v to C, and add a new vertex w to A in one of the triangular faces f incident to v (making w adjacent to every vertex on face f).

Finally, we claim that every face contains an element of D, and that C induces an independent set. Assume, for a contradiction, that there are faces containing only elements of $A \cup C$, and take f to be one adjacent to a face f' containing an element of D. Since every face in $\mathcal{M}$ is a triangle, this yields a 4-cycle (u, v_1, v_2, v_3) where $u \in D$, $v_1, v_2, v_3 \in A \cup C$ and each of (v_3, u, v_1) and (v_1, v_2, v_3) induces a face. If $v_1, v_3 \in C$, then we can delete the edge v_1v_3 and add an edge v_2u in the resulting face of length 4, which contradicts the maximality of H. So assume, by symmetry, that $v_1 \in A$. Since A induces an independent set, $v_2, v_3 \in C$. Consider the other face $f'' = (v_2, v_3, v_4)$ incident to v_2v_3. Note that $v_4 \neq v_1$, since $v_1 \in A$, so $d(v_1) = 3$. Also, $v_4 \neq u$, since $v_4 \in A$ and $u \in D$. Let u' denote the third neighbor of v_4. Now we delete the edges v_1v_3 and v_2v_3 and add the two edges uv_4 and $u'v_1$, which contradicts the maximality of H. If $u' = u$, then in the modified graph u is a cut-vertex. Now we can proceed by induction, as in the first paragraph of the proof (possibly after modifying the embedding so that the single face of length 2 is the outer face). Consequently, every face contains an element of D. If there is an edge v_1v_2 between two vertices in C, then the two incident triangles must each contain a vertex in D, respectively u_1 and u_2; now we delete the edge v_1v_2 and add an edge u_1u_2, which contradicts the maximality of H. Thus, C induces a stable set.

Let H' denote $H[C \cup D]$, the plane multigraph induced by the vertices of C and D. In fact, H' is a plane map, since deleting the vertices of A does not create any face of length two, and every face of $\mathcal{M}[H']$ is a triangle given that every face of $\mathcal{M}$ is a triangle and every vertex in A has degree exactly 3 in H. Since every face in $\mathcal{M}$ contains a vertex in A, there is in fact a bijection between A and the faces of $\mathcal{M}[H']$, so for every vertex $u \in C \cup D$ we have $d_{H'}(u) = \frac{d_H(u)}{2}$. Note that every face of $\mathcal{M}[H']$ is a triangle and contains a vertex in C, since no vertex in A has all three neighbors in D.

Let H'' denote $H[D]$, the plane multigraph induced in H by the vertices of D. Again, there is in fact a bijection between C and the faces of $\mathcal{M}[H'']$, so for every vertex $u \in D$ we have $d_{H''}(u) = \frac{d_{H'}(u)}{2}$. Note however that by the hypotheses on H, no two faces of degree 2 are adjacent. Therefore, by deleting exactly one edge in every face of degree 2, we obtain a plane map H''' where for every vertex $u \in D$ we have $d_{H'''}(u) \geq \frac{d_{H''}(u)}{2}$. Since H''' is a plane map, Euler's formula applies, so there are $v_1, v_2 \in D$ such that for each $i \in \{1, 2\}$ we have $d_{H'''}(u_i) \leq 5$. As a consequence, $d_D(u_i) = d_{H''}(u_i) \leq 10$ and $d_H(u_i) \leq 8d_{H'''}(u_i) \leq 40$. □

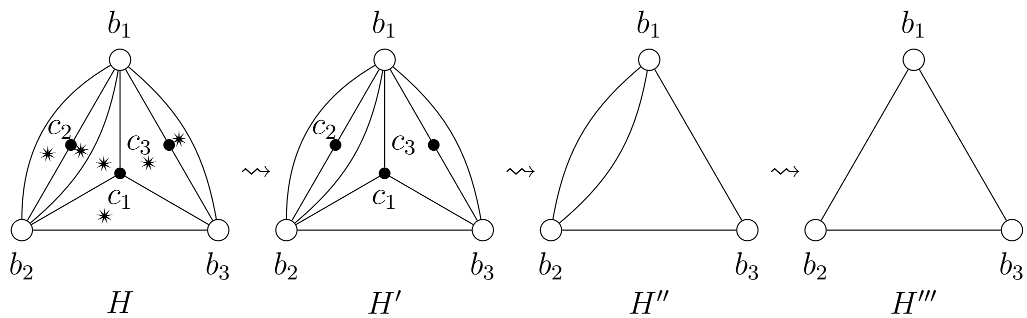

Figure 4: The evolution, in the proof of Lemma 3.5, from H to H''' of a subgraph of H. Here $b_1, b_2, b_3 \in D$, $c_1, c_2, c_3 \in C$. Each $*$ represents a vertex in A adjacent to all three vertices incident to the face. (Black vertices have all incident edges drawn, but white vertices may have more incident edges.)

Recall that $S' = V(G') \setminus B$.

Lemma 3.7 *No vertex in S' is incident to 3 or more consecutive faces of length 2 in G'.*

Proof. Assume, for a contradiction, that there is an edge uv in G', with $u \in S'$, such that at least 3 consecutive faces have boundary (u, v). First consider the case where $v \in S'$. In the construction of G' from G, an edge is added between u and v only when there is a vertex of degree 2 adjacent to both u and v that is suppressed. Hence, regardless of whether uv belongs to $E(G)$ or is formed from the suppression of a vertex of degree 2 adjacent to u and v, there exists a cycle in G of length at most 4, contradicting that G has girth at least five.

So we may assume that $v \in B$. Let $T_1 = S_1 \cap N(v)$, that is the set of small neighbors of v with exactly one big neighbor (which must be v). In the construction of G', an edge is added between u and v only if either there is a neighbor of u in T_1, or if there is a vertex of degree 2 adjacent to both u and to a vertex in T_1. Let U_1 denote the set of vertices of degree 2 that are adjacent to u and also to a vertex in T_1. Note that each copy of uv in G' corresponds to a path of length at most 3 in G with all vertices in $\{u, v\} \cup U_1 \cup T_1$. Thus, $uv \notin E(G)$, since this would create a 4-cycle in G, contradicting that G has girth at least 5.

Since uv has multiplicity at least four, and the copies of uv form 3 consecutive faces of degree 2 in G', we know $4 \leq |U_1| + |N(u) \cap T_1|$; further, there exist four vertices, $w_1, \ldots, w_4$, in $T_1 \cup U_1$ that are consecutive in the cyclic neighborhood of u in G. Since G has girth at least five, $|N(u) \cap N(v)| \leq 1$, so $|N(u) \cap T_1| \leq 1$. Thus, at least one of w_2 and w_3 is in U_1; by symmetry, assume $w_2 \in U_1$. Let x_2 be the neighbor of w_2 in G distinct from u. Note that $x_2 \in T_1$. Since G has girth at least five, x_2 is neither adjacent to a neighbor of v nor to a neighbor of a neighbor of v. Regardless of whether w_3 belongs to U_1 or T_1, it follows by planarity that w_2 and v are the only neighbors of x_2. Therefore $d(x_2) = 2$, which is a contradiction to Lemma 3.3, since $w_2 \notin N(B)$. ☐

Now we use Lemma 3.7 to strengthen the final conclusion of Lemma 3.6.

Corollary 3.8 *There exist $u, v \in B \cap V(G')$ such that there are at least $\frac{\sqrt{k}}{10} - 13$ consecutive faces of degree two with boundary (u, v).*

Proof. Let v be as in Lemma 3.6. By Lemma 3.7, each small neighbor of v in G'' accounts for at most 3 consecutive edges in G incident to v. Thus, the remaining $|N_{G''}(v) \cap B|$ big neighbors of v account for at least $d_G(v) - 3(40)$ edges. Since v is big, $d_G(v) \geq \sqrt{k}$. By Pigeonhole, some big neighbor of v, say u, accounts for at least $(\sqrt{k} - 120)/d_{G''[B]}(v)$ consecutive edges incident to v in G; and the number of consecutive faces with boundary (u, v) is one less. Since $d_{G''[B]}(v) \leq 10$, some big neighbor u shares with v at least $\frac{\sqrt{k}}{10} - 13$ consecutive faces of length 2. ☐

If $u, v \in B \cap V(G')$ are such that at least r consecutive faces $f'_1, \ldots, f'_r$ of G' have boundary (u, v), then these faces are an *r-region* R' of G'. Analogously, an *r-region* R of G is a set of faces which contract to an r-region R' in G'. We define $V(R)$ as $(\bigcup_{i=1}^{r} V(f_i)) \setminus \{b_1, b_2\}$.

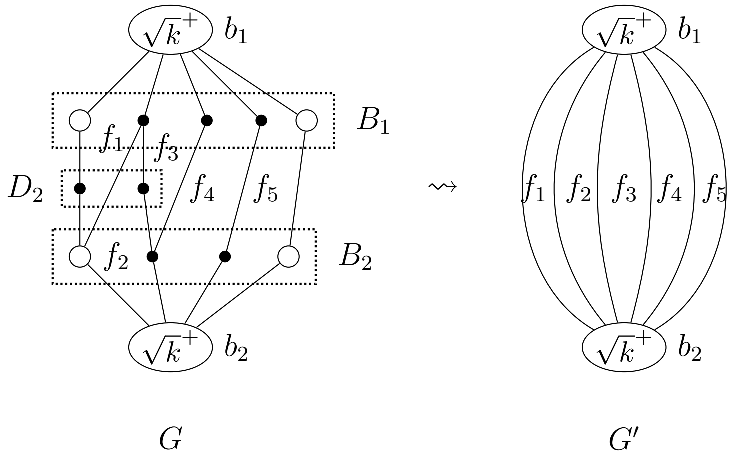

Figure 5: A 5-region in G and the corresponding 5-region in G'.

Observation 3.9 *If R is an r-region of G, then $V(R) = B_1 \cup B_2 \cup D_2$, where B_1, B_2, and D_2 are disjoint vertex sets such that $B_1 \subseteq N(b_1)$ and $B_2 \subseteq N(b_2)$ for some $b_1, b_2 \in B$; further, D_2 is an independent set of degree two vertices, each of which has one neighbor in B_1 and the other neighbor in B_2.*

Proof. Let R be an r-region of G. By definition, there exist $b_1, b_2 \in B \cap V(G')$ such that the r-region R' consists of at least r consecutive faces in G', each with boundary (b_1, b_2). Recall that G' is formed from G by suppressing the degree 2 vertices in $S \setminus N(B)$ and contracting each edge joining S_1 and B. By Lemma 3.3, these suppressed degree 2 vertices form an independent set. Thus, each copy of b_1b_2 in R in G' corresponds to a path of length 3 or 4 joining b_1 and b_2 in G. Each such path P must contain a vertex from each of B_1 and B_2. If P contains another vertex w, then w must be suppressed in forming G', so w must be a degree 2 vertex with a neighbor in each of B_1 and B_2. This proves the observation. □

Hereafter, we use B_1, B_2, D_2, b_1, b_2, and $V(R)$ as defined in the previous observation.

Lemma 3.10 *If R is an r-region of G, then B_1 and B_2 are independent sets, and each $v \in B_1 \cup B_2$ satisfies $|N(v) \cap V(R)| \leq 3$.*

Proof. The fact that B_1 and B_2 are independent sets follows from the assumption that G has girth at least five. Now choose $v \in B_1 \cup B_2$ and suppose, for a contradiction, that $|N(v) \cap V(R)| \geq 4$. Without loss of generality, we may assume that $v \in B_1$. Recall that $V(R) \subseteq B_1 \cup B_2 \cup D_2$. Since B_1 is independent, $|N(v) \cap B_1| = 0$; thus $|N(v) \cap (B_2 \cup D_2)| \geq 4$. Since G has girth at least five, $|N(v) \cap B_2| \leq 1$, so $|N(v) \cap D_2| \geq 3$. Hence, by planarity, there exists $u \in N(v) \cap D_2$ such that if w is the other neighbor of u, then $vw \in E(G')$ (actually v gets contracted into b_1 and w gets contracted into b_2 when forming G') and vw is incident with two faces, each of length two, in region R'. Since G has girth at least five, it follows that w has degree two in G. But now u and w are adjacent vertices of degree two, yet $u \notin N(B)$, which contradicts Lemma 3.3. $\square$

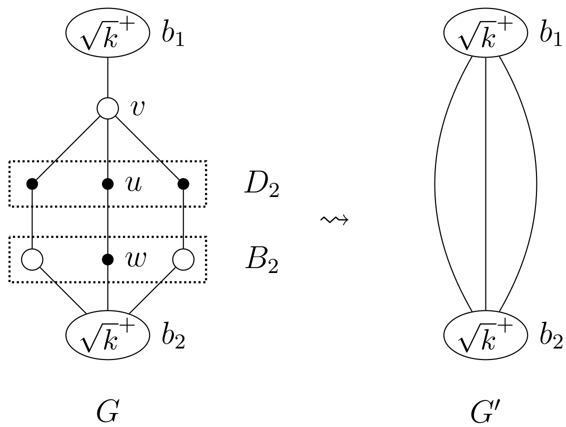

Figure 6: An illustration of the proof of Lemma 3.10.

To complete the proof of Theorem 1.3, we need one more reducible configuration; in Lemma 3.13, we show that an r-region is reducible, if $r \geq 152$. Before that, we need two lemmas about list-coloring. The first played a key role in Galvin's proof [9] that $\chi'_\ell(G) = \Delta(G)$ for every bipartite graph G (here χ'_ℓ denotes the edge list chromatic number).

A *kernel* in a digraph D is an independent set F of vertices such that each vertex in $V(D) \setminus F$ has an out-neighbor in F. A digraph D is *kernel-perfect* if for every $A \subseteq V(D)$, the digraph $D[A]$ has a kernel. To prove our next result, we will need the following lemma of Bondy, Boppana, and Siegel (see [1, p. 129] and [9, p. 155]). For completeness, we include an easy proof.

Lemma 3.11 *Let D be a kernel-perfect digraph with underlying graph G. If L is a list-assignment of $V(G)$ such that for all $v \in V(G)$,*

$$|L(v)| \geq d^+(v) + 1,$$

then G is L-colorable.

Proof. We use induction on $|V(G)|$. Choose some color $c \in \cup_{v \in V(G)} L(v)$. Let A_c be the set of vertices with color c in their lists. By assumption, $D[A_c]$ contains a kernel, F_c. Use color c on each vertex of F_c. Now let $D' = D \setminus F_c$ and $L'(v) = L(v) - c$ for each $v \in V(D')$. By induction, the remaining uncolored digraph D' can be colored from its lists L'; we must only check that D' and L' satisfy the hypothesis of the lemma. Since D is kernel-perfect, so is D'. Further, each vertex of D' lost at most one color from its list (namely, c). More precisely, each vertex of $A_c \setminus F_c$ lost one color from its list and each other vertex lost no colors. Fortunately, since F_c is a kernel for A_c, we get $d^+_{D'}(v) \leq d^+_D(v) - 1$ for each $v \in A_c \setminus F_c$. Thus, $|L'(v)| \geq d^+_{D'}(v) + 1$, for every $v \in V(D')$, as desired. $\square$

We now use Lemma 3.11 to prove the following lemma, which we will use to show that large regions are reducible for square $(\Delta+2)$-choosability.

Lemma 3.12 *Let H be covered by two disjoint cliques B_1 and B_2, L be a list-assignment for $V(H)$, and $S_1 \subseteq B_1$ and $S_2 \subseteq B_2$ be such that*

- *if $v \in B_i$, then $|N(v) \cap V(B_{3-i})| \leq 3$,*
- *if $v \in B_i \setminus S_i$, then $|L(v)| \geq |B_i|$,*
- *if $v \in S_i$, then $|L(v)| \geq |B_i| - 4$.*

Now if $|B_1| \geq 42$, $|B_2| \geq 42$, $|S_1| \leq 8$, and $|S_2| \leq 8$, then H is L-colorable.

Proof. We construct a kernel-perfect orientation D of H satisfying Lemma 3.11 as follows. Let $x_1, x_2, \ldots, x_{|B_1|}$ be an ordering of the vertices of B_1 and $y_1, y_2, \ldots, y_{|B_2|}$ be an ordering of the vertices of B_2 such that

- $x_i \in S_1$ iff $1 \leq i \leq |S_1|$, and
- $y_i \in S_2$ iff $1 \leq i \leq |S_2|$, and
- $N_{B_2}(x_a) \cap N_{B_2}(N_{B_1}(y_b)) = \emptyset$ for each a and b such that $|B_1| - 2 \leq a \leq |B_1|$ and $|B_2| - 2 \leq b \leq |B_2|$.

It is helpful to restate the third condition in words: there is no path of length 1 or 3 that starts at one of the final 3 vertices in B_1, ends at one of the final 3 vertices in B_2, and alternates between B_1 and B_2. We claim that such an ordering exists. To see this, let the vertices of S_1 be $x_1, \ldots, x_{|S_1|}$ in any order and similarly for S_2. Now it suffices to ensure the third condition holds. Note that $|B_1| - 3|S_2| \geq 18$. Suppose there exists $u \in B_1 \setminus N(S_2)$ with $d_{B_2}(u) = 3$. Choose $N_{B_2}(u)$ to be the three final vertices of B_2, and call this set Z.

Now $|N_{B_1}(Z) \setminus \{u\}| \leq 6$, so $|N_{B_2}(N_{B_1}(Z)) \setminus Z| \leq 6(2) = 12$ and $|N_{B_1}(N_{B_2}(N_{B_1}(Z)) \setminus Z)| \leq 12(2) + 6 = 30$. Since $|B_1| - |S_1| - 30 - |\{u\}| \geq 3$, we can choose the desired 3 final vertices of B_1; call this set W. If no such u exists, then there exist 3 vertices $v_1, v_2, v_3 \in B_1 \setminus N(S_2)$ such that $d_{B_2}(v_i) \leq 2$ for all $i \in \{1, 2, 3\}$. Now swap the roles of B_1 and B_2 and let $Z = \{v_1, v_2, v_3\}$. The analysis is essentially the same, except that now we have no vertex u. This proves the claim that such an ordering exists.

Let D be obtained from G by directing the edges of G as follows. For each edge with both endpoints in B_1 or both endpoints in B_2, direct the edge from the vertex with higher index to the vertex with lower index. For each edge between B_1 and B_2, direct the edge in both directions, unless one endpoint is among the final three vertices of B_1 or B_2; in that case, only direct the edge into the vertex among the final three (recall that no edge has one

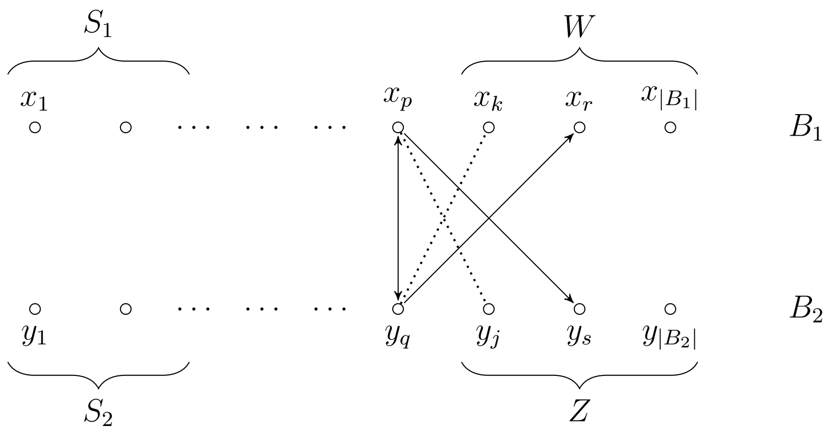

Figure 7: The proof of Lemma 3.12, constructing the orientation D of H, shows that this situation cannot occur, due to our choices of W and Z.

endpoint among the final three vertices of B_1 and the other endpoint among the final three vertices of B_2).

We claim that D is a kernel-perfect orientation. Let $A \subseteq V(H)$. Let $p = \min\{i : x_i \in A\}$ and $q = \min\{j : y_j \in A\}$. If $A \cap V(B_2) = \emptyset$, then x_p is a kernel of A as desired. Similarly if $A \cap V(B_1) = \emptyset$, then y_q is a kernel of A as desired. So we may assume that $A \cap V(B_1) \neq \emptyset$ and $A \cap V(B_2) \neq \emptyset$. If $x_p y_q \notin E(H)$, then $\{x_p, y_q\}$ is a kernel of A as desired. So we may suppose that $x_p y_q \in E(H)$.

Let $r = \min\{k : x_k \in A, x_k \notin N(y_q)\}$ and $s = \min\{\ell : y_\ell \in A, y_\ell \notin N(x_p)\}$. Now $\{x_p, y_s\}$ is a kernel of A, unless there exists j with $q \leq j < s$ such that $x_p y_j$ is either not an edge of H or is only directed from x_p to y_j. Given the choice of s, it must be that $x_p y_j$ is only directed from x_p to y_j. Thus, we conclude that y_j is among the final 3 vertices of B_2. Now, we instead take as our kernel $\{y_q, x_r\}$. This is a kernel unless there exists k with $p \leq k < r$ such that either $x_k y_q$ is not an edge or it is only directed from y_q to x_k. Given our choice of s, we know that $x_k y_q$ is an edge. But if $x_k y_q$ is only directed from y_q to x_k, then x_k is among the final 3 vertices of B_1. However, this is impossible, since now the path $x_k y_q x_p y_j$ contradicts the third condition. Thus, D is kernel-perfect, as desired.

Finally, we claim that $|L(v)| \geq d^+_D(v)+1$ for all $v \in V(H)$. First suppose that $v \in S_1 \cup S_2$. Now v has at most seven out-neighbors within its clique and at most 3 out-neighbors in the other clique, so $d^+_D(v) \leq 10$. Since $|B_1| \geq 15$ and $|B_2| \geq 15$, we have $|L(v)| \geq |B_i| - 4 \geq 11 \geq d^+_D(v) + 1$. Next, suppose that $v \in (B_1 \cup B_2) \setminus (S_1 \cup S_2)$, but v is not among the final 3 vertices of either B_i. By symmetry, we can assume that $v \in B_1$. Since v has no out-neighbors among the final 3 vertices of B_1, it has at most $|B_1| - 4$ out-neighbors in B_1. Since v has at most 3 out-neighbors in B_2, we have $|L(v)| \geq |B_1| = (|B_1| - 4) + 3 + 1 \geq d^+_D(v) + 1$. Now suppose that v is among the final 3 vertices of B_1 or B_2; by symmetry, assume that $v \in B_1$. Since all out-neighbors of v are in B_1, we get $d^+_D(v) \leq |B_1| - 1$; thus, $|L(v)| \geq d^+_D(v) + 1$. □

Lemma 3.13 *G does not have an r-region for any $r \geq 152$.*

Proof. Suppose, to the contrary, that G has such an r-region R, with $r \geq 152$. Let B_1, B_2, D_2, b_1, and b_2 be as in Observation 3.9. Let v_1 and v_2 be adjacent vertices of $B_1 \cup B_2 \cup D_2$ such that every vertex within distance 2 in G of v_1 or v_2 is in $\{b_1, b_2\} \cup N(b_1) \cup N(b_2) \cup V(R)$. By the minimality of G, we can L-color $(G - v_1v_2)^2$; call this coloring φ. Now we uncolor many of the vertices in $V(R)$ and extend the coloring to G using Lemma 3.12, as well as greedily coloring vertices of D_2 last. The details forthwith.

Let S be the set of vertices in $B_1 \cup B_2$ that are incident with a face of G not in R. Let $T = N(b_1) \cap N(b_2)$, and note that $|T| \leq 1$, since G has girth at least 5. Let $B_1' = B_1 \setminus (N[S] \cup T)$ and $B_2' = B_2 \setminus (N[S] \cup T)$. Note that B_1' and B_2' are independent sets in G but are cliques in G^2. Let $H = G^2[B_1' \cup B_2']$. For each $v \in V(H)$, let $L'(v) = L(v) \setminus \{c : \varphi(w) = c$ for some $w \in N_{G^2}(v) \setminus (V(H) \cup D_2)\}$. Let $S_1 = B_1' \cap N(N[S] \cap (B_1 \cup B_2))$ and $S_2 = B_2' \cap (N(N[S] \cap (B_1 \cup B_2)))$.

By Lemma 3.10, it follows that $|N_{G^2}(v) \cap B_{3-i}| \leq 3$ for each $i \in \{1, 2\}$ and every $v \in B_i'$. To color H by Lemma 3.12, we must verify that S_1 and S_2 are small enough and that B_1', B_2', and all of the lists L' are big enough. Note that $|L'(v)| \geq k$ for every $v \in (B_1' \cup B_2' \setminus (S_1 \cup S_2))$, since each $v \in (B_i' \setminus S_i)$ loses at most one color for each vertex in $(\{b_1, b_2\} \cup N(b_i)) \setminus (V(H) \cup D_2)$.

For each $i \in \{1, 2\}$, each $v \in B_i$ has at most three neighbors in B_{3-i}. Thus, $|N[S] \cap (B_2 \setminus S)| \leq 2|S \cap B_1| = 4$. So $|S_1| = |B_1' \cap N(N[S] \cap (B_1 \cup B_2))| = |B_1' \cap (N(N[S] \cap B_2))| \leq 2|N[S] \cap (B_2 \setminus S)| \leq 8$. Similarly, $|S_2| \leq 8$. Each vertex $v \in S_1$ has at most three colored neighbors (in G^2) in $B_2 \setminus B_2'$. So, v loses at most three more colors than in the analysis for vertices in $B_1' \setminus S_1$. Hence, each $v \in S_1$ has $|L'(v)| \geq k - 3$; the same is true for each $v \in S_2$.

Now we show that B_1' and B_2' are big enough. The number of edges of G' incident with the region R' is $|R'| + 1$. By Lemma 3.10, every vertex of B_1 or B_2 is in at most three of those edges, so $|B_1| \geq (|R| + 1)/3$ and $|B_2| \geq (|R| + 1)/3$; we can actually get better bounds using planarity, but we omit that argument to keep the proof simpler. Now $|S \cap B_1| = 2$ and $|N(S) \cap B_1| \leq 3|S \cap B_2| \leq 6$, so $|N[S] \cap B_1| \leq 8$. Thus $|B_1'| \geq (|R| + 1)/3 - (N[S] \cup T) \geq (152 + 1)/3 - (8 + 1) = 42$. Similary, $|B_2'| \geq 42$.

Thus, we can use Lemma 3.12 to extend the coloring to $V(H)$. After coloring $V(H)$, for each vertex $x \in D_2$, we can color it arbitrarily from its list, since $|L(x)| \geq k + 2$ and $d_{G^2}(x) \leq 2\sqrt{k}$. Hence, G^2 has an L-coloring, a contradiction. □

By Corollary 3.8, G contains some r-region with $\frac{\sqrt{k}}{10} - 13 \leq r$. By Lemma 3.13, G contains no r-region with $r > 151$. Thus we have $\frac{\sqrt{k}}{10} - 13 \leq 151$. Simplifying gives $k \leq 1{,}640^2 = 2{,}689{,}600$. Thus, when $\Delta \geq 1{,}640^2 + 1$ we reach a contradiction, which proves our main result.

By relying more heavily on planarity, we can reduce the value of Δ_0. However, that approach adds numerous complications, which we prefer to avoid.

4 A Coloring Algorithm and Extending to Paintability

In this section, we explain how the proof of Theorem 1.3 yields an efficient algorithm to color G^2 from its lists. Further, we show how to extend this algorithm to paintability. Essentially, we construct a vertex order σ such that we can consider the vertices of G in order σ and color them greedily from their lists, but there is a wrinkle. If vertices appear together in an r-region, for $r \geq 152$, then we consider them simultaneously, and color them as in the proof of Lemma 3.12.

Our proof of Theorem 1.3 in fact shows that every planar graph G with girth at least 5 and maximum degree $\Delta \geq 2,689,601$ contains at least one of the following four reducible configurations: (i) a vertex v with $d_G(v) \leq 1$ or (ii) an edge uv with $u \notin N[B]$ and $v \notin N[B]$ or (iii) an edge uv with $d(u) = d(v) = 2$ and either $u \notin N(B)$ or $v \notin N(B)$ or (iv) an r-region with $r \geq 152$.

In each case, by induction we construct the desired order σ' for some smaller graph, and use the presence of the reducible configuration to extend σ' to an order σ for G. More precisely, we use induction on $|V(G)|$ and, subject to that, $|E(G)|$ to construct the desired vertex order. If G has (i) a vertex v with $d_G(v) \leq 1$, then, by hypothesis, there exists a desired order σ' for $G - v$. To get the desired order σ for G, we simply append v to σ'. If G has (ii) an edge uv with $u \notin N[B]$ and $v \notin N[B]$, then, by hypothesis, there exists a desired order σ' for $G - uv$. To form the desired order σ for G, we remove u and v from σ' and append them at the end. If G has (iii) an edge uv with $d(u) = d(v) = 2$ and $v \notin N(B)$, then, by hypothesis, there exists a desired order σ' for $G - uv$. To form the desired order σ for G, remove u and v from σ' and append u, v at the end.

So suppose instead that G contains (iv) an r-region R with $r \geq 152$. Now there exist adjacent vertices $u, v \in B_1 \cup B_2 \cup D_2$ such that every neighbor in G^2 of u and v is in $V(R) \cup \{b_1, b_2\}$. By hypothesis, there exists the desired order σ' for $G - uv$. Now construct subgraph H as in Lemma 3.13. To form σ from σ', remove $V(H)$ and append $V(H)$ at the end; with this order, we have the caveat that we consider all vertices of $V(H)$ simultaneously. More precisely, we color them as specified in Lemma 3.12. This completes the algorithm.

The game of *k-paintability* (also called *online k-list-coloring*) is played between two players, Lister and Painter. On round i, Lister presents a set S_i of uncolored vertices. Painter responds by choosing some independent set $I_i \subseteq S_i$ to receive color i. If Painter eventually colors every vertex of the graph, then Painter wins. If instead Lister presents some uncolored vertex on k rounds, but Painter never colors it, then Lister wins. The *paint number* $\chi_p(G)$ is the minimum k such that Painter can win regardless of how Lister plays. Let G be a planar graph with girth at least five. We show that if G has maximum degree $\Delta \geq 2,689,601$, then $\chi_p(G^2) \leq \Delta + 2$.

To adapt our proof of Theorem 1.3 to paintability, we just slightly modify the coloring algorithm given above. On each round i, Lister presents a set of uncolored vertices S_i. Now Painter constructs the independent set I_i greedily by considering the vertices of S_i in the order σ constructed above. To apapt the algorithm to paintability, we only need to explain how to handle the vertices of the subgraph H derived from an r-region with $r \geq 152$. The first observation is that Lemma 3.11 also holds for paintability, with essentially the same proof. On each round i, we let $A_c = S_i$, that is, A_c is the set of vertices listed by Lister on round i. The rest of the proof is identical. Now we adapt the proof of Lemma 3.13 as follows. On round i, for each vertex of H in S_i, we consider it to be in A_c only if none of its neighbors have already been added to the independent set I_i that Painter is in the process of building. Let F_c be a kernel of $D[A_c]$. Add F_c to what is currently in I_i, and now continue processing the remaining vertices of S_i in the order σ. This completes the proof.

We conclude by noting that this vertex order σ gives rise to a digraph D such that G^2 is the underlying graph of D, each vertex $v \in G$ has $d^+_D(v) \leq k + 1$, and D is kernel perfect (this digraph D implies the paintability result in the previous paragraph, via Lemma 3.11). To form D from the vertex order σ, we begin with G^2 and direct the edges as follows. If some pair of vertices, v_i and v_j, adjacent in G^2 appear in σ in the same r-region (with $r \geq 152$), then we direct v_iv_j as in Lemma 3.12. We direct each other edge v_iv_j of G^2 as $v_i \to v_j$, if $i > j$. Now the proof that D is kernel perfect is the same as in the previous paragraph. Given a subset of $V(D)$, we construct a kernel greedily, by considering vertices in order σ and handling the vertices of each r-region as in Lemma 3.12.

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