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A Guide to the Discharging Method

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Abstract

We provide a "how-to" guide to the use and application of the Discharging Method. Our aim is not to exhaustively survey results that have been proved by this technique, but rather to demystify the technique and facilitate its wider use. Along the way, we present some new proofs and new problems.

1 Introduction

The Discharging Method has been used in graph theory for more than 100 years. Its most famous application is the proof of the Four Color Theorem, stating that graphs embeddable in the plane have chromatic number at most 4. Nevetheless, the method remains a mystery to many graph theorists. Our purpose in this guide is to explain its use in order to make the method more widely accessible. Although we mention many applications, such as stronger versions of results proved here, cataloguing applications is not our aim. For a survey of applications in coloring of plane graphs we refer the reader to Borodin [45].

Discharging is a tool in a two-pronged approach to inductive proofs. It can be viewed as an amortized counting argument used to prove that a global hypothesis guarantees the existence of some desirable local configuration. In an application of the resulting structure theorem, one shows that each such local configuration cannot occur in a minimal counterexample to the desired conclusion. More precisely, a configuration is *reducible* for a graph property if it cannot occur in a minimal graph failing that property. This leads to the phrase "an unavoidable set of reducible configurations" to describe the overall process.

The relationship between the average and the minimum in a set of numbers provides a trivial example of such global/local implications: if the average in a set of numbers is less than k (global hypothesis), then some number in the set is less than k (local conclusion). Many analogous structure theorems in graph theory state that a bound on the average vertex degree forces some sparse local structure.

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Discharging enables vertex degrees to be reallocated to reach a global bound. For example, each vertex may start with a "charge" equal to its degree. To show that average degree less than b forces the presence of a configuration in a specified set S of sparse local configurations, we show that their absence allows charge to be moved (via "discharging rules") so that the final charge at each vertex is at least b, violating the hypothesis.

The discharging argument yields a structure theorem quite separate from the induction step showing reducibility of the configurations. Thus the unavoidable set resulting from a particular sparseness condition may be reusable to prove other results. However, usually the set of configurations is tailored to the desired application: all must be shown to be reducible for the property stated as the conclusion of the theorem.

We present a variety of classical applications, some with new proofs. Although we emphasize discharging arguments, we include many reducibility arguments to show the techniques used in applying the discharging results. For clarity and simplicity, many of the results we prove are weaker than the best known results by using more restrictive hypotheses. In stronger results, the domain is larger and may include graphs containing no configuration in the set forced by the more restrictive hypothesis. Thus the stronger results generally require more detailed case analysis, with more configurations that must be proved reducible.

Another motivation is that the Discharging Method often yields a fast inductive algorithm to construct a witness for the desired conclusion, such as a good coloring. Iterative application of the discharging argument yields a sequence of reductions to successively smaller graphs. After a good coloring of the base graph is produced, the original graph is built back up, and all the intermediate graphs are given good colorings using the reducibility arguments. The process tends to be fast because typically the next reducible configuration can be guaranteed to be found in the neighborhood of an earlier one, so the next reduction can be found in amortized constant time (see Section 6 of [92]).

The overall idea of discharging proofs is simple, and the proofs are usually easy to follow, though they may have many details. The mystery arises in the choice of reducible configurations, the rules for moving charge, and how to find the best hypothesis for the structural lemma. We aim to explain the interplay among these and suggest how such proofs are discovered. In keeping with our instructional intent, we pose additional statements as exercises.

As suggested above, we begin in Section 2 by using discharging to prove structure theorems about sparse graphs, motivated by applications to coloring problems. A bound on the average degree forces sparse local configurations, but to use the structure theorem in an inductive proof we also require the same bound in all subgraphs. The *maximum average degree*, written $\mathrm{mad}(G)$, is $\mathrm{mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$.

Section 3 expands the discussion to list coloring. Reducibility arguments may involve coloring vertices in a good order, which works as well for list coloring as for ordinary coloring. Section 4 applies a similar approach to edge-coloring, where arbitrarily large structures may

appear in the structure theorems proved by discharging. We study restrictions on $\mathrm{mad}(G)$ that yield strong bounds on edge-coloring. Included is a recent enhancement of discharging by Woodall [246] in which charge is moved iteratively instead of all at once.

Many results about coloring or structure of planar graphs (or planar graphs with large girth) have been proved by discharging. Euler's Formula implies that (every subgraph of) a planar graph with girth at least g has average degree less than $\frac{2g}{g-2}$. Many results proved for planar graphs with girth at least g in fact hold whenever $\mathrm{mad}(G) < \frac{2g}{g-2}$, regardless of planarity, often with the same proof by discharging. Others, such as those in Section 5, truly depend on the planarity of the graph by using also the sparseness of the planar dual, assigning charge to both the faces and the vertices. In Sections 5 and 6 we use discharging to prove (known) partial results toward various open problems about planar graphs.

In Section 7, we present discharging results on several variations of coloring. We consider colorings satisfying stronger requirements (acyclic coloring, star coloring, and linear coloring) and weaker requirements ("improper" colorings). In Section 8, we briefly mention other problems in which discharging has been used.

2 Structure and coloring of sparse graphs

We start at the very beginning, with very sparse graphs. Graphs with low average degree have a vertex with small degree, but we seek more structural information.

Let $\overline{d}(G)$ denote the average degree of the vertices in G. If $0 < \overline{d}(G) < 2$, then some vertex has degree at most 1. In fact, G must have at least two such vertices, but they may be far apart. However, if $\overline{d}(G) < 2 - \epsilon$ with $\epsilon > 0$, then G must have an isolated vertex or vertices of degree 1 within distance $\frac{2}{\epsilon}$ of each other. More precisely, G must have a tree component with fewer than $\frac{2}{\epsilon}$ vertices, since an n-vertex tree has average degree $2 - \frac{2}{n}$.

Similarly, if $\overline{d}(G) < k$, then only one vertex of degree less than k is guaranteed (subdivide an edge in a k-regular graph), but $\overline{d}(G) < k-\epsilon$ guarantees more. Consider $k = 3$. If $\overline{d}(G) \geq 2$, then G may have no vertex of degree at most 1. However, if $\overline{d}(G) < 2+\epsilon$ and $\delta(G) \geq 2$, then G must have many "consecutive" vertices of degree 2. We prove this structural result by discharging and then apply it to a coloring problem. We first introduce standard terminology that allows discharging arguments to be expressed concisely.

Definition 2.1. For convenience, a *j-vertex*, *j^+-vertex*, or *j^--vertex* is a vertex with degree equal to j, at least j, or at most j, respectively. Similarly, a *j-neighbor* of v is a j-vertex that is a neighbor of v. An *ℓ-thread* in a graph G is a path of length $\ell + 1$ in G whose ℓ internal vertices have degree 2 in the full graph G.

Under this definition, an ℓ-thread contains two $(\ell - 1)$-threads when $\ell \geq 1$. Also, a cycle of length at most $\ell + 1$ contains no ℓ-thread. Write $d_G(v)$ (or simply $d(v)$) for the degree of a vertex v in a graph G. Let $\Delta(G) = \max_{v \in V(G)} d_G(v)$ and $\delta(G) = \min_{v \in V(G)} d_G(v)$.

Proposition 2.2. *If $\overline{d}(G) < 2 + \frac{1}{3t-2}$ and G has no 2-regular component, then G contains a 1^--vertex or a $(2t-1)$-thread.*

Proof. Let $\rho = \frac{1}{2}\frac{1}{3t-2}$. Give each vertex v its degree $d(v)$ as initial charge. If neither configuration occurs in G, then we redistribute charge to leave each vertex with charge at least $2 + 2\rho$. If G has no 1^--vertex, then $\delta(G) \geq 2$. We then redistribute charge as follows.

(R1) Each 2-vertex takes ρ from each end of the maximal thread containing it.

Each 2-vertex receives charge 2ρ and ends with $2 + 2\rho$. With $(2t-1)$-threads forbidden, each 3^+-vertex v loses at most $d(v)(2t-2)\rho$. For a lower bound on its final charge, we find

$$d(v) - d(v)(2t-2)\rho \geq 3\left[1 - \frac{t-1}{3t-2}\right] = 2 + \frac{1}{3t-2} = 2 + 2\rho.$$

We have proved that forbidding the specified configurations requires $\overline{d}(G) \geq 2 + \frac{1}{3t-2}$. □

The idea is simple; if no desired configuration occurs, then charge can be redistributed to contradict the hypothesis. The details are also easily checked. The mystery is where the discharging rule and the hypothesis on $\overline{d}(G)$ come from. The secret is that the discharging rule is found before one even knows what the hypothesis on $\overline{d}(G)$ will be and is used to discover that hypothesis.

Remark 2.3. For discharging proofs when $\overline{d}(G) < 2+2\rho$, only 2-vertices need to gain charge (after restricting to $\delta(G) \geq 2$), but we must ensure that 3^+-vertices don't lose too much. The natural sources of charge for the 2-vertices are the nearest vertices of larger degree.

To select ρ, we seek the weakest hypothesis that avoids taking too much charge from 3^+-vertices. With ℓ-threads forbidden, each 3^+-vertex loses at most $(\ell-1)\rho$ along each incident thread. For a j-vertex, we thus need $j - j(\ell-1)\rho \geq 2 + 2\rho$, which simplifies to $\rho \leq \frac{j-2}{j(\ell-1)+2}$. With ℓ fixed (and $j \geq 3$), the bound is tightest when $j = 3$. Therefore, setting $\rho = \frac{1}{3\ell-1}$ makes the proof work and also gives the weakest hypothesis where it works. We used $\ell = 2t-1$ in Proposition 2.2 for consistency with the intended application, but the statement and argument are valid for all ℓ.

The structure theorem is sharp. From the proof, all vertices in a sharpness example should have degree 2 or 3. To generate infinitely many such examples, replace each edge of any 3-regular n-vertex graph with an $(\ell-1)$-thread. Long threads do not occur, and there are $\ell\frac{3n}{2}$ edges and $n + (\ell-1)\frac{3n}{2}$ vertices; the average degree is $2 + \frac{2}{3\ell-1}$.

Discovering a discharging argument is fun in itself, but the value is in applications. To use structural results in inductive proofs, we must require the hypothesis of the structural lemma to hold also in subgraphs.

A graph is *d-degenerate* if every subgraph has a vertex of degree at most d; this is an "every subgraph" analogue of $\delta(G) \leq d$. A *k-coloring* of a graph G labels vertices using a

set of k colors; a coloring is *proper* if adjacent vertices always have distinct colors. A graph is *k-colorable* if it has a proper k-coloring, and the *chromatic number* $\chi(G)$ is the least such k. Every $(k-1)$-degenerate graph is k-colorable, by induction on the number of vertices.

Using bounds on average degree analogously to obtain good colorings requires all subgraphs to satisfy the same bound. Hence we restrict the maximum average degree over all subgraphs, denoted by $\mathrm{mad}(G)$. Note that $\mathrm{mad}(G) < k$ implies that G is $(k-1)$-degenerate. Thus we already have $\chi(G) \le k$ when $\mathrm{mad}(G) < k$. Even when $\mathrm{mad}(G) = k-1$, we cannot improve the bound on $\chi(G)$, due to the complete graph K_k. Hence to obtain stronger coloring results when $\mathrm{mad}(G) < k$, we consider a more refined notion of chromatic number. Let $\mathbb{Z}_p$ denote the set of congruence classes of integers modulo p; here p is any positive integer.

Definition 2.4. A *homomorphism* from a graph G to a graph H is a map $\phi\colon V(G) \to V(H)$ such that $uv \in E(G)$ implies $\phi(u)\phi(v) \in E(H)$. Such a map is an *H-coloring*; the vertices of G are "colored" by vertices of H. Let $K_{p:q}$ be the graph with vertex set $\mathbb{Z}_p$ in which vertices are adjacent when they differ by at least q (see Figure 1). A *(p,q)-coloring* of G is a homomorphism from G into $K_{p:q}$. A graph having a (p,q)-coloring is *(p,q)-colorable*. The *circular chromatic number* of G, written $\chi_c(G)$, is $\inf\{\frac{p}{q}\colon G \text{ is } (p,q)\text{-colorable}\}$.

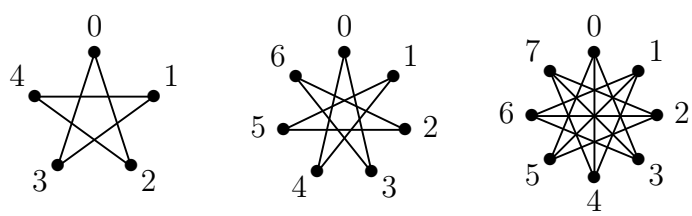

Figure 1: $K_{5:2}$, $K_{7:3}$, and $K_{8:3}$

The term "circular coloring" suggests viewing the colors as equally-spaced points on a circle. A (p,q)-coloring ϕ then uses colors in $\{0,\dots,p-1\}$ so that $q \le |\phi(u)-\phi(v)| \le p-q$ when $uv \in E(G)$. A $(k,1)$-coloring is just a proper k-coloring. Zhu [258] surveyed basic facts about circular coloring, which include that $\chi_c(G)$ is a well-defined rational number ("inf" becomes "min") and that $\lceil \chi_c(G) \rceil = \chi(G)$. Thus $\chi_c(G)$ gives more information than $\chi(G)$. Also, if G has a (p,q)-coloring and $\frac{p'}{q'} \ge \frac{p}{q}$, then G also has a (p',q')-coloring.

Note that $K_{(2t+1):t}$ is isomorphic to C_{2t+1}. Thus $\chi_c(G) \le 2 + \frac{1}{t}$ if and only if there is a homomorphism from G into C_{2t+1}, which cannot happen if G has a shorter odd cycle. Let $g_o(G)$ denote the length of a shortest odd cycle in G, with $g_o(G) = \infty$ when G is bipartite. Requiring $g_o(G) \ge 2t+1$ gives us a chance to prove $\chi_c(G) \le 2 + \frac{1}{t}$.

Proposition 2.5. *Fix $t \in \mathbb{N}$. If $g_o(G) \ge 2t+1$ and $\mathrm{mad}(G) < 2 + \frac{1}{3t-2}$, then $\chi_c(G) \le 2 + \frac{1}{t}$.*

Proof. If $G = K_1$ or G is a cycle, then $\chi_c(G) \le 2 + \frac{1}{t}$. With this as basis, we use induction on $|V(G)|$ to obtain a homomorphism from G into C_{2t+1}. When G is neither K_1 nor a

cycle, Proposition 2.2 implies that G has either a 1^--vertex or a $(2t-1)$-thread. Note that $g_o(G') \geq 2t+1$ and $\mathrm{mad}(G') < 2 + \frac{1}{3t-2}$ whenever G' is an induced subgraph of G.

If G has a 1^--vertex u, then let ϕ be a homomorphism of $G-u$ into C_{2t+1}, as guaranteed by the induction hypothesis. To extend ϕ to G, when u is isolated we can give it any color. When u has a neighbor v, we can assign u either neighbor of $\phi(v)$ in C_{2t+1}.

Otherwise, G contains a thread P with endpoints v and w that has internal vertices $u_1, \ldots, u_{2t-1}$. Let $G' = G - \{u_1, \ldots, u_{2t-1}\}$. Since $\mathrm{mad}(G') < 2 + \frac{1}{3t-2}$, the induction hypothesis yields a homomorphism ϕ from G' to C_{2t+1}. To extend ϕ to the desired coloring of G, map the internal vertices of P to the internal vertices of a walk of length $2t$ from $\phi(v)$ to $\phi(w)$ in C_{2t+1}. This extension exists because any two vertices of C_{2t+1} are joined by a path of even length at most $2t$. To follow a path of length $2s$ in $2t$ steps, repeat one edge $2t-2s+1$ times in succession. □

We need a $(2t-1)$-thread in the induction step because with shorter threads there are choices for the colors at the endpoints that do not permit extension along the thread. For example, if the endpoints of a $(2t-2)$-thread have the same color, then a C_{2t+1}-coloring cannot be extended along the thread.

Note again that the structural lemma uses only a bound on $\overline{d}(G)$. Only the application requires the bound also for all subgraphs. In any hereditary family $\mathcal{G}$, a bound on $\overline{d}(G)$ that holds for all $G \in \mathcal{G}$ immediately yields the same bound on $\mathrm{mad}(G)$.

The structural result in Proposition 2.2 is sharp, but Proposition 2.5 is not. We can improve it by considering more configurations, with the same proof idea. Let the *weak 2-neighbors* of a vertex v be the 2-vertices lying on threads incident to v.

Lemma 2.6. *If $\overline{d}(G) < 2 + \frac{1}{2t-1}$ and G has no 2-regular component, then G contains (1) a 1^--vertex, or (2) a 3-vertex with at least $4t-3$ weak 2-neighbors, or (3) a 4^+-vertex incident to a $(2t-1)$-thread.*

Proof. Let $\rho = \frac{1}{2}\frac{1}{2t-1}$. Assign each vertex v initial charge $d(v)$. We may assume $\delta(G) \geq 2$. Redistribute charge using the same rule as before.

(R1) Each 2-vertex takes ρ from each end of the maximal thread containing it.

As in Proposition 2.2, each 2-vertex finishes with $2+2\rho$. If no 3-vertex has $4t-3$ weak 2-neighbors, then each 3-vertex v loses charge at most $(4t-4)\rho$ and hence retains at least $3 - \frac{2t-2}{2t-1}$, which equals $2+2\rho$.

Now let v be a 4^+-vertex. If v has no incident $(2t-1)$-thread, then v gives charge to at most $2t-2$ vertices on each incident thread. The minimum remaining charge $d(v)[1-(2t-2)\rho]$ is minimized when $d(v)=4$. We compute $4[1-(2t-2)\rho] = 4 - 2\frac{2t-2}{2t-1} = 2+4\rho$. (There is not enough excess charge to force a longer thread; fortunately, this is just long enough!)

Every vertex finishes with charge at least $2+2\rho$, so avoiding the specified configurations requires $\mathrm{mad}(G) \geq 2 + \frac{1}{2t-1}$. □

This structural result is driven by the application, where a 3-vertex with $4t-3$ weak 2-neighbors is reducible. Balancing the needs of 2-vertices and 3-vertices then governs the choice of ρ to make the result apply to the largest family. When the reducible configurations at 1^--vertices and 3-vertices do not arise, some thread is long enough to be reducible.

Lemma 2.6 yields a better coloring bound than Proposition 2.5, because we now have a more flexible way of extending circular colorings of subgraphs. We need a well-known lemma about extension of circular colorings, illustrated in Figure 2

Lemma 2.7. *Assume $p > 2q$, and let P be an ℓ-thread with endpoints x and y in a graph G. In a (p,q)-coloring ϕ, a fixed choice of $\phi(x)$ can be extended along P with at most $\max\{0, p-1-(\ell+1)(p-2q)\}$ values in $\mathbb{Z}_p$ forbidden as $\phi(y)$.*

Proof. Let $u_0 = x$ and $u_{\ell+1} = y$, so $P = \langle u_0, u_1, \ldots, u_\ell, u_{\ell+1}\rangle$. We claim that the colors allowed at u_i include $1 + i(p-2q)$ consecutive colors in $\mathbb{Z}_p$, true by definition for $i = 0$. If the colors from a through b (modulo p) are allowed at u_i, then the colors from $a+q$ through $b+(p-q)$ are allowed at u_{i+1}. The size of the new interval exceeds the size of the previous interval by $p-2q$. When $\ell+1 \geq \frac{p-1}{p-2q}$, there is no restriction at y from this thread. □

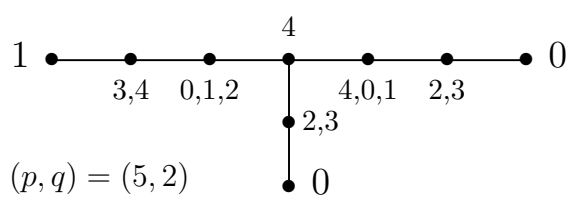

Figure 2: Extension of $(5,2)$-coloring along three threads; allowed colors shown

We now adopt the language of "minimal counterexample" for inductive proofs. Most applications of structure theorems proved by discharging are proved for hereditary graph classes. Proving reducibility of forced configurations then completes an inductive proof, since it forbids the existence of a minimal counterexample to the claim.

Theorem 2.8. *If $g_o(G) \geq 2t+1$ and $\mathrm{mad}(G) < 2 + \frac{1}{2t}$, then $\chi_c(G) \leq 2 + \frac{1}{t}$.*

Proof. Let G be a minimal graph satisfying the hypotheses but not the conclusion; the hypotheses hold for all subgraphs of G. The claim holds for cycles, so we may forbid 2-regular components. Lemma 2.6 then guarantees one of three configurations in G. We already showed in Theorem 2.5 that 1^--vertices and $(2t-1)$-threads are reducible for $\chi_c(G) \leq 2 + \frac{1}{t}$.

It remains only to show that a minimal counterexample cannot contain a 3-vertex v with at least $4t-3$ weak 2-neighbors. If G is such a graph, then form G' from G by deleting v and all its weak 2-neighbors. Extending a C_{2t+1}-coloring of G' along the threads incident to v completes a C_{2t+1}-coloring of G if some color in $\mathbb{Z}_{2t+1}$ is not forbidden at v.

Let l_i be the number of 2-vertices along the ith thread at v. By Lemma 2.7 with $p = 2t+1$ and $q = t$, this thread forbids at most $(2t) - (l_i + 1)1$ colors from use at v. Together, the three threads forbid at most $(6t) - (l_1 + l_2 + l_3 + 3)$ colors. Since $\sum l_i \geq 4t - 3$, at most $2t$ colors are forbidden, so some color remains available for a simultaneous extension along the three threads to complete a C_{2t+1}-coloring of G. $\square$

To see that Theorem 2.8 improves Proposition 2.5, note that the conclusion (G is $(2+\frac{1}{t})$-colorable) is the same in both, but the hypothesis on $\mathrm{mad}(G)$ is weaker in Theorem 2.8 ($\mathrm{mad}(G) < 2 + \frac{1}{2t}$) than in Proposition 2.5 ($\mathrm{mad}(G) < 2 + \frac{1}{3t-2}$). Weakening the hypothesis introduces graphs avoiding the unavoidable set in Proposition 2.2, but then another configuration is forced that is also reducible for the desired conclusion.

The bound on $\mathrm{mad}(G)$ in Theorem 2.8 is still not sharp for $\chi_c(G) \leq 2 + \frac{1}{t}$. In particular, Borodin et al. [52] proved for triangle-free graphs that $\mathrm{mad}(G) < \frac{12}{5}$ implies $\chi_c(G) \leq \frac{5}{2}$, while Theorem 2.8 with $t = 2$ requires $\mathrm{mad}(G) < \frac{9}{4}$ to obtain $\chi_c(G) \leq \frac{5}{2}$. Sharpness of their result follows from the case $t = 2$ of the following construction.

Example 2.9. Let G_t consist of two cycles of length $2t + 1$ sharing a single edge, plus a $(2t-2)$-thread joining the vertices opposite the shared edge on the two cycles (see Figure 3). Note that $G_1 = K_4$, and $\overline{d}(G_2) = \frac{12}{5}$. If $\chi_c(G_t) \leq 2 + \frac{1}{t}$, then G_t must have a homomorphism into C_{2t+1}. However, once the colors are chosen on the edge shared by the two $(2t + 1)$-cycles in G_t, the colors on the remaining two 3-vertices are forced to be the same. The homomorphism cannot be extended to all of G_t, since there is no homomorphism from C_{2t-1} into C_{2t+1}. In fact, $\chi_c(G_t) = 2 + \frac{1}{t-1/2}$, so $\chi_c(G_2) = \frac{8}{3}$. In general, $\overline{d}(G_t) = 2 + \frac{2}{3t-1}$.

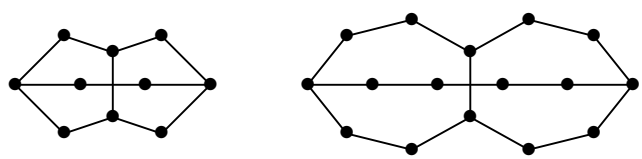

Figure 3: The graphs G_2 and G_3 of Example 2.9

Example 2.9 shows that the following conjecture is best possible.

Conjecture 2.10. *If* $g_o(G) \geq 2t + 1$ *and* $\mathrm{mad}(G) < 2 + \frac{2}{3t-1}$, *then* $\chi_c(G) \leq 2 + \frac{1}{t}$.

The conjecture is trivial for $t = 1$; [52] proved it for $t = 2$. The proof uses "long-distance" discharging, where charge can move along special long paths. A weaker version of Conjecture 2.10 is a special case of a result by Borodin et al. [65]: if G has girth at least $6t - 2$ and $\mathrm{mad}(G) < 2 + \frac{3}{5t-2}$, then $\chi_c(G) \leq 2 + \frac{1}{t}$. This uses larger configurations.

The result in [65] is motivated by the dual of a conjecture of Jaeger [138]. Jaeger conjectured that every $4t$-edge-connected graph has "circular flow number" at most $2+\frac{1}{t}$. When G

is planar, making this statement for the dual graph G^* yields $\chi_c(G) \le 2+\frac{1}{t}$ when G has girth at least $4t$. Lovász, Thomassen, Wu, and Zhang [173] proved the weaker form of Jaeger's Conjecture replacing $4t$ by $6t$. Thus $\chi_c(G) \le 2+\frac{1}{t}$ when G is planar with girth at least $6t$. By Euler's Formula, $\mathrm{mad}(G) < \frac{2g}{g-2}$ when G is planar with girth g. Thus $\mathrm{mad}(G) < 2+\frac{2}{2t-1}$ when G is planar with girth at least $4t$. Conjecture 2.10 in some sense proposes a trade-off: by further restricting to $\mathrm{mad}(G) < 2+\frac{2}{3t-1}$, the girth requirement can be relaxed to $g_o(G) \ge 2t+1$ and still yield $\chi_c(G) \le 2+\frac{1}{t}$, even without requiring planarity.

We have discussed only very sparse graphs and very small $\chi_c(G)$, but the problem can be studied in general. Always $\chi(G) \ge \omega(G)$, where $\omega(G)$ is the maximum number of pairwise adjacent vertices in G, called the *clique number* of G. The *circular clique number*, written $\omega_c(G)$, is $\max\{\frac{p}{q}\colon\ K_{p:q} \subseteq G\}$; note that $\chi_c(G) \ge \omega_c(G)$.

Problem 2.11. Among graphs G with $\omega_c(G) \le s$, what is the largest ρ such that $\mathrm{mad}(G) < \rho$ implies $\chi_c(G) \le s$?

The answer ρ satisfies $\lfloor s \rfloor \le \rho < s$. The question generalizes the statement that $(k-1)$-degenerate graphs are k-colorable but K_{k+1} is not. If $\mathrm{mad}(G) < \lfloor s \rfloor$, then G is $\lfloor s \rfloor$-colorable and hence also s-colorable. If $\mathrm{mad}(G) = s$, then G may be $K_{p:q}$ with $\frac{p}{q} = s$. Problem 2.11 has not been studied much, but discharging should permit some progress on it.

Exercise 2.1. Let G be a graph with $\delta(G) = 2$ and $\mathrm{mad}(G) < 3$. Prove that G has a 2-vertex with a 5^--neighbor. Prove that this is sharp in the sense that the conclusion may fail when $\mathrm{mad}(G) = 3$.

Exercise 2.2. Given $0 \le j < k$, let G be a graph with $\delta(G) = k$. Determine the largest ρ such that $\overline{d}(G) < k+\rho$ guarantees that G has a k-vertex having more than j neighbors of degree k.

Exercise 2.3. Show that Lemma 2.6 is sharp. For each $t \in \mathbb{N}$ construct infinitely many examples with average degree $2+\frac{1}{2t-1}$ in which none of the specified configurations occurs.

Exercise 2.4. (Cranston–Kim–Yu [93]) Let G be a connected graph with at least four vertices. Prove that if $\overline{d}(G) < \frac{5}{2}$ and $\delta(G) \ge 2$, then G contains a 2-thread or a 3-vertex having three 2-neighbors, one of which has a second 3-neighbor.

Exercise 2.5. (Cranston–Jahanbekam–West [91]) Prove that if $\overline{d}(G) < \frac{5}{2}$ and G is connected, then G contains a 3^--vertex with a 1-neighbor, a 4^--vertex with two 2^--neighbors, or a 5^+-vertex v with at least $\frac{d(v)-1}{2}$ 2^--neighbors. (Comment: These configurations are reducible for the "1, 2-Conjecture" of Przybyło and Woźniak [190]. Although that proves the conjecture when $\mathrm{mad}(G) < \frac{5}{2}$, [190] proved the stronger result that the conjecture holds for 3-colorable graphs.)

Exercise 2.6. Prove that if $\mathrm{mad}(G) < k+\rho$ with $0 < \rho \le \frac{k}{k+1}$, then G contains a $(k-1)^-$-vertex, two adjacent k-vertices, or a $(k+1)$-vertex with more than $(\frac{1}{\rho}-1)k$ neighbors having degree k. Construct sharpness examples with $\mathrm{mad}(G) = k+\rho$ when $\rho = \frac{1}{2}$ and when $\rho = \frac{k}{k+1}$ (the latter may have maximum degree $k+1$ or $k+2$).

Exercise 2.7. Prove that if $\Delta(G) = k \ge 3$ and $\mathrm{mad}(G) < k - \frac{2}{k^2+1}$, then G contains one of the following configurations: (C1) a $(k-2)^-$-vertex, (C2) two adjacent $(k-1)$-vertices, (C3) a k-vertex with two $(k-1)$-neighbors, (C4) two adjacent k-vertices each having a $(k-1)$-neighbor, or (C5) a k-vertex having three k-neighbors such that each is adjacent to a $(k-1)$-vertex.

3 List Coloring

Reducibility arguments for coloring often involve deleting some parts of a graph and then choosing colors for the missing pieces as they are reinserted. Suitable choices can be made if there are enough available colors; it does not really matter what the colors are. In this situation, the arguments often extend to yield stronger results about coloring from lists.

Definition 3.1. A *list assignment* on a graph G is a map L giving each $v \in V(G)$ a set $L(v)$ of colors called its *list*. In a *k-uniform* list assignment, each list has size k. Given a list assignment L on G, an *L-coloring* of G is a proper coloring ϕ of G such that $\phi(v) \in L(v)$ for all $v \in V(G)$. A graph G is *k-choosable* if G is L-colorable whenever each list has size at least k (we may assume L is k-uniform). The *list chromatic number* of G, written $\chi_\ell(G)$, is the least k such that G is k-choosable.

Since the lists could be identical, always $\chi_\ell(G) \geq \chi(G)$. Thus proving $\chi_\ell(G) \leq b$ strengthens a result that $\chi(G) \leq b$. For example, Brooks' Theorem states that $\chi(G) \leq \Delta(G)$ when G is a connected graph that is not a complete graph or an odd cycle; Erdős, Rubin, and Taylor [105] proved the same bound for $\chi_\ell(G)$. Although $\chi_\ell(G)$ may be larger for 2-colorable graphs (Exercise 3.1), cycles of even length are well behaved; we will need this fact.

Lemma 3.2. *Even cycles are 2-choosable.*

Proof. We show that C_{2t} is L-colorable when every list has size 2. If the lists are identical, then choose the colors to alternate. Otherwise, there are adjacent vertices x and y such that $L(x)$ contains a color c not in $L(y)$. Use c on x, and then follow the path $C_{2t} - x$ from x to y to color the vertices other than x: at each new vertex, choose a color from its list that was not chosen for the previous vertex. Such a choice is always available, and they satisfy every edge because the colors chosen on x and y differ. □

Coloring and list-coloring have been studied extensively for squares of graphs. Given a graph G, let G^2 be the graph obtained from G by adding edges to join vertices that are distance 2 apart in G. The neighbors of a vertex v in G form a clique with v in G^2, so always $\chi(G^2) \geq \Delta(G) + 1$. Kostochka and Woodall [158] conjectured that always $\chi_\ell(G^2) = \chi(G^2)$. This was proved in special cases, but Kim and Park [149] disproved it in general. They used orthogonal families of Latin squares to construct a graph G for prime p such that G^2 is the complete $(2p-1)$-partite graph $K_{p,\ldots,p}$; on such graphs, $\chi_\ell - \chi$ is unbounded.

Thus sufficient conditions for $\chi_\ell(G^2) = \Delta(G)+1$ hold only on special classes but establish a strong property. We present such a result to show how a discharging proof is discovered. The Discharging Method often begins with configurations that are easy to show reducible. A discharging proof of unavoidability of a set of such configurations starts by forbidding them. When discharging, we may encounter a situation that does not guarantee the desired final

charge on some vertices. Instead of trying to adjust the discharging rules, we may try to add this configuration to the unavoidable set, allowing us to assume that it does not occur. This approach succeeds if we can show that the new configuration is reducible.

We use $N_G(v)$ for the neighborhood of a vertex v in a graph G, with $N_G[v] = N_G(v) \cup \{v\}$.

Lemma 3.3. *Fix $k \geq 4$. If $\Delta(G) \leq k$, and G is a smallest graph such that $\chi_\ell(G^2) > k+1$, then G does not contain the following configurations:*
(A) a 1^--vertex,
(B) a 2-thread joining a $(k-1)^-$-vertex and a $(k-2)^-$-vertex,
(C) a cycle of length divisible by 4 composed of 3-threads whose endpoints have degree k.

Proof. Let L be a $(k+1)$-uniform assignment on G; Figure 4 shows (B) and (C).

If (A) occurs at a 1^--vertex v, then let $G' = G - v$. The L'-coloring of G'^2 extends to an L-coloring of G^2, because at most k colors need to be avoided at v.

If (B) occurs, then G has a path $\langle x, u, v, y \rangle$ such that $d(u) = d(v) = 2$, $d(x) \leq k-1$, and $d(y) \leq k-2$. With distance 3 between x and y, we have $(G - \{u,v\})^2 = G^2 - \{u,v\}$. Let $G' = G - \{u,v\}$. By minimality, G'^2 has an L'-coloring ϕ, where L' is the restriction of L to $V(G')$. In G, the color on u must avoid the colors on $\{x,y\} \cup N_{G'}(x)$. Since $d(x) \leq k-1$ and $|L(u)| = k+1$, a color is available for u. Now the color on v must avoid those on $\{x,y,u\} \cup N_{G'}(y)$. Since $d(y) \leq k-2$ and $|L(u)| = k+1$, a color is available for v.

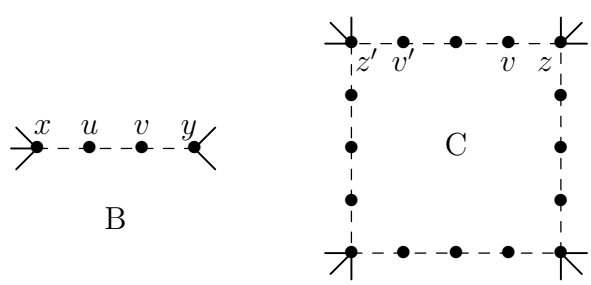

Figure 4: Reducible configurations in G for $\chi_\ell(G^2) \leq k+1$ (with $k = 5$)

If (C) occurs, then obtain G' from G by deleting the 2-vertices on the given cycle C. Again G'^2 is the subgraph of G^2 induced by $V(G')$. Let v be a deleted vertex having a k-neighbor z in G. The color on v must avoid those on z and all $k-2$ neighbors of z in G'. Since $|L(v)| = k+1$, at least two colors are available for v. These neighbors of k-vertices on C induce an even cycle in G^2. By Lemma 3.2, we can extend the coloring of G' to these vertices. Finally, the 2-vertices at the centers of the 3-threads have only four neighbors in G^2, all of which are now colored. Since $k \geq 4$, a color remains available at each such vertex. $\square$

In a discharging argument, we often say that a vertex is *happy* when its final charge satisfies the desired inequality.

Theorem 3.4 ([35, 95]). *If* $\Delta(G) \le 6$ *and* $\mathrm{mad}(G) < \frac{5}{2}$, *then* $\chi_\ell(G^2) = 7$.

Proof. Let G be a minimal counterexample. Let $k = 6$. By Lemma 3.3(A), we may assume $\delta(G) \ge 2$. By Lemma 3.3(B), G has no 4-thread (or longer), and 3-threads have k-vertices at both ends. By Lemma 3.3(C), the union of the 3-threads is an acyclic subgraph H.

Let each leaf of H *sponsor* its incident 3-thread, delete the edges of the sponsored 3-threads, and repeat. When a component of what remains has just one 3-thread, pick one endpoint as the sponsor. In this way, each 3-thread in G receives one of its endpoints as a sponsoring k-vertex, and each k-vertex is chosen at most once as a sponsor.

We now seek discharging rules to prove that if $\mathrm{mad}(G) < \frac{5}{2}$ and $\delta(G) \ge 2$, then some configuration of type (B) or (C) in Lemma 3.3 must occur. This will not quite work; we will need to add more configurations to the set, but they will be reducible. A vertex is *high* if its degree is 5 or 6; *medium* if it is 3 or 4. Each vertex v has initial charge $d(v)$; we seek final charge at least $\frac{5}{2}$. We treat the internal vertices of a j-thread together; they need to receive at least $\frac{j}{2}$. Here by "j-thread" we refer only to j-threads whose endpoints are 3^+-vertices.

(R1) Each high vertex gives $\frac{1}{2}$ to each neighbor.
(R2) Each 3-thread takes $\frac{1}{2}$ from its sponsoring 6-vertex.
(R3) Each 2-thread takes $\frac{1}{2}$ from each medium vertex incident to it.
(R4) Each 1-thread takes $\frac{1}{4}$ from each endpoint if they are both medium vertices.

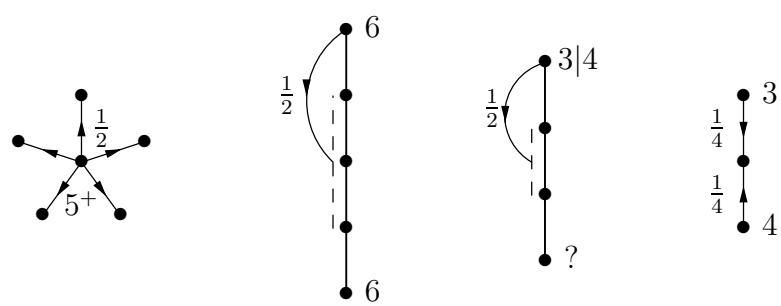

Figure 5: Discharging rules (R1)–(R4) for Theorem 3.4

High vertices can afford to give $\frac{1}{2}$ to each neighbor, and 6-vertices can afford to give an extra $\frac{1}{2}$ to one sponsored 3-thread. Group the 2-vertices by threads. Since 3-threads end at 6-vertices, one being a sponsor, each 3-thread receives $\frac{3}{2}$ and is happy. Similarly, 2-threads are happy by (R1) and (R3), and 1-threads are happy by (R1) and (R4) (see Figure 5).

It remains to consider medium vertices. A 4-vertex v is unhappy only if it loses more than $\frac{3}{2}$; this requires that all its neighbors are 2-vertices and at least three of the incident threads are 2-threads. We show that this configuration is reducible. Define G' from G by deleting v and its neighbors on three incident 2-threads; note that G'^2 is the subgraph of G^2 induced by $V(G')$. Since $|N_{G^2}(v) \cap V(G')| = 5$, we can extend the L'-coloring of G'^2 to

v. When we restore the deleted 2-neighbors of v, the numbers of vertices whose colors they must avoid are $4, 5, 6$, respectively, so at each step a color is available.

Since it gives at most $\frac{1}{2}$ to each thread, an unhappy 3-vertex v has no high neighbor and loses more than $\frac{1}{2}$. It may give charge at least $\frac{1}{4}$ to each of three threads or give $\frac{1}{2}$ to one thread and at least $\frac{1}{4}$ to another.

In the first case, let $N_G(v) = \{x_1, x_2, x_3\}$, and let $G' = G - N_G[v]$. The neighbor of x_i other than v has degree at most 4. As we restore $N_G(v)$, the number of vertices whose colors they must avoid are $4, 5, 6$, so at each step a color is available. We can then replace v; it must avoid the colors on six vertices.

In the second case, v has two 2-neighbors. Let z be the medium neighbor of v, let x be the neighbor on a thread receiving $\frac{1}{2}$ from v (it is a 2-thread), and let y be the remaining neighbor (y is a 2-vertex whose other neighbor is a 4^--vertex). With $S = \{v, x, y\}$, let $G' = G - S$; again $G'^2 = G^2 - S$. Restore v, then y, then x. As each is restored, its color is chosen from its list to avoid the colors on at most six other vertices. $\square$

Cranston and Škrekovski [95] proved more generally that if $\Delta(G) \geq 6$ and $\mathrm{mad}(G) < 2 + \frac{4\Delta(G)-8}{5\Delta(G)+2}$, then $\chi_\ell(G^2) = \Delta(G) + 1$. Thus when $\mathrm{mad}(G)$ is sufficiently small compared to $\Delta(G)$, the trivial lower bounds on $\chi(G^2)$ and $\chi_\ell(G^2)$ are tight. With a similar but shorter proof, Bonamy, Lévêque, and Pinlou [35] proved the less precise statement that for each positive ϵ, there exists k_ϵ such that $\chi_\ell(G^2) = \Delta(G)+1$ for $\Delta(G) \geq k_\epsilon$ when $\mathrm{mad}(G) < \frac{14}{5} - \epsilon$.

Even for planar graphs and ordinary coloring, $\mathrm{mad}(G) < 4$ does not yield $\chi(G) \leq \Delta(G^2) + c$ for any constant c. Note that girth 4 implies $\mathrm{mad}(G) < 4$ when G is planar. Consider the 3-vertex multigraph in which each pair has multiplicity k; this is called the *fat triangle*. Subdividing each edge once yields a planar graph with girth 4 and maximum degree $2k$ whose square has chromatic number $3k$ (see Figure 6). Nevertheless, [35] obtained a function c such that if $\mathrm{mad}(G) < 4 - \epsilon$, then $\chi_\ell(G^2) \leq \Delta(G) + c(\epsilon)$.

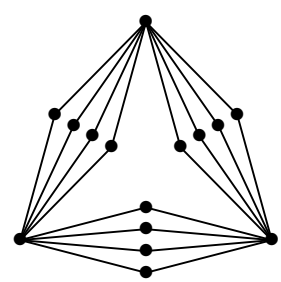

Figure 6: Construction with girth 4 and $\chi(G^2) = 3k$ (here $k = 4$)

When G is planar, larger girth restricts $\mathrm{mad}(G)$ more tightly. Motivated by the subdivided fat triangle, Wang and Lih [233] conjectured that for $g \geq 5$, there exists k_g such that $\Delta(G) \geq k_g$ implies $\chi(G^2) = \Delta(G) + 1$ when G is a planar graph with girth at least g. The

conjecture is false for $g \in \{5,6\}$; [50] and [100] both contain infinite sequences of planar graphs with girth 6, growing maximum degree, and $\chi(G^2) = \Delta(G) + 2$.

However, the Wang–Lih Conjecture holds and can be strengthened to list coloring when $g \geq 7$. Ivanova [137] proved $\chi_\ell(G^2) = \Delta(G) + 1$ for planar G having girth at least 7 and $\Delta(G) \geq 16$ (improving on $\Delta(G) \geq 30$ from [50]), and she also showed that the thresholds $10, 6, 5$ on $\Delta(G)$ are sufficient when G has girth at least $8, 10, 12$, respectively.

For girth 6, Dvořák, Král', Nejedlý, and Škrekovski [100] proved $\chi(G^2) \leq \Delta(G) + 2$ for planar G with $\Delta(G) \geq 8821$ (they also conjectured $\chi(G^2) \leq \Delta(G) + 2$ for girth 5 when $\Delta(G)$ is large enough). For girth 6, Borodin and Ivanova [54] improved $\Delta(G) \geq 8821$ to $\Delta(G) \geq 18$; they also showed that $\Delta(G) \geq 24$ yields $\chi_\ell(G^2) \leq \Delta(G) + 2$ [55].

Bonamy, Lévêque, and Pinlou [34] proved $\chi_\ell(G^2) \leq \Delta(G) + 2$ when $\Delta(G) \geq 17$ and $\mathrm{mad}(G) < 3$, regardless of planarity. As we have noted, $\mathrm{mad}(G) < \frac{2g}{g-2}$ when G is a planar graph with girth at least g, so the result of [34] is *stronger* than saying that $\chi_\ell(G^2) \leq \Delta(G)+2$ for all planar graphs with girth at least 6 and maximum degree at least 17.

Now consider again the result of Cranston and Škrekovski [95]. Reducing the bound on $\mathrm{mad}(G)$ from 3 to $2 + \frac{4\Delta(G)-8}{5\Delta(G)+2}$ yields $\chi_\ell(G^2) = \Delta(G) + 1$ rather than $\chi_\ell(G^2) \leq \Delta(G) + 2$, even for the larger family where $\Delta(G) \geq 6$. Furthermore, as $\Delta(G)$ grows, the needed bound on $\mathrm{mad}(G)$ tends to $\frac{14}{5}$, which is the bound guaranteed for planar graphs with girth at least 7. Hence it seems plausible that $\chi_\ell(G^2) = \Delta(G) + 1$ for planar graphs with girth at least 7 even when $\Delta(G) \geq 6$. For fuller understanding of this parameter, we suggest a problem.

Problem 3.5. Among the family of graphs such that $\Delta(G) \geq k$, what is the largest value $b_{j,k}$ such that $\mathrm{mad}(G) < b_{j,k}$ implies $\chi_\ell(G^2) \leq \Delta(G) + j$?

Next we weaken the requirements. A coloring where vertices at distance 2 have distinct colors but adjacent vertices need not is an *injective coloring* (the coloring is injective on each vertex neighborhood). The *injective chromatic number*, written $\chi^i(G)$, is the minimum number of colors needed, and the *injective choice number*, $\chi^i_\ell(G)$, is the least k such that G has an injective L-coloring when L is any k-uniform list assignment.

From the definition, always $\chi^i(G) \leq \chi(G^2)$ and $\chi^i_\ell(G) \leq \chi_\ell(G^2)$. The trivial lower bound on $\chi^i(G)$ is $\Delta(G)$ rather than $\Delta(G) + 1$. We seek results like those above, with a bound on $\chi^i(G)$ or $\chi^i_\ell(G)$ that is one less than the corresponding bound for $\chi(G^2)$ or $\chi_\ell(G^2)$. Again when $\mathrm{mad}(G)$ is small relative to $\Delta(G)$, the value is close to the lower bound. In [34], for example, it is noted that the proof there also yields $\chi^i_\ell(G) \leq \Delta(G) + 1$ when $\Delta(G) \geq 17$ and $\mathrm{mad}(G) < 3$. Similarly, the proof in [95] yields $\chi^i_\ell(G) = \Delta(G)$ under the conditions there.

Nevertheless, the analogue of Problem 3.5 for injective coloring remains largely open. When $j = 0$, rather tight bounds on $\mathrm{mad}(G)$ suffice. Cranston, Kim, and Yu [93] proved that $\chi^i(G) = \Delta(G)$ when $\mathrm{mad}(G) < \frac{42}{19}$ and $\Delta(G) \geq 3$. Sharpness is not known, even for $\Delta(G) = 3$. Subdividing one edge of K_4 yields a graph H such that $\chi'(H) > \Delta(H)$, and then subdividing every edge of H yields a bipartite graph G such that $\chi^i(G) > \Delta(G)$ and

$\mathrm{mad}(G) = \frac{7}{3}$. The largest b such that $\mathrm{mad}(G) < b$ implies $\chi^i(G) = \Delta(G)$ when $\Delta(G) = 3$ is not known; it is at least $\frac{42}{19}$ and at most $\frac{7}{3}$.

To yield $\chi^i_\ell(G) \le \Delta(G) + 1$, it suffices to have $\mathrm{mad}(G) \le \frac{5}{2}$ when $\Delta(G) \ge 3$ [93]. For $\Delta(G) \ge 4$ this is fairly easy (it uses Exercise 2.4); for $\Delta(G) \ge 6$ it follows from [95].

To yield $\chi^i_\ell(G) \le \Delta(G) + 2$, it suffices to have $\mathrm{mad}(G) < \frac{36}{13}$ when $\Delta(G) = 3$ [94]; we will see that this is sharp. For $\Delta(G) \ge 4$, it suffices to have $\mathrm{mad}(G) < \frac{14}{5}$ [94]; the cases $\Delta(G) \in \{4, 5\}$ are difficult, and sharpness is not known. Note that when $\Delta(G) \ge 4$ the allowed values of $\mathrm{mad}(G)$ are larger than when $\Delta(G) = 3$; the loosest condition on $\mathrm{mad}(G)$ that suffices for a given bound on $\chi^i(G) - \Delta(G)$ should grow (somewhat) as $\Delta(G)$ grows.

We use one of these results to further explore how discharging arguments are found. In the discharging process, charge may travel distance 2.

Theorem 3.6. ([94]) *If* $\Delta(G) \le 3$ *and* $\mathrm{mad}(G) < \frac{36}{13}$, *then* $\chi^i_\ell(G) \le 5$.

Proof. We present the discharging argument and leave the reducibility of the configurations in the resulting unavoidable set to Exercise 3.8. We claim that every graph G with $\Delta(G) = 3$ and $\overline{d}(G) < \frac{36}{13}$ contains one of the following configurations: a 1^--vertex, adjacent 2-vertices, a 3-vertex with two 2-neighbors, or adjacent 3-vertices each having a 2-neighbor.

If none of these configurations occurs, then $\delta(G) \ge 2$. With initial charge equal to degree, only 2-vertices need charge; all other vertices are 3-vertices. A way to allow 2-vertices to reach charge $\frac{36}{13}$ without taking too much from 3-vertices is as follows:

(R1) Every 2-vertex takes $\frac{3}{13}$ from each neighbor.
(R2) Every 2-vertex takes $\frac{1}{13}$ via each path of length 2 from a 3-vertex.

Each 3-vertex v having a 2-neighbor gives it $\frac{3}{13}$. Since no two 2-vertices are adjacent, and adjacent 3-vertices cannot both have 2-neighbors, v loses no other charge. Each 3-vertex w having no 2-neighbor loses at most $\frac{1}{13}$ along each incident edge, because its 3-neighbors do not have two 2-neighbors. (Under (R2), a 3-vertex opposite a 2-vertex x on a 4-cycle gives $\frac{2}{13}$ to x.) Thus every 3-vertex ends with charge at least $\frac{36}{13}$.

A 2-vertex gains $\frac{3}{13}$ from each neighbor, and it also gains $\frac{1}{13}$ along each of the two other edges incident to each neighbor (see Figure 7). Hence it gains $\frac{10}{13}$ and reaches charge $\frac{36}{13}$. (With no adjacent 3-vertices having 2-neighbors, no 2-vertex lies on a triangle.)

We have shown that $\overline{d}(G) \ge \frac{36}{13}$ when the specified configurations do not occur. □

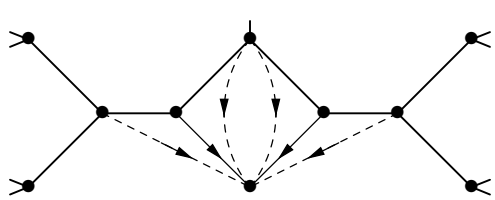

Figure 7: Discharging rules for Theorem 3.6; dashes move $\frac{1}{13}$

Remark 3.7. The proof of Theorem 3.6 allows every vertex to end with charge exactly $\frac{36}{13}$. This can happen, making the structure theorem sharp. In fact, here also the coloring result is sharp. Deleting one vertex from the Heawood graph (the incidence graph of the Fano plane) yields a graph H with $\overline{d}(H) = \frac{36}{13}$, $\Delta(H) = 3$, and $\chi^i(G) = 6$.

The discharging rules in Theorem 3.6 follow naturally from the bound on $\mathrm{mad}(G)$ and the forbidden configurations, but how are those found? To discover the structure theorem, first study the coloring problem to find reducible configurations. A 1^--vertex and two adjacent 2-vertices are easy to show reducible. With a bit more thought, a 3-vertex with two 2-neighbors is reducible. These configurations form an unavoidable set for $\mathrm{mad}(G) < \frac{8}{3}$, using the discharging rule that each 2-vertex takes $\frac{1}{3}$ from each neighbor. That yields the desired conclusion when $\mathrm{mad}(G) < \frac{8}{3}$, but we can do better.

After adding the reducible configuration consisting of two adjacent 3-vertices having 2-neighbors, we seek the loosest bound on $\mathrm{mad}(G)$ under which this larger set is unavoidable. It will exceed $\frac{8}{3}$. The 2-vertices can take charge only from 3-vertices, but when $\mathrm{mad}(G) > \frac{8}{3}$ their neighbors cannot afford to give enough to satisfy them. When two adjacent 3-vertices with 2-neighbors are forbidden, the 2-vertices can also gain charge along paths of length 2.

Now we have the "avenues" of discharging. Let each 2-vertex take a from each neighbor and b along each path of length 2. Now 2-vertices end with $2 + 2a + 4b$, 3-vertices having 2-neighbors end with $3 - a$, and 3-vertices without 2-neighbors end with as little as $3 - 3b$. We seek a and b to maximize the minimum of $\{2+2a+4b, 3-a, 3-3b\}$. If $3-a$ and $3-3b$ are not equal, then the value can be improved, so take $a = 3b$. Now $\min\{2 + 10b, 3 - 3b\}$ is maximized when $2 + 10b = 3 - 3b$, or $b = \frac{1}{13}$. Hence the proof works when $\mathrm{mad}(G) < \frac{36}{13}$ and fails for any larger bound (as also implied by the sharpness example.).

Exercise 3.1. (Erdős–Rubin–Taylor [105], Vizing [228]) Prove that the complete bipartite graph $K_{m,m}$ is not k-choosable when $m \geq \binom{2k-1}{k}$. Determine the values of r such that $K_{k,r}$ is k-choosable.

Exercise 3.2. (Cranston–Kim [92]) Apply Exercise 2.7 to prove that if $\Delta(G) \leq 3$ and $\mathrm{mad}(G) \leq \frac{14}{5}$, then $\chi_\ell(G^2) \leq 7$.

Exercise 3.3. (Kim–Park [150]) Prove that if $\delta(G) \geq 2$ and $\overline{d}(G) < \frac{4k}{k+2}$ with $k \geq 4$, then G has a 3^--vertex with a $(k-1)^-$-neighbor. Guarantee a 2-vertex with a $(k-1)^-$-neighbor when $k \leq 6$. Conclude that if $\mathrm{mad}(G) < \frac{4k}{k+2}$ with $k \geq 4$ (and no components are 5-cycles if $k = 4$), then from any lists of size at least k a proper coloring of G can be chosen so that every vertex with degree at least 2 has neighbors with distinct colors. Show also that this is sharp: there exists G with $\mathrm{mad}(G) = \frac{4k}{k+2}$ and an assignment of k-lists from which no such coloring can be chosen.

Exercise 3.4. In Problem 3.5, prove that $b_{1,k} \geq 2$. Show that equality holds when $k \in \{2, 3\}$.

Exercise 3.5. (Cranston–Erman–Škrekovski [90]) Prove that a cycle of length divisible by 3 with vertices whose degrees cycle repeatedly through $2, 2, 3$ is reducible for 5-choosability of G^2. Use discharging to conclude that if $\Delta(G) \leq 4$ and $\mathrm{mad}(G) < 16/7$. then $\chi_\ell(G^2) \leq 5$.

Exercise 3.6. (Cranston–Erman–Škrekovski [90]) Prove that if $\Delta(G) \le 4$ and $\overline{d}(G) < \frac{18}{7}$, then G contains one of: (C1) a 1^--vertex, (C2) two adjacent 2-vertices, (C3) a 3-vertex with three 2-neighbors, or (C4) a four-vertex path alternating between 2-vertices and 3-vertices. Conclude that if $\Delta(G) \le 4$ and $\mathrm{mad}(G) < \frac{18}{7}$, then $\chi_\ell(G^2) \le 7$.

Exercise 3.7. (Cranston–Erman–Škrekovski [90]) Prove that if $\Delta(G) \le 4$ and $\overline{d}(G) \le \frac{10}{3}$, then G contains one of: (C1) a 1^--vertex, (C2) a 2-vertex with a 3^--neighbor, (C3) a 3-vertex with two 3-neighbors, or (C4) a 4-vertex with a 2-neighbor and a 3^--neighbor. Construct infinitely many graphs with average degree $\frac{10}{3}$ and maximum degree 4 that contain no such configuration. Prove that if $\Delta(G) \le 4$ and $\mathrm{mad}(G) < \frac{10}{3}$, then $\chi_l(G^2) \le 12$.

Exercise 3.8. (Cranston–Kim–Yu [94]) Complete the proof of Theorem 3.6 by showing that those configurations are reducible for $\chi^i(G) \le 5$ in the family of graphs with $\Delta(G) \le 3$.

Exercise 3.9. (Cranston–Kim–Yu [94]) Prove that if $\overline{d}(G) < \frac{14}{5}$ and $\Delta(G) \ge 6$, then G contains one of the following configurations: (C1) a 1^--vertex, (C2) adjacent 2-vertices, (C3) a 3-vertex with neighbors of degrees $2, a, b$, where $a + b \le \Delta(G) + 2$, or (C4) a 4-vertex having four 2-neighbors, one of which has other neighbor of degree less than $\Delta(G)$. Argue that none of these configurations can appear in a minimal graph G such that $\Delta(G) \ge 6$ and $\chi^i(G) > \Delta(G) + 2$. Reducibility of the first two configurations and part of (C3) is already requested in Exercise 3.8.

4 Edge-coloring and List Edge-coloring

A *proper edge-coloring* assigns colors to the edges of a graph G so that incident edges receive distinct colors. The *edge-chromatic number*, written $\chi'(G)$, is the minimum number of colors in such a coloring. Since the colors on edges with a common endpoint must be distinct, always $\chi'(G) \ge \Delta(G)$. Vizing [224, 226] and Gupta [126] proved one of the most famous theorems in graph coloring. It gives an upper bound for $\chi'(G)$ when G is a multigraph (allowing multiedges) and specializes to the following for graphs (no loops or multiedges).

Theorem 4.1 (Vizing's Theorem)**.** *If G is a graph, then $\Delta(G) \le \chi'(G) \le \Delta(G) + 1$.*

Recognition of $\chi'(G) = \Delta(G)$ is NP-complete, so we seek sufficient conditions for equality.

Conjecture 4.2 (Vizing's Planar Graph Conjecture [225, 227])**.** *If G is a planar graph and $\Delta(G) \ge 6$, then $\chi'(G) = \Delta(G)$.*

Both conditions in Vizing's Conjecture are needed. The complete graph K_7 is 6-regular but not planar. Each color can be used on at most three edges, so $\chi'(K_7) \ge \frac{21}{3} = 7$. Similarly, obtain G from a 5-regular planar graph with $2k$ vertices by subdividing one edge. Since G has $5k + 1$ edges, and at most k edges can receive the same color, $\chi'(G) \ge 6$. This difficulty does not arise for $\Delta(G) \ge 6$, because regular planar graphs have degree at most 5.

Vizing [225] proved Conjecture 4.2 for $\Delta(G) \ge 8$, using Vizing's Adjacency Lemma (VAL). It is common to say that G is *Class 1* if $\chi'(G) = \Delta(G)$, *Class 2* otherwise. An

edge-critical graph G is then a Class 2 graph such that $\chi'(G-e) = \Delta(G)$ for all $e \in E(G)$. In fact, VAL implies that every edge-critical graph has at least three vertices of maximum degree, so $\Delta(G) = \Delta(G-e)$. Note also that every Class 2 graph contains an edge-critical graph with the same maximum degree.

Theorem 4.3 (Vizing's Adjacency Lemma [225])**.** *If x and y are adjacent in an edge-critical graph G, then at least* $\max\{1+\Delta(G)-d(y), 2\}$ *neighbors of x have degree $\Delta(G)$.*

Using VAL, Vizing proved the conjecture for $\Delta(G) \geq 8$ via counting arguments about vertices of various degrees. The proof is clearer in the language of discharging, which was not then in use. Luo and Zhang [174] used VAL and discharging to prove $\chi'(G) = \Delta(G)$ for the larger family of graphs G with $\mathrm{mad}(G) \leq 6$ and $\Delta(G) \geq 8$. We present a slightly simpler proof of a slightly weaker result, requiring $\mathrm{mad}(G) < 6$. In fact, Miao and Sun [175] proved $\chi'(G) = \Delta(G)$ also when $\Delta(G) \geq 8$ and $\mathrm{mad}(G) < \frac{13}{2}$. Their result (and that of [174]) uses additional adjacency lemmas. Here VAL takes the place of reducibility arguments.

Theorem 4.4 ([174])**.** *If G is a graph with* $\mathrm{mad}(G) < 6$ *and* $\Delta(G) \geq 8$*, then* $\chi'(G) = \Delta(G)$.

Proof. Let G be a minimal counterexample, and let $k = \Delta(G)$. Since $\chi'(G) > k$ requires an edge-critical subgraph with the same maximum degree, we may assume that G is edge-critical. Since VAL gives each vertex at least two k-neighbors, $\delta(G) \geq 2$. We use discharging with initial charge $d(v)$; it suffices to show that each vertex ends with charge at least 6.

(R1) If $d(v) \leq 4$, then v takes $\frac{6-d(v)}{d(v)}$ from each neighbor.
(R2) If $d(v) \in \{5, 6\}$, then v takes $\frac{1}{4}$ from each 6^+-neighbor.

For $v \in V(G)$, let j be the least degree among vertices in $N_G(v)$. If $j < k$, then v has at least $k+1-j$ neighbors of degree k, by VAL. Hence $k+1-j \leq d(v)-1$, which yields $j \geq 10 - d(v)$ since $k \geq 8$. Note that 7^+-vertices take no charge.

If $d(v) \leq 4$, then $j \geq 6$, so v loses no charge, and (R1) sends enough to make v happy.

If $d(v) = 5$, then $j \geq 5$. Furthermore, $j = 5$ yields $k-4$ neighbors with degree k. Since $k \geq 8$, charge at least $4(\frac{1}{4})$ comes to v, no charge is given away, and v is happy.

The remaining cases are all similar but require individual checking. We show representative cases in Figure 8.

If $d(v) = 6$, then $j \geq 4$. At most $j-3$ neighbors have degree less than k. For $j \in \{4, 5, 6\}$, v gives at most $\frac{2}{4}, \frac{2}{4}, \frac{3}{4}$ and receives at least $\frac{5}{4}, \frac{4}{4}, \frac{6}{4}$, respectively, ending happy.

If $d(v) = 7$, then $j \geq 3$. At most $j-2$ neighbors have degree less than k. For $j \in \{3, 4, 5, 6\}$, v gives at most $\frac{3}{3}, \frac{4}{4}, \frac{3}{4}, \frac{4}{4}$, respectively, and remains happy.

If $d(v) \geq 8$, then $j \geq 2$. At most $j-1$ neighbors have degree less than k. For $j \in \{2, 3, 4, 5, 6\}$, v gives at most $\frac{4}{2}, \frac{6}{3}, \frac{6}{4}, \frac{4}{4}, \frac{5}{4}$, respectively, and remains happy. $\square$

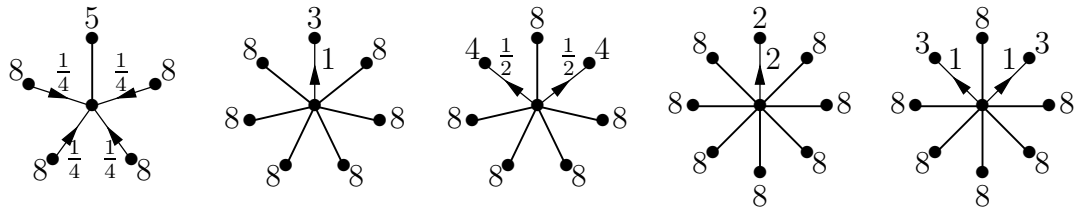

Figure 8: Some cases in Theorem 4.4 ending with 0 charge

Sanders and Zhao [203] showed that Conjecture 4.2 also holds for planar graphs when $\Delta(G) = 7$. In [205], they proved $\chi'(G) = \Delta(G)$ for graphs with maximum degree at least 7 that embed in a surface of nonnegative Euler characteristic. Since the proof above uses only $\mathrm{mad}(G) < 6$, it holds also for graphs in the projective plane. Graphs on the torus (or Klein bottle) also satisfy $\mathrm{mad}(G) < 6$ unless they triangulate the surface, in which case $\overline{d}(G) = 6$.

Although Conjecture 4.2 remains open when $\Delta(G) = 6$, it has been proved for various classes of planar graphs with certain subgraphs forbidden, such as short cycles with chords (see [72, 232, 240]). Note that $\mathrm{mad}(G) < 6$ is not sufficient when $\Delta(G) = 6$; planarity really is needed. Although K_7 is forbidden by $\mathrm{mad}(G) < 6$, consider the graph G obtained from K_7 by subdividing one edge with a new 2-vertex v; we have $\Delta(G) = 6$ and $\mathrm{mad}(G) < 6$. In a proper edge-coloring of G, only two colors can appear four times (using edges at v); hence six colors can cover only 20 edges, but G has 22 edges.

We next consider coloring both vertices and edges.

Definition 4.5. A *total coloring* of a graph G is a coloring of $V(G) \cup E(G)$ such that any two adjacent or incident elements have distinct colors. The *total chromatic number* $\chi''(G)$ is the minimum number of edges in a *total coloring* of G.

In total coloring, the edges cause the most difficulty; already a proper edge-coloring of a simple graph G needs $\Delta(G)$ or $\Delta(G) + 1$ colors. Behzad [30] and Vizing independently conjectured an upper bound; see [216, p. 261–262] for the history.

Conjecture 4.6 (Total Coloring Conjecture)**.** *If G is a simple graph, then $\chi''(G) \le \Delta(G)+2$.*

In Lemma 3.2, we showed that even cycles are 2-choosable. Since cycles are isomorphic to their line graphs, we can also choose a proper edge-coloring from lists of size 2 on the edges. This yields the Total Coloring Conjecture for 4-colorable Class 1 graphs.

Theorem 4.7. *If $\chi(G) \le 4$ and $\chi'(G) = \Delta(G)$, then $\chi''(G) \le \Delta(G) + 2$.*

Proof. Color $V(G)$ properly using colors $1, 2, 3, 4$. Use colors $3, \ldots, \Delta(G) + 2$ to form a proper edge-coloring of G. Uncolor all edges having colors 3 and 4; we will recolor them. We claim that on each such edge, at least two colors among $\{1, 2, 3, 4\}$ are available for use.

The colored edges exclude no colors, because now none of them have any color in $\{1, 2, 3, 4\}$. The only vertices that can exclude a color from use on an edge are its endpoints.

Now lists of size 2 are available on the edges. The graph consisting of edges that had colors 3 or 4 has maximum degree at most 2, so it consists of paths and even cycles, and a proper edge-coloring can be chosen from the lists. □

Using Theorem 4.7, the results on Vizing's Planar Graph Conjecture, and the 4-colorability of planar graphs (discussed briefly in Section 5), the Total Coloring Conjecture has almost been proved for planar graphs. The cases with $\Delta(G) \le 5$ were proved by ad hoc arguments. Since $\chi'(G) = \Delta(G)$ when G is planar with $\Delta(G) \ge 7$, Theorem 4.7 applies in that case. Thus only the case $\Delta(G) = 6$ remains unsettled for planar graphs. The bound is sharp, since $\chi''(K_4) = 5$. However, Borodin [39] proved $\chi''(G) = \Delta(G) + 1$ when G is a simple planar graph with $\Delta(G) \ge 14$, equaling the trivial lower bound (see Exercise 5.6).

Proper edge-coloring of G is equivalent to proper coloring of the line graph $L(G)$. Since the line graph has a clique of size $\Delta(G)$, Vizing's Theorem states that the optimization problem of proper coloring behaves much better when restricted to line graphs. The same phenomenon seems to occur with the list version of the problem.

Definition 4.8. An *edge-list assignment* L assigns lists of available colors to the edges of a graph G. Given an edge-list assignment L, an *L-edge-coloring* of G is a proper edge-coloring ϕ such that $\phi(e) \in L(e)$ for all $e \in E(G)$. A graph G is *k-edge-choosable* if G is L-edge-colorable whenever each list has size at least k. The *list edge-chromatic number* of G, written $\chi'_\ell(G)$, is the least k such that G is k-edge-choosable.

Conjecture 4.9 (List Coloring Conjecture)**.** *$\chi'_\ell(G) = \chi'(G)$ for every graph G.*

This conjecture was posed independently by many researchers. It was first published by Bollobás and Harris [32], but it was independently formulated earlier by Albertson and Collins in 1981 and by Vizing as early as 1975 (both unpublished). Vizing also posed the weaker conjecture that always $\chi'_\ell(G) \le \Delta(G) + 1$. Kahn [144] proved the conjecture asymptotically: $\chi'_\ell(G) \le (1 + o(1))\chi'(G)$.

The List Coloring Conjecture was proved for bipartite multigraphs by Galvin [119], showing that $\chi'_\ell(G) = \Delta(G)$. Vizing [226, 227] conjectured that $\chi'(G) = \Delta(G)$ when $\mathrm{mad}(G) < \Delta(G) - 1$. Based on the List Coloring Conjecture, Woodall [246] conjectured that $\mathrm{mad}(G) < \Delta(G) - 1$ also implies $\chi'_\ell(G) = \Delta(G)$. In this direction, it is known that $\chi'_\ell(G) = \Delta(G)$ when $\mathrm{mad}(G) < \sqrt{2\Delta(G)}$. The result is implicit in [67], using the tool below. Woodall [246] reinterpreted the argument in the language of discharging.

Theorem 4.10 (Borodin–Kostochka–Woodall [67])**.** *If L is an edge-list assignment on a bipartite multigraph G such that $|L(uv)| \ge \max\{d_G(u), d_G(v)\}$ for all $uv \in E(G)$, then G has an L-edge-coloring.*

Woodall [246] introduced an exciting new technique of moving charge in successive stages rather than all at once. When the average degree is large, vertices with very small degree need a lot of charge. It may be too complicated to specify exactly where it all comes from. Hence he allows charge to shift in phases, which we call "iterated discharging".

We start with the reducible configurations. The *weight* of an edge in G (or of any subgraph H) is the sum of the degrees (in G) of its vertices. When the weight of a subgraph satisfies a desired bound, we say that the subgraph is *light* in G.

Proposition 4.11. *Edges of weight at most $k+1$ are reducible for k-edge-choosability.*

Proof. A light edge e is incident to at most $k-1$ edges. If $G-e$ is k-edge-choosable and $|L(e)| \geq k$, then an L-edge-coloring of $G-e$ extends to an L-edge-coloring of G. □

For the other reducible configuration, we need a definition.

Definition 4.12. In a multigraph G, an *i-alternating subgraph* is a bipartite submultigraph F with partite sets U and W such that $d_F(u) = d_G(u) \leq i$ when $u \in U$ and $d_G(w) - d_F(w) \leq \Delta(G) - i$ when $w \in W$. Note that cycles in F alternate between W and i^--vertices in U.

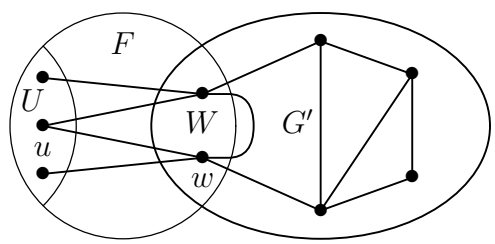

Figure 9: A 2-alternating subgraph F

Lemma 4.13 ([67, 246])**.** *i-alternating subgraphs are reducible for the property that edge-choosability equals maximum degree.*

Proof. Let L be a $\Delta(G)$-uniform edge-list assignment for such a multigraph G. Let F be an i-alternating subgraph of G, and let $G' = G - E(F)$ (see Figure 9). Choose an L-edge-coloring of G', and delete the chosen colors from the lists of their incident edges in F. We claim that the lists remain large enough to apply Theorem 4.10 to F.

For $uw \in E(F)$, no colors have been lost to edges incident at u, since all edges incident to u lie in F. The number of colors lost to edges incident to w, by definition, is at most $d_G(w) - d_F(w)$. Since $d_G(w) \leq \Delta(G)$, the list on uw retains at least $d_F(w)$ colors. Also $d_F(u) \leq i \leq \Delta(G) - (d_G(w) - d_F(w))$, so the list on uw also retains at least $d_F(u)$ colors. Now Theorem 4.10 applies to complete the L-edge-coloring of G. □

To avoid technicalities, we make the bound on $\mathrm{mad}(G)$ slightly tighter than needed.

Theorem 4.14 ([67, 246])**.** *If* $\mathrm{mad}(G) \le \sqrt{2\Delta(G)} - 1$*, then* $\chi'_\ell(G) = \Delta(G)$*.*

Proof. Let $x = \sqrt{2\Delta(G)} - 1$. It suffices to show that every graph G with average degree at most x contains a edge with weight at most $\Delta(G) + 1$ or an i-alternating subgraph with $i \le x$. Suppose that G contains neither. An edge incident to a 1-vertex would be light, so we may assume $\delta(G) \ge 2$.

Give each vertex initial charge equal to its degree. In phase i of discharging, for $2 \le i \le \lfloor x \rfloor$, each i^--vertex receives charge 1 from a neighbor. We design these phases so that every vertex ends with charge at least $\lceil x \rceil$.

To begin phase i, let $U = \{v\colon d_G(v) \le i\}$, and let W be the set of all vertices having neighbors in U. Note that U is independent (no light edge). Let F be the subgraph with vertex set $U \cup W$ containing all edges incident to U. Since G has no i-alternating subgraph, there exists $w \in W$ such that $d_F(w) \le d_G(w) + i - \Delta(G) - 1 \le i - 1$. Move charge 1 from this vertex w to each of its neighbors in U.

Now delete $\{w\} \cup (N(w) \cap U)$ from F. Each deleted vertex in U has received charge 1, and w lost at most $i - 1$. Iterate. What remains of U and W at each step cannot form an i-alternating subgraph, so we continue to find the desired vertex until U is empty.

Since each vertex with degree at most i receives a unit of charge in phase i, vertices with degree less than $\sqrt{2\Delta(G)}$ have their charge increased to at least $\sqrt{2\Delta(G)}$ (and they never lose charge). Since there is no light edge, vertices with larger degree j lose charge only on rounds i with $i \ge \Delta(G)+2-j$. Hence such a vertex loses charge at most $\sum_{i=\Delta(G)+2-j}^{\lfloor x \rfloor}(i-1)$. With each reduction of 1 in j, the amount of lost charge declines by more than 1, so it suffices to show that vertices with degree $\Delta(G)$ keep sufficient charge. Their lost charge is bounded by $\frac{1}{2}x(x-1)$, so they keep charge at least $\frac{3}{2}\sqrt{2\Delta(G)} - 1$, which is more than enough. $\square$

Woodall also gave an example to show that the discharging argument is essentially sharp, meaning that more reducible configurations will be needed to weaken the hypothesis on $\mathrm{mad}(G)$. By using VAL and further adjacency lemmas to study the relationship between criticality for Class 1 and $\mathrm{mad}(G)$, Sanders and Zhao [204] proved that $\mathrm{mad}(G) < \frac{1}{2}\Delta(G)$ suffices to make G Class 1, and Woodall [245] proved that $\mathrm{mad}(G) < \frac{2}{3}\Delta(G)$ suffices. The list version seems to be much harder, and the more restrictive requirement of $\mathrm{mad}(G) < \sqrt{2\Delta(G)}$ in Theorem 4.14 is a first step.

Exercise 4.1. Let G be a graph with maximum degree at least 8 that embeds on the torus. By a closer examination of the proof of Theorem 4.4, prove that $\chi'(G) = \Delta(G)$ except possibly when G is obtained from a 6-regular triangulation H of the torus by inserting vertices of degree 3 into one-third of the faces in H, chosen so that each vertex in H lies on exactly two of the chosen faces, and making each new vertex adjacent to the vertices of H on its face. It suffices to show that otherwise every vertex ends with charge at least 6 and some vertex ends with larger charge.

Exercise 4.2. Prove that if $\Delta(G) \le 6$ and $\overline{d}(G) < \frac{7}{2}$, then G contains an isolated vertex, an edge with weight at most 7, or a cycle alternating between 2-vertices and 6-vertices. Conclude that if $\Delta(G) \le 6$ and $\mathrm{mad}(G) < \frac{7}{2}$, then G is 6-edge-choosable and 7-total-choosable.

Exercise 4.3. (Borodin–Kostochka–Woodall [67]) Adapt the proofs of Lemma 4.13 and Theorem 4.14 to show that if $\mathrm{mad}(G) \le \sqrt{2\Delta(G)} - 1$, then $\chi''_\ell(G) = \Delta(G) + 1$.

5 Planar Graphs and Coloring

The Discharging Method was developed in the study of planar graphs. Although many results on planar graphs (especially with girth restrictions) extend to all graphs satisfying the resulting bound on $\mathrm{mad}(G)$, in this section we return to the historical roots and emphasize results where planarity is needed. Since planarity is very restrictive, for planar graphs one can sometimes prove stronger results than hold for $\mathrm{mad}(G) < 6$. As examples, we first pursue the problems discussed in Section 4 (other examples appear in the exercises).

Borodin [41] confirmed the List Coloring Conjecture for planar graphs with large maximum degree, proving that $\chi'_\ell(G) = \Delta(G)$ when $\Delta(G) \ge 14$. This later was strengthened to $\Delta(G) \ge 12$ by Borodin, Kostochka, and Woodall [67]. Borodin [41] also confirmed Vizing's weaker conjecture in the broader class of planar graphs with $\Delta(G) \ge 9$, showing that then $\chi'_\ell(G) \le \Delta(G) + 1$. Bonamy [33] obtained this conclusion also for $\Delta(G) = 8$, by a much longer proof using 11 reducible configurations.

The bound $\Delta(G) + 1$ has also been proved for planar graphs with $\Delta(G) \ge 6$ having no two 3-faces sharing an edge [89]. It was proved in [143] for $\Delta(G) \le 4$ (including nonplanar graphs), and when $\Delta(G) = 5$ it is known for planar graphs with no 3-cycle [255], no 4-cycle [89], or no 5-cycle [233]. The proofs for $\Delta(G) = 5$ all use discharging.

The distinctive feature of discharging for planar graphs is that charge can also be assigned to faces, which are vertices in the dual graph. The dual graph G^* is also planar, so $\mathrm{mad}(G^*) < 6$ and we can use discharging on both G and G^*. Using their interaction is more effective and leads to three common (and natural) ways to assign charge on planar graphs.

Proposition 5.1. *Let $V(G)$ and $F(G)$ be the sets of vertices and faces in a plane graph G, and let $\ell(f)$ denote the length of a face f. The following equalities hold for G.*

$$\begin{array}{ll} \sum_{v\in V(G)}(d(v)-6) + \sum_{f\in F(G)}(2\ell(f)-6) = -12 & \text{vertex charging} \\ \sum_{v\in V(G)}(2d(v)-6) + \sum_{f\in F(G)}(\ell(f)-6) = -12 & \text{face charging} \\ \sum_{v\in V(G)}(d(v)-4) + \sum_{f\in F(G)}(\ell(f)-4) = -8 & \text{balanced charging} \end{array}$$

Proof. Euler's Formula for connected planar graphs is $n - m + p = 2$, where n, m, and p count the vertices, edges, and faces ("points" in the dual). Multiply Euler's Formula by -6 or -4 and split the term for edges to obtain the three formulas below.

$$-6n + 2m + 4m - 6p = -12; \quad -6n + 4m + 2m - 6f = -12; \quad -4n + 2m + 2m - 4f = -8.$$

Substitute $\frac{1}{2}\sum_{v\in V(G)} d(v)$ for the first occurrence of m and $\frac{1}{2}\sum_{f\in F(G)} \ell(f)$ for the second in each equation, and then collect the contributions by vertices and by faces. $\square$

Here the initial charges assigned to vertices or faces are not the degree or length, but rather an adjustment of those quantities that uses the interaction between vertices and faces. Setting charge to degree is in some sense more intuitive, but this adjustment facilitates using the dual graph also. A vertex or face now is "happy" when it reaches nonnegative charge. When specified configurations are assumed not to occur, making every vertex and face happy provides a contradiction in the same way as when charges are not shifted.

For triangulations, such as in the Four Color Problem, vertex charging is appropriate. All the faces have charge 0, and often they can be ignored. For 3-regular planar graphs, face charging is appropriate, with each vertex given charge 0. With balanced charging when G and its dual G^* are simple, 3-vertices and 3-faces are the only objects needing charge; those with degree or length at least 5 have spare charge to give away.

We begin with the recent use of balanced charging to prove the result of Borodin [41] on Vizing's List Edge-Coloring Conjecture for planar graphs with large degree. Balanced charging is natural when neither the graphs nor their duals are triangulations. An interesting aspect of the proof is a reservoir or "pot" of charge that can flow to or from vertices or faces without regard to their location. In this proof, the pot facilitates moving charge from maximum-degree vertices to 3-vertices; we need not name specific recipients.

Theorem 5.2 ([41]). *If G is a planar graph and $\Delta(G) \ge 9$, then $\chi'_\ell(G) \le \Delta(G) + 1$.*

Proof. (Cohen and Havet [84]) Let G be a minimal counterexample, with an edge-list assignment L such that each list has size $\Delta(G) + 1$ and G has no L-edge-coloring. An edge with weight at most $\Delta(G) + 2$ is reducible, by Proposition 4.11. Hence we may assume that $\delta(G) \ge 3$ and that every neighbor of a j-vertex has degree at least $\Delta(G) + 3 - j$. Let $k = \Delta(G)$; since $k \ge 9$, the degree-sum of any two adjacent vertices is at least 12.

We use balanced charging, with initial charge equal to degree or length minus 4. Initially, the pot of charge is empty. The discharging rules must make each vertex and face happy and keep the charge in the pot nonnegative to contradict the assumption of a counterexample.

(R1) Every 3-vertex takes 1 from the pot, and every k-vertex gives $\frac{1}{2}$ to the pot.
(R2) Each 3-face takes $\frac{1}{2}$ from each incident 8^+-vertex and $\frac{j-4}{j}$ from each incident j-vertex with $j \in \{5, 6, 7\}$.

To ensure positive charge in the pot, we prove $n_k > 2n_3$, where n_j is the number of j-vertices in G. The edges incident to 3-vertices form a bipartite graph H; its partite sets are the 3-vertices and the k-vertices. If H has a cycle C, then C has even length, since H is bipartite. By the minimality of the counterexample, $G - E(C)$ has an L-edge-coloring. Each edge of C is incident to $\Delta(G) - 1$ edges that have now been colored, so there remain at least

two available colors on each edge (see Figure 10). Since even cycles are 2-edge-choosable (by Lemma 3.2 and cycles being isomorphic to their line graphs), the L-edge-coloring extends to G. Since G is a counterexample, we thus may assume that H is acyclic and therefore has fewer than $n_3 + n_k$ edges. Since it also has $3n_3$ edges, we have $3n_3 < n_3 + n_k$, as desired.

For vertices, (R1) immediately makes 3-vertices happy. A j-vertex v with $j \in \{4, 5, 6, 7\}$ loses altogether at most $j - 4$, its initial charge. An 8-vertex loses at most 4, since $k \geq 9$. For $j \geq 9$, possibly sending $\frac{1}{2}$ to the pot, a j-vertex loses at most $\frac{j+1}{2}$ and is happy.

For faces, the 4^+-faces lose no charge and remain happy; we must show that each 3-face f gains at least 1. Let j be the least degree among vertices incident to f. If $j \leq 4$, then two incident 8^+-vertices give $\frac{1}{2}$ each. If $j = 5$, then two incident 7^+-vertices give at least $\frac{3}{7}$ each, plus $\frac{1}{5}$ for the 5-vertex. If $j \geq 6$, then each vertex incident to f gives at least $\frac{1}{3}$ to f. $\square$

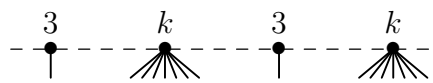

Figure 10: Excluded cycles in Theorem 5.2

This proof fits the model of discharging to produce an unavoidable set of reducible configurations. The reducible configurations are light edges (degree-sum at most $\Delta(G) + 2$) and cycles alternating between 3-vertices and $\Delta(G)$-vertices. Also, the role of the pot of charge is analogous to the use of 6-vertices sponsoring 3-threads in Theorem 3.4. We used reducibility of certain cycles in that proof to show that H is acyclic, thereby guaranteeing more 6-vertices than 3-threads. A pot could then transfer extra charge from 6-vertices to 3-threads; who sponsors whom is not important. Conversely, in Theorem 5.2 the forest H can be dismantled to obtain two $\Delta(G)$-neighbors as sponsors for each 3-vertex.

Now we consider planar graphs with larger maximum degree, large enough to yield $\chi'_\ell(G) = \Delta(G)$. As mentioned earlier, the best result known is that $\Delta(G) \geq 12$ is sufficient (Borodin, Kostochka, and Woodall [67]). We present a short proof of the earlier weaker result of Borodin [41] that $\Delta(G) \geq 14$ is sufficient. The result in [67] uses similar discharging, but it requires more reducible configurations and more detailed analysis.

A *t-alternating* cycle alternates between t-vertices and vertices of higher degree. This suggested the term "i-alternating subgraph" in Section 4, which is almost a generalization.

Lemma 5.3 ([41])**.** *If G is a simple plane graph with $\delta(G) \geq 2$, then G contains*
(C1) an edge uv with $d(u) + d(v) \leq 15$, or
(C2) a 2-alternating cycle C.

Proof. In a counterexample G, we have $d(u) + d(v) \geq 16$ for every edge uv. Both neighbors of any 2-vertex are 14^+-vertices. Since G is simple, every 2-vertex lies on a 4^+-face.

To obtain a contradiction, we use face charging, with initial charge $2d(v) - 6$ at each vertex v and $\ell(f) - 6$ at each face f. We also keep a central pot of charge (initially empty) and use the following discharging rules (see Figure 11).

(R1) Each 14^+-vertex gives charge 1 to the pot, and each 2-vertex takes 1 from the pot.
(R2) Each 4^+-vertex distributes its charge remaining after (R1) equally to its incident faces.
(R3) Each 4^+-face gives charge 1 to each incident 2-vertex.

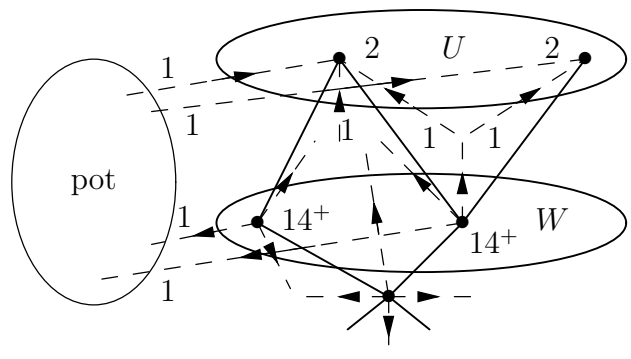

Figure 11: Discharging for Lemma 5.3

To keep the charge in the pot nonnegative, we need $|U| \le |W|$, where U and W denote the sets of 2-vertices and 14^+-vertices, respectively. Let H be the subgraph of G with vertex set $U \cup W$ and edge set consisting of all edges with endpoints in both U and W. Since (C2) does not occur in G, the components of H are trees. Also (C1) does not occur, so $2|U| = |E(H)| < |U| + |W|$. Thus $|U| < |W|$.

A 2-vertex takes 1 from the pot and 1 from an incident 4^+-face (since G is simple) and ends happy. A 3-vertex starts and ends with no charge. By (R2), a 4^+-vertex also ends with charge 0. Hence all vertices are happy.

Faces give charge to 2-vertices and take charge from 4^+-vertices. Under (R2), a face takes charge $\frac{2j-6}{j}$ or $\frac{2j-7}{j}$ from an incident j-vertex when $j \ge 4$, the latter when $j \ge 14$. Thus the value is at least $\frac{1}{2}$ when $j \ge 4$, at least 1 when $j \ge 6$, and at least $\frac{3}{2}$ when $j \ge 12$.

If a face f has no incident 3^--vertices, then it receives at least $\frac{1}{2}\ell(f)$; its final charge is at least $\frac{3}{2}\ell(f) - 6$, which is nonnegative when $\ell(f) \ge 4$. When f is a 3-face or a face incident to some 3^--vertex, let k be the least degree among the vertices incident to f. Prohibiting (C1) gives f two incident $(16-k)^+$-vertices.

A 3-face needs to receive charge at least 3. When $k \ge 6$, it receives at least 1 from each incident vertex. When $k = 2$, the other incident vertices have degree at least 14, and each provides $\frac{3}{2}$. When $3 \le k \le 5$, it receives at least $\frac{2k-6}{k} + 2 \cdot \frac{26-2k}{16-k}$, which is at least 3.

A 4^+-face f needs 2 (or maybe less) to become happy. If $k \ge 3$ or f has exactly one incident 2-vertex, then f receives at least 3 and gives away at most 1. If f has at least two

incident 2-vertices, then each is followed on f (in a consistent direction) by a 14^+-vertex, which contributes at least $\frac{3}{2}$. These pairs net at least $\frac{1}{2}$ each for f. If G has no 2-alternating cycle, then f has another incident 14^+-vertex that has not been counted, which provides more than enough charge to f. □

Theorem 5.4 ([41]). *If G is a plane graph with $\Delta(G) \geq 14$, then $\chi'_\ell(G) = \Delta(G)$.*

Proof. Let G be a minimal counterexample, having no L-edge-coloring from edge-list assignment L. If G has a 1-vertex with incident edge e, then $G - e$ has an L-edge-coloring, and it extends to e. Thus $\delta(G) \geq 2$. By Lemma 5.3, G has an edge uv with $d(u) + d(v) \leq 15$ or a 2-alternating cycle C. In the first case, we can extend an L-edge-coloring of $G - uv$, since $|L(uv)| \geq 14$ and at most 13 colors are restricted from use on uv. In the other case, by minimality $G - E(C)$ has an L-edge-coloring. Since each list has size at least $\Delta(G)$, each edge of C has at least two colors remaining available, and the 2-choosability of even cycles allows us to extend the edge-coloring. □

We close this section with a new proof of a classical result that appeared in the survey [45] with the traditional proof. Steinberg [215] conjectured that every planar graph without 4-cycles or 5-cycles is 3-colorable. Results on this family can be compared with the family where $\mathrm{mad}(G) < 4$; see Exercise 5.7.

Many papers have used discharging to prove 3-colorability under various excluded-cycle conditions. For example, Borodin et al. [51] proved that excluding cycles of lengths 4 through 7 suffices. Earlier, Borodin [43] and Sanders and Zhao [200] proved that excluding 4-cycles and faces of lengths 5 through 9 is sufficient. The traditional proof uses balanced charging, but face charging yields a somewhat simpler proof.

Lemma 5.5 ([43]). *Every plane graph G with $\delta(G) \geq 3$ has two 3-faces with a common edge, or a j-face with $4 \leq j \leq 9$, or a 10-face whose vertices all have degree 3.*

Proof. Let G be a plane graph with $\delta(G) \geq 3$ having none of the listed configurations. Use face charging: assign charge $2d(v) - 6$ to each vertex v and charge $\ell(f) - 6$ to each face f. The total charge is -12.

Since no faces have lengths 4 through 9, the only objects with initial negative charge are triangles; they begin with charge -3. Each triangle takes 1 from each neighboring face. To repair faces that may lose too much, each face f takes 1 from each incident 4^+-vertex lying on at least one triangle sharing an edge with f (see Figure 12).

We have made 3-faces happy, and 3-vertices remain at charge 0. Other vertices remain happy because 3-faces do not share edges. For $j \geq 4$, a j-vertex loses charge at most $\lfloor \frac{j}{2} \rfloor$ and ends with at least $\lceil \frac{3j}{2} \rceil - 6$, which is nonnegative for $j \geq 4$.

Now consider a j-face f for $j \geq 10$. It loses 1 for every path along its boundary such that the neighboring faces are triangles and the endpoints have degree 3; f gives 1 to each

of those faces but regains 1 from each intervening vertex. If an endpoint of a maximal such path has degree at least 4, then there is no net loss. Hence the net loss for f is at most $\lfloor \frac{j}{2} \rfloor$, and the final charge is at least $\lceil \frac{j}{2} \rceil - 6$, which is nonnegative when $j \geq 11$.

Hence negative charge can occur only at 10-faces. A 10-face f must lose more than 4 to become negative. This requires five paths through which f loses 1. The paths must be single edges sharing no vertices, and all the vertices incident to f must have degree 3. □

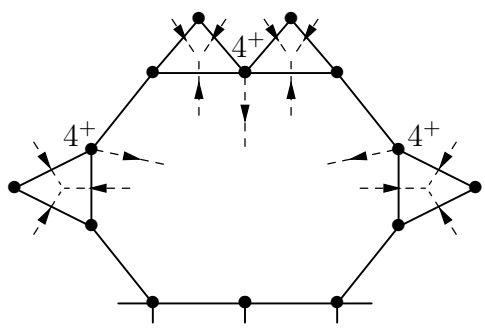

Figure 12: Discharging for Lemma 5.5

Theorem 5.6 ([43, 200])**.** *Every plane graph having no 4-cycle and no j-face with $5 \leq j \leq 9$ is 3-colorable.*

Proof. A smallest counterexample G must be 4-critical, and hence it has minimum degree at least 3 and is 2-connected. Since there is no 4-cycle, no two 3-faces share an edge. By Lemma 5.6, we may thus assume that G is embedded with at least one 10-face C, whose vertices all have degree 3. Let f be a proper 3-coloring of $G - V(C)$. Since each vertex on C has exactly one neighbor outside C, two colors remain available at each vertex of C. Since even cycles are 2-choosable, the coloring can be completed. □

Exercise 5.1. Let G be a simple plane graph with $\delta(G) \geq 3$. Prove that G has a 3-vertex on a 5^--face or a 5^--vertex on a triangle.

Exercise 5.2. (Lebesgue [168]) Strengthen the previous exercise by proving that every plane graph G with $\delta(G) \geq 3$ contains a 3-vertex on a 5^--face, a 4-vertex on a 3-face, or a 5-vertex with four incident 3-faces. (Comment: Lebesgue phrased the proof only for 3-connected plane graphs.)

Exercise 5.3. (Cranston [89]) Let G be a plane graph with $\Delta(G) \geq 7$. Prove that G has either two 3-faces with a common edge or an edge with weight at most $\Delta(G) + 2$. Conclude that if G is a plane graph with $\Delta(G) \geq 7$ and no two 3-faces sharing an edge, then G is $(\Delta(G) + 1)$-edge-choosable. (Hint: Use balanced charging. Comment: Cranston proved that the same conditions are also sufficient when $\Delta(G) \geq 6$, which implies several earlier results.)

Exercise 5.4. Prove that every plane triangulation with minimum degree 5 has two 3-faces sharing an edge such that the non-shared vertices have degree-sum at most 11. (Hint: Use vertex charging; 6-vertices that give charge to 5-neighbors will need charge from 7^+-neighbors. Comment: Albertson [6] used this configuration in a proof that $\alpha(G) \geq \frac{2n}{9}$ when G is an n-vertex planar graph with no separating triangle, without using the Four Color Theorem or the language of discharging.)

Exercise 5.5. (Dvořak, Kawarabayashi, and Thomas [101]) Let C be the outer boundary in a 2-connected triangle-free plane graph G that is not a cycle. If C has length at most 6, and every vertex not on C has degree at least 3, then G contains a bounded 4-face or a proper 5-face, where a 5-face is *proper* if (at least) four of its vertices have degree 3 and are not on C. (Comment: This result was used in [101] to give a new proof of Grötzsch's Theorem [122] that triangle-free planar graphs are 3-colorable. The proof in [101] used vertex charging, but using face charging is simpler.)

Exercise 5.6. (Borodin [39]) Using Lemma 5.3, adapt the proof of Theorem 5.4 to show that $\chi''_\ell(G) = \Delta(G) + 1$ when G is a planar graph with $\Delta(G) \geq 14$.

Exercise 5.7. Let G be a plane graph having no 4-cycle and no face with length in $\{4, \ldots, k\}$. Use discharging to prove that the average face length in G is at least $6 - \frac{18}{k+4}$. Conclude that $\mathrm{mad}(G) < 3 + \frac{9}{2k-1}$. In particular, $\mathrm{mad}(G) < 4$ when G is a plane graph having no 4-face or 5-face.

6 Light Subgraphs of Planar Graphs

We have noted that in inductive proofs for $\chi'_\ell(G) \leq k$, a light edge uv (that is, satisfying $d_G(u) + d_G(v) \leq k + 1$) is reducible, since extending a proper edge-coloring of $G - uv$ to G only requires the color on uv to avoid the colors on $k - 1$ other edges. Planar graphs are also guaranteed to have light copies of other subgraphs, often by discharging.

The Four Color Theorem begins with light subgraphs — specifically, light vertices. A planar graph has a 5^--vertex. After showing that 4^--vertices are reducible for proper 4-coloring, attention turns to triangulations with minimum degree 5. Wernicke [243] guaranteed an edge with weight at most 11 in such a graph; in Lemma 6.1 we will prove more. Further light subgraphs (and other configurations) are generated via discharging rules, and we seek reducible configurations. Unfortunately, the details become rather complicated.

The initial proof by Appel and Haken (working with Koch) involved 1936 reducible configurations. The unavoidable set was generated by hand, but the details of proving reducibility were so lengthy that they were checked by computer. The publication comprised nearly 140 pages in two papers [13, 20] plus over 400 pages of microfiche [14, 15]. With further explanations, this became a 741-page book [19] (see also [18]).

Some researchers objected to the use of computers. Robertson, Sanders, Seymour, and Thomas [196] looked for a simpler proof; they wound up using the same approach. Their unavoidable set had only 633 configurations and 32 discharging rules, but they still needed a computer. With a faster computer and simpler arguments, their proof ran in only 20 minutes instead of the original 1200 hours.

Not all parts of these proofs are hopelessly complicated. Appel and Haken [17] published a shorter paper with simpler discharging to prove that for planar triangulations with minimum degree 5 but no adjacent 5-vertices there is an unavoidable set of 47 configurations likely to be reducible; in [21] there is an even shorter such list, whose reducibility they showed.

Stronger results later followed. Robertson, Seymour, and Thomas [197] used the Four Color Theorem (in a paper that won the 1994 Fulkerson Prize) to prove the case $k = 6$ of Hadwiger's Conjecture: graphs not having K_6 as a minor are 5-colorable (the case $k = 5$ is equivalent to the Four Color Theorem, as shown by Wagner [229]).

Using the planar dual, Tait [217] showed that 4-colorability of planar triangulations is equivalent to 3-edge-colorability of 3-regular 2-edge-connected planar graphs. Planar graphs contain no subdivision of the (nonplanar) Petersen graph. Hence Tutte's conjecture that 3-regular 2-edge-connected graphs containing no Petersen-subdivision are 3-edge-colorable is stronger than the Four Color Theorem. In 2001, Robertson, Sanders, Seymour, and Thomas announced a proof of this. First Robertson, Seymour, and Thomas [198] reduced the conjecture to proving it for two special classes of graphs: "apex" graphs and "doublecross" graphs. However, [198] relies on an unpublished manuscript [199]. Later, 3-edge-colorability was proved for the two special classes by Sanders, Seymour, and Thomas, but this also has not been published. In 2012, Seymour described this proof, with its reliance on discharging and many reducible configurations, as "Repeat the proof of the 4CT (twice)".

The Four Color Theorem uses discharging and reducibility as our examples do, but its details remain too lengthy for discussion here. Instead, we will simply assume its correctness and resume giving simpler results to explain various ideas and techniques that arise in discharging proofs and their applications.

We continue with light subgraphs, a subject surveyed by Jendrol' and Voss [140]. We focus on light edges, which have been thoroughly studied. Edges are small enough that the best bounds on the weight of light edges in various settings can be determined by discharging, and yet they have many applications.

The best-known result on light edges is Kotzig's Theorem [159]: every 3-connected planar graph has an edge of weight at most 13. Borodin [39] later extended it. A *normal plane map* is a plane multigraph such that every vertex degree and face length is at least 3; Euler's Formula holds also in this setting. Jendrol' [139, 140] gave a short proof of the extension that every normal plane map G has an edge with weight at most 11 or a 3-vertex with a 10^--neighbor. Most applications of the lemma (including in this paper) use only this statement. In fact, Borodin proved more. We extend the proof by Jendrol' to obtain Borodin's stronger statement, which we will apply in Theorem 7.15. Discharging yields a short proof.

Lemma 6.1 (Borodin [39])**.** *Every normal plane map G has an edge with weight at most 11 or a 4-cycle through two 3-vertices and a common 10^--neighbor.*

Proof. If any face has length more than 3, then adding a chord does not create light edges

(it may create multi-edges). Hence we may assume that every face has length 3. Assume that G has no light edge, and give initial charges by vertex charging.

(R1) Every 5^- vertex v takes charge $\frac{6-d(v)}{d(v)}$ from each 7^+-neighbor.

Since G is a triangulation without light edges, a k-vertex sends charge to at most $\lfloor \frac{k}{2} \rfloor$ vertices. Since a 7-vertex sends charge only to 5-vertices, it loses at most $3 \cdot \frac{1}{5}$ and remains positive. An 8-vertex sends charge only to 4^+-vertices and loses at most $4 \cdot \frac{1}{2}$, remaining nonnegative. When $d(v) \geq 11$, neighbors of v may have degree 3 and take charge 1, but the final charge is at least $\left\lceil \frac{d(v)}{2} \right\rceil - 6$, again nonnegative.

Finally, suppose $d(v) \in \{9, 10\}$. Since faces have length 3, the neighbors of v form a closed walk of length $d(v)$ when followed in order, and each 3-vertex appears only once in this walk. With light edges and the specified 4-cycles forbidden, 3-vertices must be separated by at least three steps along this walk.

Hence a 9-vertex has at most three 3-neighbors, and when it has three it ends with charge 0, since it loses no other charge. Similarly it may have two 3-neighbors and up to two 4-neighbors, reaching charge 0, or one 3-neighbor and up to three 4-neighbors, losing charge at most $\frac{5}{2}$. A 10-vertex may also have three 3-neighbors, or two 3-neighbors and three 4-neighbors, or other combinations losing less charge. The total charge lost is at most $\frac{7}{2}$, and the final charge is positive. □

Further results on light edges for planar graphs with large girth appear in Lemma 6.11 and in Exercises 6.4–6.7. We will apply light edges in Theorems 6.3 and 6.10.

We discussed total coloring in Section 4. Now consider coloring vertices and faces instead of vertices and edges. Ringel [195] conjectured the following theorem, proved by Borodin [37] using discharging. The discharging argument produced a set of 35 configurations, all then shown to be reducible (without computers). Borodin [42] later simplified the proof.

Theorem 6.2 (Borodin [37, 42])**.** *The vertices and faces of any plane graph G can be colored using six colors so that elements adjacent or incident in G or its dual receive different colors.*

Beyond total coloring, an *entire coloring* of a plane graph G colors its vertices, edges, and faces so that any two elements adjacent or incident in G or its dual have distinct colors. Let $\chi'''(G)$ be the least number of colors needed (also χ_{VEF} is used). Kronk and Mitchem [161] introduced entire coloring and conjectured $\chi'''(G) \leq \Delta(G) + 4$ for every simple plane graph G. Equality holds for K_4 (see Figure 13). It seems unknown whether there are infinitely many sharpness examples; perhaps fewer colors suffice when $\Delta(G)$ is sufficiently large.

This conjecture has been proved. It was proved for $\Delta(G) \leq 3$ by Kronk and Mitchem [162], for $\Delta(G) \geq 7$ by Borodin [44], for $\Delta(G) = 6$ by Sanders and Zhao [202] (corrected in [241]), and for $\Delta(G) \in \{4, 5\}$ by Wang and Zhu [241]. Using Lemma 6.1 and the Four Color Theorem, it is easy to prove the conjecture for $\Delta(G) \geq 11$ (see [38]; also Exercise 6.10 develops another proof for $\Delta(G) \geq 7$). The crucial step is a list version of total coloring.

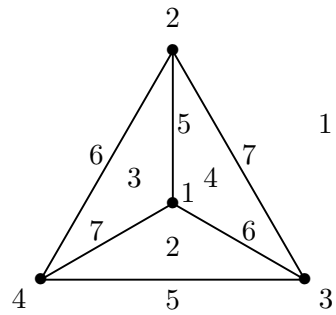

Figure 13: Entire coloring of K_4

Theorem 6.3. *If G is a simple plane graph with $\Delta(G) \geq 11$, and L assigns lists of size $\Delta(G)$ to the vertices and size $\Delta(G)+2$ to the edges, then G has a total L-coloring.*

Proof. A minimal counterexample G comes with a list assignment L from which no suitable coloring can be chosen. If $\delta(G) \geq 3$, then Lemma 6.1 applies. Considering the possibility $\delta(G) \leq 2$ and the possible conclusions from Lemma 6.1, we have three cases.

Case 1: G has an edge uv such that $d(u) \leq 2$.
Case 2: G has an edge uv such that $d(u) = 3$ and $d(v) \leq 10$.
Case 3: G has an edge uv such that $d(u) \leq 5$ and $d(v) \leq 7$.

In each case, minimality of G allows $G - uv$ to be colored from the lists. Let $e = uv$. Delete the color on u; we will recolor it later after choosing a color for e.

In each case, we bound the number of colors that can now be on elements incident to e. It is at most $\Delta(G)+1$ in Case 1, at most $2+1+9$ in Case 2, and at most $4+1+6$ in Case 3. Since $|L(e)| = \Delta(G)+2 \geq 13$, a color remains available for e.

Finally, we color u to avoid the colors on incident edges and neighboring vertices. Since $d(u) \leq 5$, at most 10 colors need to be avoided, and $|L(u)| = \Delta(G) \geq 11$. □

Corollary 6.4. *For simple plane graphs G with $\Delta(G) \geq 11$ and no cut-edge, $\chi'''(G) \leq \Delta(G)+4$.*

Proof. First, properly color the faces using four colors (by the Four Color Theorem), leaving $\Delta(G)+2$ colors at each edge and at least $\Delta(G)$ at each vertex. Now apply Theorem 6.3. □

We have mentioned that structure theorems should have multiple applications. Our next structure theorem has this property.

Lemma 6.5. *Every planar graph has a 5^--vertex with at most two 12^+-neighbors.*

Proof. Adding an edge cannot give any vertex the desired property (although it can take the property away); hence it suffices to prove the claim when G is a triangulation.

Assume that G has no such vertex, so $\delta(G) \geq 3$. Assign charge by degrees; $\overline{d}(G) < 6$. Every 5^--vertex has at least three 12^+-neighbors. Let each 5^--vertex u take $\frac{6-d(u)}{3}$ from each

12^+-neighbor. Now 5^--vertices are happy, and j-vertices with $6 \le j \le 11$ lose no charge, so it suffices to show that 12^+-vertices do not lose too much.

Let v be a 12^+-vertex. Since G is a triangulation, the neighbors of v form a cycle C. Let H be the subgraph of C induced by its vertices having degree at most 5 in G. Each 5^--vertex w has at least three 12^+-neighbors, so $d_H(w) \le d_G(w) - 3$. If all neighbors of v have degree 5, then v loses $\frac{d(v)}{3}$ and finishes with $\frac{2}{3}d(v)$, which is at least 8.

Otherwise, the components of H are paths (bold in Figure 14). Combine such a k-vertex path P with the next vertex on C, which receives no charge from v. If $k = 1$, then v gives at most 1 to these two vertices. If $k > 1$, then v gives at most $\frac{1}{3}$ to internal vertices of P (degree at least 5) and at most $\frac{2}{3}$ to its endpoints (degree at least 4). Hence the $k+1$ vertices receive at most $0 + 2(\frac{2}{3}) + (k-2)(\frac{1}{3})$ from v. This equals $\frac{k+2}{3}$, which is less than $\frac{k+1}{2}$ when $k > 1$. Hence v loses in total at most $\frac{d(v)}{2}$, leaving at least 6 when $d(v) \ge 12$. $\square$

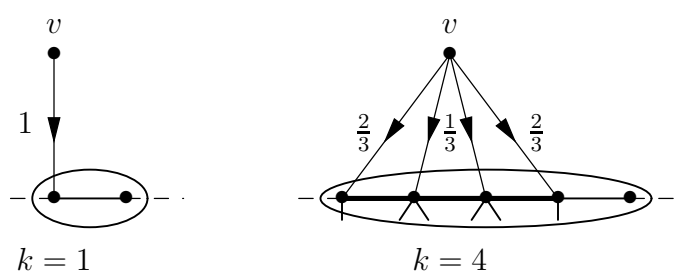

Figure 14: Final case for Lemma 6.5

Lemma 6.5 can be viewed as strengthening the guarantee of light vertices: there must be a light vertex with not too much heaviness among its neighbors. It can be used to strengthen results that follow from the basic existence of light vertices (that is, 5^--vertices) in planar graphs, a fact that follows from Euler's Formula and does not need discharging. For example, consider the *arboricity* of a graph G, written $\Upsilon(G)$, defined to be the minimum number of forests whose union is G. Nash-Williams [187] famously proved that for every graph G, the trivial lower bound is the exact value.

Theorem 6.6 ([187]). *Always* $\Upsilon(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1}$.

Although there is a short general proof using matroids, for planar graphs the existence of 5^--vertices yields a direct inductive construction of decompositions into three forests. We include this in Exercise 6.11 to illustrate a technique for reducibility with triangulations.

Lemma 6.5 was improved by Balogh, Kochol, Pluhar, and Yu [22] to guarantee a 5^-vertex having at most two 11^+-neighbors, proved by a much longer discharging argument than in Lemma 6.5. The result is sharp in that "11" cannot be replaced by "10" (the graph obtained by adding a vertex of degree 3 in each face of the icosahedron has three 10-neighbors for each 5^--vertex). From this result they proved that every planar graph decomposes into three

forests, among which one has maximum degree at most 8. Lemma 6.5 allows the degree of the third forest to be bounded by 9 (Exercise 6.11).

We also apply Lemma 6.5 to the problem of coloring the square of a planar graph, where there is another well-known conjecture (the original conjecture was more general).

Conjecture 6.7 (Wegner's Conjecture [242])**.** *If G is planar, then $\chi(G^2) \le \lfloor \frac{3}{2}\Delta(G) \rfloor + 1$ for $\Delta(G) \ge 8$; also $\chi(G^2) \le \Delta(G) + 5$ for $4 \le \Delta(G) \le 7$ and $\chi(G^2) \le 7$ for $\Delta(G) \le 3$.*

Wegner gave sharpness constructions; fixing $\Delta(G)$, these are planar graphs of diameter 2 (so $\chi(G^2) = |V(G)|$) with the most vertices. The general situation uses graphs studied by Erdős and Rényi [106], shown on the left in Figure 15. The other graphs there with maximum degree k have diameter 2 with $k + 5$ vertices for $4 \le k \le 7$, taken from [130] (the half-edges in the graph for $k = 6$ meet at the eleventh vertex). Wegner found the three leftmost graphs.

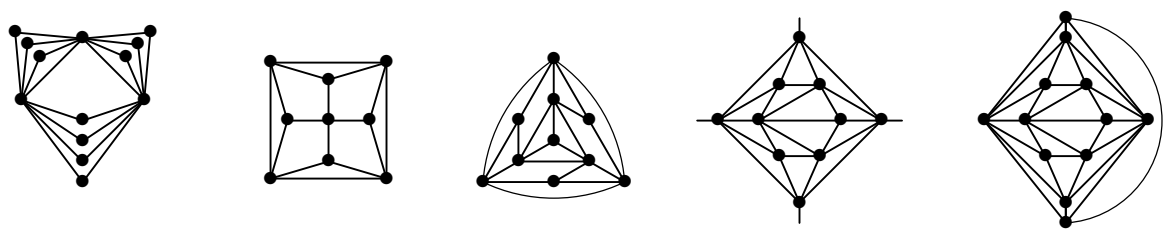

Figure 15: Large graphs with diameter 2 and fixed maximum degree

For upper bounds, van den Heuvel and McGuinness [131] proved $\chi(G^2) \le 2\Delta(G) + 25$, but their argument also yields $\chi_\ell(G^2) \le 2\Delta(G) + 25$. We present a weaker version of this, showing that $\chi(G^2) \le 2\Delta(G)+34$ when G is planar. To keep our additive constant small, we use an enhanced version of Lemma 6.1. Instead of letting each 5^--vertex v take $\frac{6-d(v)}{d(v)}$ from each 7^+-neighbor, change the rule for 5-vertices to take $\frac{1}{4}$ from each 7^+-neighbor. Now a 5-vertex becomes happy when it has at least four 7^+-neighbors. With 4^--neighbors forbidden, a 7-vertex lost at most $\frac{3}{5}$ before, now at most $\frac{3}{4}$, so again it remains happy. Hence we conclude the following.

Lemma 6.8. *Every normal plane map G has a 3-vertex with a 10^--neighbor, or a 4-vertex with a 7^--neighbor, or a 5-vertex with two 6^--neighbors.*

Our first bound on $\chi(G^2)$ helps when $\Delta(G)$ is small. It slightly refines an idea from [142].

Theorem 6.9. *If G is a planar graph, then* $\chi_\ell(G^2) \le \begin{cases} \Delta(G)^2 + 1 & \text{when } \Delta(G) \le 5, \\ 7\Delta(G) - 7 & \text{when } \Delta(G) \ge 6. \end{cases}$

Proof. For $\Delta(G) \le 5$, the claim is just the trivial upper bound from $\Delta(G^2)$, so we may assume $\Delta(G) \ge 6$.

Index the vertices from v_n to v_1 as follows. Having chosen $v_n, \ldots, v_{i+1}$, let $G_i = G - \{v_n, \ldots, v_{i+1}\}$. If $\delta(G_i) \le 3$, then let v_i be a vertex of minimum degree; otherwise let v be a vertex as guaranteed by Lemma 6.8. Let $S_i = \{v_1, \ldots, v_i\}$. We choose colors for vertices in the order $v_1, \ldots, v_n$ so that the coloring of S_i satisfies all the constraints in the full graph G^2 from pairs of vertices in S_i.

Let $j = |N(v_i) \cap S_i|$; by the choice of the ordering, $j \le 5$. The other neighbors of v_i occur later and are not yet colored. However, v_i must avoid the colors on the neighbors in S_i of these vertices; there may be up to $4(\Delta(G) - j)$ such colors. For $u \in N(v_i) \cap S_i$, all neighbors of u may lie in S_i, so there may be as many as $d(u)$ colors that v_i must avoid due to u.

If $j \le 3$, then the number of colors v_i must avoid is at most $4(\Delta(G)-j)+j\Delta(G)$. If $j = 4$, then the bound is $7\Delta(G) - 9$, since v_i has a 7^--neighbor in S_i. If $j = 5$, then it is $7\Delta(G) - 8$, since v_i has two 6^--neighbors in S_i. Hence always the bound is at most $7\Delta(G) - 8$. □

Using Lemma 6.5 instead of Lemma 6.8, a vertex with j earlier neighbors must avoid at most $6\Delta(G) + 7j - 2$ colors. Hence $\chi_\ell(G^2) \le 6\Delta(G) + 33$. We strengthen this bound.

Theorem 6.10. *If G is a planar graph, then $\chi_\ell(G^2) \le 2\Delta(G) + 34$.*

Proof. Theorem 6.9 provides upper bounds that are at most $2\Delta(G) + 34$ when $\Delta(G) \le 8$. Hence we may assume $\Delta(G) \ge 9$.

Let G be a minimal counterexample, with list assignment L from which no such coloring can be chosen. We will form G' by contracting an edge of G incident to a 5^--vertex v, viewed as absorbing v into the other endpoint u; the new vertex retains the list assigned to u. All the constraints forcing vertices of G to have distinct colors are present also in G', so any proper coloring of the square of G' can also be used on $V(G')$ in G. If $\Delta(G')$ is small enough, then the induction hypothesis applies to properly color the square of G' from lists of the desired size, and the task is only to show that few enough other vertices are within distance 2 of v in G, leaving a color available for v to complete an L-coloring of G^2.

If $\delta(G) \le 2$, then let v be a vertex of minimum degree. The contracted vertex has degree at most $\Delta(G)$. At most $2\Delta(G)$ vertices are within distance 2 of v, which leaves a color available for v. Hence we may assume $\delta(G) \ge 3$, so Lemmas 6.8 and 6.5 apply.

Case 1: $9 \le \Delta(G) \le 13$. In this case we prove $\chi_\ell(G^2) \le 52 \le 2\Delta(G) + 34$. Let v be a vertex as guaranteed by Lemma 6.8: a 3-vertex with a 10^--neighbor, a 4-vertex with a 7^--neighbor, or a 5-vertex with two 6^--neighbors. The specified edge is the edge to be contracted, and the degree of the new vertex is at most 11. Thus $\Delta(G') \le 13$, and the induction hypothesis yields a proper L-coloring of the square of G' from lists of size 52 (if $\Delta(G') = 8$, then $\chi_\ell(G'^2) \le 2\Delta(G') + 34 \le 52$). The bound on the number of other vertices

within distance 2 of v is $2\Delta(G)+10$ for $d(v)=3$, $3\Delta(G)+7$ for $d(v)=4$, and $3\Delta(G)+12$ for $d(v)=5$. Since $\Delta(G)\le 13$, the value in each case is at most 51.

Case 2: $\Delta(G)\ge 14$. Let v be a vertex as guaranteed by Lemma 6.5; note that $d(v)\le 5$. Since $d(v)\ge 3$ and v has at most two 12^+-neighbors, v has an 11^--neighbor; contract such an edge. The contracted vertex has degree at most 14, so the induction hypothesis applies. Also, the number of vertices within distance 2 of v in G is bounded by $2\Delta(G)+33$. □

The improved upper bound of $2\Delta(G)+25$ in [131] uses a slightly stronger version of Lemma 6.5 to improve the argument for large $\Delta(G)$ (see Exercise 6.13). Their main additional work was proving a second lemma specifically for graphs with small maximum degree.

Havet, van den Heuvel, McDiarmid, and Reed [127] proved for planar graphs that $\chi_\ell(G^2)\le(\frac{3}{2}+o(1))\Delta(G)$ as $\Delta(G)\to\infty$, by probabilistic methods. Hence we also seek bounds below $2\Delta(G)$ when $\Delta(G)$ is "small". Borodin et al. [47] proved for planar graphs that $\chi_\ell(G^2)\le 59$ when $\Delta(G)\le 20$ and $\chi_\ell(G^2)\le\max\{\Delta(G)+39,\lceil\frac{9}{5}\Delta(G)\rceil+1\}$ when $\Delta(G)>20$. In particular, if $\Delta(G)\ge 47$, then $\chi_\ell(G^2)\le\lceil\frac{9}{5}\Delta(G)\rceil+1$. More generally, they proved that G^2 is k-degenerate, where $k=\max\{\Delta(G)+38,\lceil\frac{9}{5}\Delta(G)\rceil\}$. For the coloring problem alone, Molloy and Salavatipour [178] proved $\chi(G^2)\le\frac{5}{3}\Delta(G)+78$ for all planar G. For results in terms of $\mathrm{mad}(G)$ (without planarity), see the exercises in Section 3.

We have noted that many results for planar graphs with girth at least g extend to the family of graphs G with $\mathrm{mad}(G)<\frac{2g}{g-2}$ (especially when the proof uses vertex charging). In some cases, planarity permits a stronger result, meaning that obtaining the same conclusion using only a bound on $\mathrm{mad}(G)$ requires $\mathrm{mad}(G)<b$ for some b smaller than $\frac{2g}{g-2}$.

For example, consider a special case of Exercise 3.3: every graph G with $\mathrm{mad}(G)<\frac{8}{3}$ and $\delta(G)\ge 2$ has a 2-vertex with a 3^--neighbor. This applies to planar graphs with girth at least 8. The result in terms of $\mathrm{mad}(G)$ is sharp, since subdividing every edge of a 4-regular graph yields a graph G with $\mathrm{mad}(G)=\frac{8}{3}$ having no such vertex. However, the conclusion holds for planar graphs with girth 7, which allow $\mathrm{mad}(G)$ to be larger, so in this case planarity is needed to obtain the best result. The argument we present illustrates typical difficulties that may arise when discovering discharging arguments.

Lemma 6.11. *Every planar graph G with girth at least 7 and $\delta(G)\ge 2$ has a 2-vertex with a neighbor of degree at most 3.*

Proof. Assume that G has no such configuration. Use face charging, giving each vertex v initial charge $2d(v)-6$ and each face f initial charge $\ell(f)-6$. The total charge is -12. Since g has girth at least 7, the only objects with negative initial charge are 2-vertices. Let each 2-vertex take $\frac{1}{2}$ from each neighbor and each incident face. To complete the proof, we check that all vertices and faces end with nonnegative charge.

The discharging rule ensures that 2-vertices end with charge 0. Since 3-vertices have no 2-neighbors, their charge remains 0. For $j\ge 4$, a j-vertex may lose $\frac{1}{2}$ along each edge and

ends with charge at least $2j - 6 - \frac{j}{2}$, which is nonnegative.

A j-face has at most $\left\lfloor \frac{j}{2} \right\rfloor$ incident 2-vertices, since 2-vertices are not adjacent. Hence a j-face has final charge at least $j - 6 - \frac{1}{2}\left\lfloor \frac{j}{2} \right\rfloor$, which is nonnegative for $j \geq 8$. To help the 7-faces, we add another discharging rule. When adjacent 4^+-vertices form an edge e, direct the charge $\frac{1}{2}$ that each could send to a 2-neighbor so that instead the two faces bounded by e each receive $\frac{1}{2}$. Now when a 7-face gives away $\frac{3}{2}$ to three 2-vertices, it recovers $\frac{1}{2}$ from the two adjacent 4^+-vertices on its boundary and ends with charge 0. $\square$

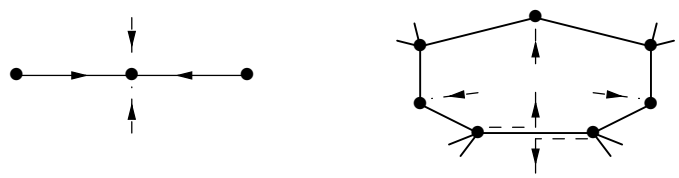

Figure 16: Discharging for Lemma 6.11

This proof illustrates both "redirection" of transmitted charge and the phenomenon of designing discharging rules initially to make deficient vertices happy but discovering later that additional rules are then needed to repair other vertices that lost too much. It turns out that balanced charging, where 2-vertices and 3-vertices both need charge, yields a simpler discharging proof of this result; see Exercise 6.5.

Lemma 6.11 yields a stronger result for planar graphs with girth 7 than is possible for the corresponding bound on $\mathrm{mad}(G)$. A graph G is *dynamically k-choosable* if for every k-uniform list assignment L, there is a dynamic L-coloring of G, meaning a proper L-coloring with the additional property that no neighborhood with size at least 2 is monochromatic. By showing that the configuration in Lemma 6.11 is reducible (Exercise 6.12), Kim and Park [150] showed that every planar graph with girth at least 7 is dynamically 4-choosable. This result is sharp, since subdividing every edge of a non-4-choosable planar graph with girth 3 yields a planar graph with girth 6 that is not dynamically 4-choosable.

We close this section with one of the most famous results on light subgraphs: the existence of light triangles in planar graphs with minimum degree 5. This was conjectured for triangulations by Kotzig and proved in stronger form by Borodin; see Exercise 6.3 for a sharpness example. We sketch the proof because an interesting wrinkle in the discharging rules can again be viewed as redirecting charge. The result strengthens the early result of Franklin [118] guaranteeing weight at most 17 on some copy of P_3. Specifically, Franklin showed that if G is planar with minimum degree 5, then G has a 5-vertex with two 6^--neighbors; this is strengthened also by Lemma 6.8.

Theorem 6.12 (Borodin [40]). *If G is a simple plane graph with $\delta(G) \geq 5$, then G has a 3-face with weight at most 17, and the bound is sharp.*

Proof. (sketch) For sharpness, add a vertex in each face of the dodecahedron joined to the vertices of that face. The new vertices have degree 5, and the old ones now have degree 6. Every face has one new vertex and two old vertices for total weight 17.

To prove the bound, consider an edge-maximal counterexample G. Every vertex on a 4^+-face has degree 5, since adding a triangular chord at a 6^+-vertex would create only heavy 3-faces, producing a counterexample containing G.

Suppose there is no light triangle and use vertex charging, with charge $d(v) - 6$ on each vertex v and $2l(f) - 6$ on each face f. Move charge via the following rules:

(R1) Each 4^+-face gives $\frac{1}{2}$ to each incident vertex.
(R2) Each 7-vertex gives $\frac{1}{3}$ to each 5-neighbor.
(R3) Each 8^+-vertex gives $\frac{1}{4}$ through each incident 3-face to its 5-neighbors on that face, split equally if there are two such neighbors.

It suffices to show that final charges are nonnegative. The discharging rules are chosen so that 4^+-faces and 7^+-vertices do not give away too much charge. (Maximality of the counterexample puts a 7-vertex on seven triangles, and absence of light triangles then restricts a 7-vertex to have at most three 5-neighbors).

Hence the task is to prove that a 5-vertex v gains charge 1. When all incident faces are triangles, avoiding light triangles restricts v to have at most two 5-neighbors. It also forces the other neighbors to be 8^+-vertices when v has two 5-neighbors. In this and the remaining cases (such as when v lies on a 4^+-face), it is easy to check that v receives enough charge. □

In each discharging rule in Theorem 6.12, the charge given away is the most that object can afford to lose. Only 5-vertices need to gain charge. Hence it would be natural to have 8^+-vertices give $\frac{1}{4}$ to each 5-neighbor. However, this would give only $\frac{3}{4}$ to a 5-vertex having two 5-neighbors and three 8-neighbors. The 5-vertex needs to get more when it has two 8-neighbors on a single triangle. Guiding the charge from 8^+-neighbors through the incident triangles is a way to arrange that.

Exercise 6.1. Construct planar graphs to show that the bounds in Lemma 6.1 are sharp. That is, none of the values 10, 7, 6 can be reduced in the statement that normal plane maps have a 3-vertex with a 10^--neighbor, a 4-vertex with a 7^--neighbor, or a 5-vertex with a 6^--neighbor.

Exercise 6.2. Prove that planarity is needed in Lemma 6.1 by showing that a graph G with $\mathrm{mad}(G) < 6$ and $\delta(G) = 5$ need not have a 5-vertex with any 6^--neighbor.

Exercise 6.3. Prove that requiring minimum degree 5 in Theorem 6.12 is necessary, by constructing for each $k \in \mathbb{N}$ a planar graph with minimum degree 4 having no triangle with weight at most k.

Exercise 6.4. The argument in Remark 2.3 with $\ell = 2$ shows that $\frac{12}{5}$ is the largest ρ such that $\mathrm{mad}(G) < \rho$ guarantees adjacent 2-vertices in G. Note that $\mathrm{mad}(G) < \frac{12}{5}$ when G is planar with girth at least 12. Restriction to planar graphs permits a stronger result: Prove that planar graphs with girth at least 11 have adjacent 2-vertices, and provide a construction to show that the conclusion fails for some planar graph with girth 10.

Exercise 6.5. Use balanced charging to prove that every planar graph with girth at least 7 and minimum degree at least 2 has a 2-vertex adjacent to a 3^--vertex. Prove that the conclusion does not always hold when $\mathrm{mad}(G) < \frac{14}{5}$ (thus planarity is needed). Show that the conclusion does not hold for all planar graphs with girth 6.

Exercise 6.6. Planar graphs with girth at least 6 satisfy $\mathrm{mad}(G) < 3$, so by Exercise 2.1 each such graph has a 2-vertex with a 5^--neighbor. Show that this is sharp even for planar graphs by constructing a planar graph with girth 6 having no edge of weight at most 6.

Exercise 6.7. Determine whether a planar graph with girth at least 4 and minimum degree 3 must have a 3-vertex with a 4^--neighbor. Construct a planar graph G_k with girth 4 and minimum degree 3 in which the distance between 3-vertices is at least k. Construct a planar graph H_k with minimum degree 5 in which the distance between 5-vertices is at least k.

Exercise 6.8. Let G be a graph with $\delta(G) \geq 3$ and $\mathrm{mad}(G) < \frac{10}{3}$. Prove that G has a 3-vertex whose neighbors have degree-sum at most 10. Prove that this result is sharp even in the family of planar graphs with girth at least 5 by constructing such a graph in which no 3-vertex has three 3-neighbors. (Comment: G. Tardos constructed such a graph with 98 vertices.)

Exercise 6.9. (Borodin [44]) Prove that every planar graph with minimum degree 5 contains two 3-faces sharing an edge with weight at most 11. Show that this configuration is reducible for $\chi'''(G) \leq 12$. (Hint: Use vertex charging, with 5-vertices taking $\frac{1}{2}$ from incident 4^+-faces and the remaining needed charge from 7^+-neighbors. Comments: Borodin proved a stronger structural conclusion and used it in his proof that $\chi'''(G) \leq \Delta(G) + 4$ when $\Delta(G) \geq 7$; the proof of the statement here is simpler. Note also the similarity between the configurations here and in Exercise 5.4).

Exercise 6.10. (Cranston) Give a short proof that $\chi'''(G) \leq \Delta(G) + 4$ when G is a planar graph with $\Delta(G) \geq 7$ by using the technique in the proof of Theorem 4.7, relying on Theorem 6.2 instead of the Four Color Theorem.

Exercise 6.11. Prove inductively that every planar graph decomposes into three forests. (Hint: Reduce to triangulations, and then apply the induction hypothesis to a smaller graph obtained by deleting a light vertex and triangulating the resulting face. There are a number of cases when the deleted vertex has degree 5, depending on the usage of the two added edges.) Use Lemma 6.5 and more detailed analysis to prove that the third forests can be guaranteed to have maximum degree at most 9. (Comment: The second part of this exercise is long. See Balogh et al. [22] for maximum degree at most 8.)

Exercise 6.12. (Kim and Park [150]) Prove that among the planar graphs with girth at least 7, a minimal graph that is not dynamically 4-choosable cannot contain a 2-vertex with a 3^--neighbor. (Comment: With Lemma 6.11, this proves that every planar graph with girth at least 7 is dynamically 4-choosable. Note that $\mathrm{mad}(G) < \frac{14}{5}$ when G is planar with girth at least 7, but $\mathrm{mad}(G) < \frac{14}{5}$ is not sufficient for dynamic 4-choosability.)

Exercise 6.13. van den Heuvel and McGuinness [131] refined Lemma 6.5 to state that a planar graph G with $\delta(G) \geq 3$ has a 5^--vertex v with at most two 12^+-neighbors such that v has a 7^--neighbor if $d(v) \in \{4, 5\}$, and v has an additional 6^--neighbor if $d(v) = 5$ (see Exercise 7.14). Use this to prove that $\chi(G^2) \leq 2\Delta(G) + 25$ when $\Delta(G) \geq 12$.

7 Stronger and Weaker Colorings

In this section, we consider both restricted and relaxed versions of proper coloring. As a relaxation, we study "improper" coloring, where the subgraph induced by a color class has maximum degree at most d; proper coloring is the case $d = 0$. First we study restricted versions of coloring. For example, the union of two color classes in a proper coloring induces a bipartite graph; we may further restrict the bipartite graphs allowed to arise.

7.1 Acyclic Coloring

An *acyclic coloring* of a graph G is a proper coloring in which the union of any two color classes induces a forest. Equivalently, no cycle is 2-colored. Let $a(G)$ denote the minimum number of colors in an acyclic coloring of G.

Grünbaum [123] conjectured that planar graphs are acyclically 5-colorable, observed that the graph of the octahedron is not acyclically 4-colorable, and proved that planar graphs are acyclically 9-colorable. Improved upper bounds of 8 [176], 7 [7], and 6 [151] culminated in the result of Borodin [36] that $a(G) \leq 5$ when G is planar. The bound is sharp, even among bipartite planar graphs [153] (Exercise 7.2). Borodin's proof used discharging with some 450 reducible configurations but no computers.

We present the earlier result by Albertson and Berman [7] that planar graphs are acyclically 7-colorable; it has several interesting features. In the discharging, the strictly negative charge from Euler's Formula is used to keep the forced configurations away from the unbounded face. In the reducibility, the smaller graphs used in coloring G are modifications of subgraphs of G. Our proof simplifies some aspects of the original proof.

A *near-triangulation* is a plane graph in which every bounded face is a triangle. A vertex not incident to the unbounded face is an *internal vertex*. Our modification of Lemma 6.1 is closely tailored to the reducibility arguments in Theorem 7.2.

Lemma 7.1. *Let G be a near-triangulation with $\delta(G) \geq 3$. If the outer face has length 4 and G has at least two internal vertices, then G contains an internal 3-vertex, or an internal 4-vertex or 5-vertex with an internal 6^--neighbor, or an internal 7-vertex with two internal 4-neighbors. If the outer face has length 3, then again G contains one of these configurations, but without the restriction to internal vertices.*

Proof. For triangulations, consider the proof of Lemma 6.1. If we change the forbidden configuration at a 7-vertex to require two 4-neighbors (so 7-vertices with one 4-neighbor

are allowed), then a 7-vertex still ends with positive charge, since a 4-neighbor and two 5-neighbors together take only $\frac{9}{10}$ from it. This proves the statement for triangulations here.

Now let the outer face have length 4, and again give initial charges by vertex charging. The bounded faces have charge 0, and the unbounded face has charge 2. Hence the vertex charges sum to -14. We move charge so that if none of the specified configurations occur, then every internal vertex has nonnegative charge and the sum of the charges of the external vertices is greater than -14, yielding a contradiction. We use the same discharging rule as in Lemma 6.1, except that external vertices do not receive charge.

(R1) Every internal 5^--vertex v takes charge $\frac{6-d(v)}{d(v)}$ from each neighbor.

With the specified configurations forbidden, the discharging rule insures that all internal 6^--vertices end happy. By the same argument as for triangulations, an internal 7-vertex ends happy. As in Lemma 6.1, an internal k-vertex with $k \geq 8$ may have internal 4-neighbors, losing charge at most $\frac{2}{4}$ to at most $\lfloor \frac{k}{2} \rfloor$ neighbors, retaining nonnegative charge.

Now consider the external vertices. A 3-vertex on the outside face has only one internal neighbor, to which it loses at most $\frac{1}{2}$ since internal 3-vertices are forbidden. Since external vertices do not receive charge, an external k-vertex loses charge at most $\frac{1}{2}$ to each of at most $\lceil \frac{k-2}{2} \rceil$ neighbors. The final charge is at least $(k-6) - \frac{1}{2}\lceil \frac{k-2}{2} \rceil$, which for $k \geq 4$ is at least -2.5. Thus every external vertex ends with charge at least -3.5, with equality possible only for 3-vertices. However, if all four external vertices are 3-vertices, then G has only one internal vertex, contradicting the hypothesis. We conclude that the sum of the charges on the external vertices is strictly more than -14. □

Theorem 7.2 ([7]). *Every planar graph is acyclically 7-colorable.*

Proof. It suffices to show that G is acyclically 7-colorable when G is a simple triangulation. This is trivial for $|V(G)| \leq 7$. Let G be a simple triangulation having the fewest vertices among those that are not acyclically 7-colorable. Triangulations with at least four vertices have minimum degree at least 3. The neighbors of any internal vertex induce a cycle. For $e \in E(G)$, let $T(e)$ denote the set of two vertices forming 3-faces with the endpoints of e.

Case 1: G has a 3-vertex v. By minimality, $G - v$ has an acyclic 7-coloring. The colors it uses on $N_G(v)$ are distinct. Replace v and give it any color not used on its neighbors. Any cycle through v has at least three colors, as does any cycle in $G - v$.

Case 2: G has a 4-vertex v with a 6^--neighbor u. See Figure 17. Let $\{x, y\} = T(uv)$ and $\{z\} = N_G(v) - N_G(u)$. Let $G' = (G - v) + xy$. Since G' has fewer vertices, it has an acyclic 7-coloring ϕ. Let $S = \{\phi(u), \phi(x), \phi(y), \phi(z)\}$. If $|S| = 4$, then give v any color outside S to extend ϕ to G. If $|S| = 3$, then $\phi(z) = \phi(u)$. Since $d_G(u) \leq 6$, at most three colors outside S have been used on $N_{G'}(u)$. Hence a color outside $S \cup \phi(N_{G'}(u))$ remains available to extend ϕ to G. Again this produces no 2-colored cycle through v, since such a cycle would have to visit z and u and use a color in $N_{G'}(u)$.

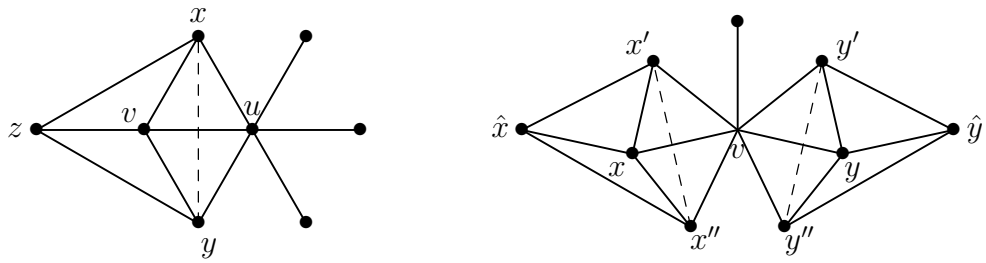

Figure 17: Cases 2 and 3 of Theorem 7.2

Case 3: G has a 7-vertex v with two 4-neighbors x and y. See Figure 17. With Case 2 forbidden, we may assume that x and y are not consecutive neighbors of v. Let $\{x', x''\} = T(xv)$ and $\{y', y''\} = T(yv)$. Let $G' = G - \{x, y\} + \{x'x'', y'y''\}$; we have deleted two vertices and inserted two edges so that G' is a smaller triangulation (the size of $\{x', x'', y', y''\}$ may be 3 or 4). By minimality, G' has an acyclic 7-coloring. Let $\hat{x}$ and $\hat{y}$ be the remaining neighbors in G of x and y, respectively.

On each of $N_G(x)$ and $N_G(y)$, the number of colors used by ϕ is 3 or 4 (like S in Case 2). If both sets have size 4, then give each of x and y any color not used by ϕ on its neighbors in G; this prevents 2-colored cycles through x and y.

If exactly one of these two sets has size 3, then we may assume that ϕ uses three colors on $N_G(x)$. Now $\phi(v) = \phi(\hat{x})$. Since x and y are not yet colored, ϕ uses at most six colors on $N_G(v) \cup \{v\}$. Choose a color outside this set for x, and choose a color for y outside the set of four colors used by ϕ on $N_G(y)$. This prevents 2-colored cycles through x and y.

Hence we may assume that $\phi(v) = \phi(\hat{x}) = \phi(\hat{y})$. Now let S be the set of colors used by ϕ on the six vertices of $N_G(v) \cup \{v\}$ already colored. If $|S| \le 5$, then two distinct colors outside S can be used on x and y. If $|S| = 6$, then recolor v to the color outside S; still there are no 2-colored cycles through v. Now $N_G(x)$ and $N_G(y)$ are each colored with four distinct colors, so ϕ can be extended to x and y to complete an acyclic 7-coloring of G.

Case 4: The configurations in Cases 1–3 do not occur. If G has a 4-cycle enclosing more than one vertex, then let C be such a 4-cycle enclosing the fewest vertices; otherwise C is the external 3-cycle. Let H be the subgraph consisting of C and everything enclosed by C.

By Lemma 7.1 and the exclusion of Cases 1–3, H has an internal 5-vertex v with an internal neighbor u having degree 5 or 6. Let $\{x, y\} = T(uv)$. Form G' from G by contracting vx and vy, letting z be the combined vertex. By the minimality of G, there is an acyclic 7-coloring ϕ of G'. Use the same coloring on G, with x and y receiving color $\phi(z)$. With v not yet colored, there are no improperly colored edges or 2-colored cycles. See Figure 18.

Let S be the set of colors used on $N_G(v)$. If $|S| = 4$, then give v a color outside S. If $|S| = 3$ (so $\phi(u)$ appears on another neighbor of v), then give v a color not in S and not on any neighbor of u. Such a color exists, since $d(u) \le 6$. Now G is properly colored, and any 2-colored cycle C' through v must also visit x and y. If C' has length more than 4, then it

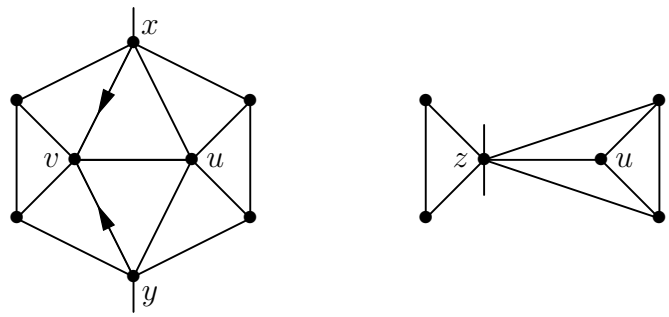

Figure 18: Case 4 of Theorem 7.2

contracts to a 2-colored cycle in G', which contradicts ϕ being an acyclic coloring of G'.

Finally, suppose that C' has length 4. Note that C' encloses either $N(v) - \{u, x, y\}$ or the vertex u and at least one neighbor of u; in each case at least two vertices. If x and y are not both on C, then C' is enclosed by C and contradicts the choice of C. If x and y are both on C, then the fourth vertex of C' may be outside H, but in this case the path $\langle x, v, y\rangle$ combines with one of the two other vertices of C to again contradict the choice of C. $\square$

As in Theorem 6.10, the use of contraction in the proof means that the proof does not provide a bound for the list version; the same holds for Borodin's proof of $a(G) \le 5$. Nevertheless, Borodin et al. [49] conjectured that $a_\ell(G) \le 5$ for every planar graph G, where $a_\ell(G)$ is the least k such that for every k-uniform list assignment L, there is an acyclic L-coloring of G. They proved (via discharging) that $a_\ell(G) \le 7$ for planar graphs. The conjecture would strengthen both the result of Borodin [36] that $a(G) \le 5$ when G is planar and the result of Thomassen [219] that $\chi_\ell(G) \le 5$ when G is planar.

Toward the conjecture, Borodin and Ivanova [57] proved $a_\ell(G) \le 5$ for planar graphs with no 4-cycles, improving [78, 83, 184, 231, 239] (most forbidding 4-cycles and something else. In [56], only special 4-cycles are forbidden. Montassier, Raspaud, and Wang [183] conjectured that $a_\ell(G) \le 4$ when G is planar with no 4-cycles and proved this in some cases; $a_\ell(G) \le 4$ holds when both 4-cycles and 5-cycles are forbidden [58, 79].

As in ordinary coloring, larger girth for planar graphs (or smaller $\mathrm{mad}(G)$) makes acyclic coloring easier. Montassier [179] and Borodin et al. [48] proved $a_\ell(G) \le 4$ and $a_\ell(G) \le 3$ for planar graphs with girth at least 5 and at least 7, respectively, improving [64, 68, 76, 80, 180]. Weaker results are fairly easy to prove. When G is planar with girth at least 7 (Exercise 6.5) or when $\mathrm{mad}(G) < \frac{8}{3}$ (Exercise 3.3), G contains a 1^--vertex or a 2-vertex with a 3^--neighbor. These configurations are reducible for $a_\ell(G) \le 4$ (Exercise 7.3), so graphs in these families are acyclically 4-choosable. For planar graphs with girth at least 5, we prove a structure theorem that yields $a_\ell(G) \le 6$; it has further applications in this section.

Lemma 7.3 (Cranston–Yu [96])**.** *If G is a planar graph with girth at least 5 and $\delta(G) \ge 2$,*

then G has a 2-vertex with a 5^--neighbor or a 5-face whose incident vertices are four 3-vertices and a 5^--vertex.

Proof. (sketch) Let G be such a graph containing none of the specified configurations. Assign charges by balanced charging; discharging will make all vertices and faces happy when the specified configurations do not occur.

(R1) Each 3^--vertex v takes $\frac{4-d(v)}{d(v)}$ from each incident face.
(R2) Each 6^+-vertex v gives $\frac{d(v)-4}{d(v)}$ to each incident face.

The rules immediately make each vertex happy (5-vertices end positive), and it remains only to check that each face ends happy. The configurations in Figure 19 show faces that end with charge 0; Exercise 7.1 requests the verification that other faces end happy. □

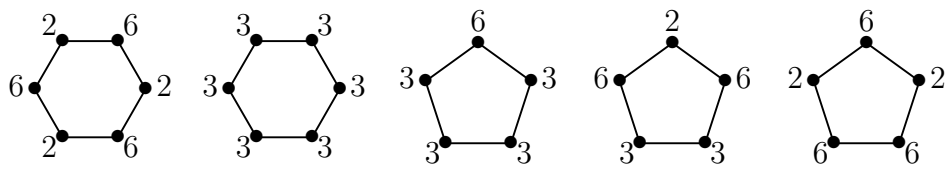

Figure 19: Sharp configurations for Lemma 7.3

Planar graphs with girth at least 6 satisfy $\mathrm{mad}(G) < 3$, and this is sufficient for acyclic 6-choosability. To strengthen the result on planar graphs, we must work harder.

Theorem 7.4. *If G is a planar graph with girth at least 5, or if $\mathrm{mad}(G) < 3$, then $a_\ell(G) \le 6$.*

Proof. Since a 1^--vertex lies in no cycle, its color need only avoid that of its (possible) neighbor. Hence a 1^--vertex is reducible for $a_\ell(G) \le 6$, and we may assume $\delta(G) \ge 2$.

Since $\mathrm{mad}(G) < 3$ and $\delta(G) \ge 2$ guarantees a 2-vertex with a 5^--neighbor (Exercise 2.1), it suffices to show that the configurations in Lemma 7.3 are reducible for $a_\ell(G) \le 6$. Let L be a 6-uniform list assignment for G.

First consider a 2-vertex v with a 5^--neighbor u. Let ϕ be an acyclic L-coloring of $G - v$. If the colors on $N_G(v)$ are distinct, then color v with a color in $L(v)$ other than those. If the colors on $N(v)$ are equal, then color v with a color not used on $N_G(v) \cup N_{G-v}(u)$. Since $|N_{G-v}(u)| \le 4$, this forbids at most five colors, and a color remains available in $L(v)$. Now there are no 2-colored cycles in $G - v$ and none through v.

For the remaining configuration, let v_1, v_2, v_3, v_4, w be the vertices on a 5-face, with each v_i of degree 3 and $d(w) \le 5$. Let x_i be the neighbor of v_i outside the 5-cycle (see Figure 20).

Let ϕ be an acyclic L-coloring of $G - \{v_2, v_3\}$. We consider three cases, depending on whether ϕ uses one color or two colors on $N_G(v_2)$ and on $N_G(v_3)$. (a) If $\phi(v_1) \ne \phi(x_2)$ and $\phi(v_4) \ne \phi(x_3)$, then choose $\phi(v_2)$ and $\phi(v_3)$ distinct and outside $\{\phi(v_1), \phi(x_2), \phi(x_3), \phi(v_4)\}$.

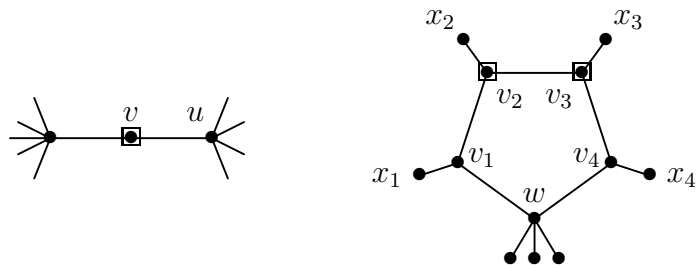

Figure 20: Reducible configurations for Theorem 7.4

(b) If $\phi(v_1) = \phi(x_2)$ but $\phi(v_4) \neq \phi(x_3)$, then choose $\phi(v_2) \notin \{\phi(w), \phi(x_1), \phi(x_2)\}$ and $\phi(v_3) \notin \{\phi(v_4), \phi(x_3), \phi(v_2), \phi(v_1)\}$. (c) If $\phi(v_1) = \phi(x_2)$ and $\phi(v_4) = \phi(x_3)$, then choose $\phi(v_2) \notin \{\phi(w), \phi(v_1), \phi(x_1), \phi(v_4)\}$ and $\phi(v_3) \notin \{\phi(v_2), \phi(v_4), \phi(w), \phi(x_4)\}$. In each case, the coloring is proper, and the new vertices lie in no 2-colored cycle. □

For general graphs, let $f(k) = \max\{a(G)\colon \Delta(G) = k\}$. Grünbaum [123] proved $f(k) \leq k^2$ and $f(3) \leq 4$ (see also [214]) and asked whether $f(k) = k + 1$ in general. Burštein [73] and Kostochka [152] independently proved $f(4) = 5$. For $f(5)$, the best result is $f(5) \leq 7$ (Kostochka–Stocker [156], improving [110] and [252]), but it is not known to be sharp. In general, the answer to Grünbaum's question is "no"; Alon, McDiarmid, and Reed [10] showed by probabilistic methods that $\Omega(\frac{k^{4/3}}{(\log k)^{1/3}}) \leq f(k) \leq 50k^{4/3}$. Kostochka and Stocker [156] gave a short constructive proof of $f(k) \leq 1 + \lfloor \frac{1}{4}(k+1)^2 \rfloor$, improving [109].

Without restriction to planar graphs, Borodin, Chen, Ivanova, and Raspaud [48] proved $a_\ell(G) \leq 3$ when G has girth at least 7 and $\mathrm{mad}(G) < \frac{14}{5}$. In general, when $\mathrm{mad}(G)$ is smaller than $\Delta(G)$, the bound on $a(G)$ in terms of $\Delta(G)$ can be improved. For example, Fiedorowicz [115] proved that $a(G) \leq 6$ when $\Delta(G) \leq 5$ and $\mathrm{mad}(G) < 4$.

Although $a(G)$ is bounded in terms of $\Delta(G)$, it is not bounded in terms of $\chi(G)$, since $a(K_{n,n}) = n + 1$. On the other hand, there is such a bound for line graphs. Let $a'(G)$ be the minimum number of colors in an acyclic edge-coloring of G, which is just an acyclic coloring of $L(G)$. Esperet and Parreau [108] proved $a'(G) \leq 4\Delta(G) - 4$, improving [177], which improved [10]. Girth restrictions permit smaller coefficients (see [108], improving [185, 188]). A stronger bound is conjectured.

Conjecture 7.5 (Fiamčik [113]; see also [11])**.** *Always $a'(G) \leq \Delta(G) + 2$.*

Alon, Sudakov, and Zaks [11] proved the existence of a constant c such that girth at least $c\Delta(G)\log\Delta(G)$ guarantees $a'(G) \leq \Delta(G) + 2$. Stiebitz et al. [216, p. 263] suggested that always $a'(G) \leq \Delta(G) + 1$, which holds for random regular graphs [187]. Of course, $\Delta(G)$ is a lower bound. For planar graphs, there is an analogue of Vizing's Planar Graph Conjecture:

Conjecture 7.6 (Cohen, Havet, and Müller [85])**.** *There is a positive integer k such that if G is a planar graph with $\Delta(G) \geq k$, then $a'(G) = \Delta(G)$.*

The existence of such k was proved in [85] for the family of planar graphs with $\mathrm{mad}(G) \le 4-\epsilon$, for any fixed ϵ. When $\mathrm{mad}(G) < \frac{10}{3}$, Basavaraju et al. [28] proved that $\Delta(G) \ge 19$ suffices. Several papers have provided pairs (k, g) such that $a'(G) = \Delta(G)$ for all planar graphs with $\Delta(G) \ge k$ and girth at least g.

Lemma 6.5 easily yields $a'(G) \le 2\Delta(G) + 33$ when G is planar (Exercise 7.5). Wang et al. [234] proved that $\Delta(G) + 7$ colors suffice for planar graphs, improving upper bounds of $2\Delta(G) + 29$ [117], $\max\{2\Delta(G) - 2, \Delta(G) + 22\}$ [136], $\Delta(G) + 25$ [85], $\Delta(G) + 12$ [28], and $\Delta(G) + 10$ [124]. The bound improves to $\Delta(G) + 3$ for girth at least 4 [27] (improving [117]), to $\Delta(G) + 2$ for girth at least 5 [70, 136], and to $\Delta(G) + 1$ for girth at least 6 [70, 253] or for girth 5 when $\Delta(G) \ge 11$ [253]).

Bounds on $a'(G)$ have been given for planar graphs under various sets of forbidden cycle lengths; one goal is to prove Conjecture 7.5 under the weakest possible restrictions. For example, $a'(G) \le \Delta(G) + 2$ for planar graphs with no 5-cycles [208], with no 4-cycles [235], and with no 3-cycles [209]; furthermore, the bound still holds when no 3-cycle is adjacent to a 4-cycle [236] and when no 3-cycle is adjacent to a 6-cycle [238].

Beyond planar graphs, Fiedorowicz [114] proved for integer t that $a'(G) \le (t-1)\Delta(G) + 2t^3 - 3t + 2$ when $\mathrm{mad}(G) < 2t$. In terms of the maximum degree only, $a'(G) \le 4$ when $\Delta(G) = 3$ except for K_4 and $K_{3,3}$ (in [24] for $\delta(G) < 3$ and in [12] for 3-regular graphs, improving [214]). If $\Delta(G) \le 4$, then $a'(G) \le 6$ ([25] for $\delta(G) < 4$ and [237] for 4-regular. Also, $a'(G) = \Delta(G)$ when G is obtained from any other graph by subdividing each edge [116].

Finally, Lai and Lih [166] extended many of these results to the acyclic edge-choosability $a'_\ell(G)$, which is the minimum k such that an acyclic edge-coloring can be chosen from any k-uniform assignment on the edges. For example, $a'_\ell(G) \le \Delta(G) + 1$ for outerplanar graphs and for non-regular graphs with $\Delta(G) = 3$ (the bound is 5 for 3-regular graphs, and it is $\Delta(G)$ for outerplanar graphs with $\Delta(G) \ge 5$ [210]). In [165], they observed that many of the bounds on $a'(G)$ for planar graphs with forbidden cycle lengths hold also in the list context; also, $\mathrm{mad}(G) < 4$ implies $a'_\ell(G) \le \Delta(G) + 5$ and $\mathrm{mad}(G) < 3$ implies $a'_\ell(G) \le \Delta(G) + 1$. Extending Conjecture 7.5, they posed the following conjectures:

Conjecture 7.7 (Lai and Lih [166])**.** *Every graph G satisfies $a'_\ell(G) = a'(G)$. Every graph G satisfies $a'_\ell(G) \le \Delta(G) + 2$.*

Exercise 7.1. Complete the proof of Lemma 7.3.

Exercise 7.2. (Grünbaum [123], Kostochka–Mel'nikov [153]) Prove that the two graphs in Figure 21 are not acylically 4-colorable. The half-edges leaving the figure on the right lead to an additional vertex having the same neighborhood as the central vertex.

Exercise 7.3. Prove that a 2-vertex with a 3^--neighbor is reducible for acyclic 4-choosability.

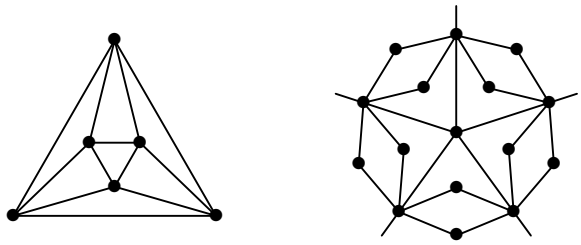

Figure 21: Graphs that are not acyclically 4-colorable

Exercise 7.4. (Basavaraju–Chandran–Kummini [29]) Prove that if G is regular, then $a'(G) \geq \Delta(G)+1$. Prove that if also $|V(G)| < 2\Delta(G)$ and $|V(G)|$ is even, then $a'(G) \geq \Delta(G)+2$. (Comment: Also $a'(K_{n,n}) \geq \Delta(G)+2$ when n is odd [29], with equality when n is an odd prime [26].)

Exercise 7.5. Use Lemma 6.5 to prove $a'_\ell(G) \leq 2\Delta(G)+24$ for planar G.

Exercise 7.6. Use the lemma of van den Heuvel and McGuinness stated in Exercise 6.10 to prove that all planar graphs are acyclically 11-choosable. (Hint: Choose a vertex v by their lemma, then triangulate $N(v)$ in $G-v$ appropriately.)

7.2 Star Coloring

Besides forbidding cycles, we may also restruct the maximum degree or diameter of subgraphs induced by pairs of color classes. We consider the latter first. The paper on acyclic coloring by Grünbaum [123] also introduced these "star colorings".

Definition 7.8. A *star coloring* is an acyclic coloring where the union of any two color classes induces a forest of stars; equivalently, no 4-vertex path is 2-colored. The *star chromatic number* $s(G)$ (also written $\chi_s(G)$) is the minimum number of colors in such a coloring. The *star list chromatic number* $s_\ell(G)$ is the least k such that a star coloring of G can be chosen from any k-uniform list assignment L.

A set I of vertices is a 2-*independent set* if the distance between any two vertices of I exceeds 2. An I,F-*partition* of a graph G, introduced by Albertson et al. [8], is a partition of $V(G)$ into sets I and F such that I is a 2-independent set and $G[F]$ is a forest.

Since every star coloring is an acyclic coloring, always $s(G) \geq a(G)$. All trees are acyclically 2-colorable, but trees with diameter at least 3 are not star 2-colorable.

Lemma 7.9. *Every forest is star 3-colorable. Also, if a graph G has an I,F-partition, then $s(G) \leq 4$. More generally, if $V(G)$ has a partition into $F, I_1, \ldots, I_k$ such that $G[F]$ is a forest and always I_j is a 2-independent set in $G[F \cup I_1 \cup \cdots \cup I_j]$, then $s(G) \leq k+3$.*

Proof. In a tree T, choose a root r and color each vertex v with $d_T(v,r)$, reduced modulo 3. Each connected 2-colored subgraph is a star, consisting of a vertex and its children.

Given the specified partition $F, I_1, \ldots, I_k$, use three colors to produce a star coloring of the forest. Use one additional color on I_j for each j. Still there is no 2-colored 4-vertex path, since the highest-indexed color appearing on it would be a 2-independent set in the subgraph induced by its vertices. □

Extensive results about star colorings and their relationships to other parameters appear in papers by Fertin, Raspaud, and Reed [112] and by Albertson et al. [8], without discharging. For example, for graphs with maximum degree k there is a relatively easy constructive upper bound $s(G) \le k^2 - k + 2$ [8] and a probabilistic argument for $s(G) \le O(k^{3/2})$ [111]. In terms of the acyclic chromatic number, [8] proved $s(G) \le a(G)(2a(G) - 1)$.

For planar graphs, [8] obtained constant bounds: always $s(G) \le 20$ when G is planar, and sometimes $s(G) = 10$ [8]. Let $s(g)$ be the maximum of $s(G)$ over planar graphs with girth at least g, so $10 \le s(3) \le 20$. In [8] are also $s(5) \le 16$ and $s(7) \le 9$, and $s(g) \ge 4$ for all g. In addition, $s(5) \ge 6$ [163], $s(7) \ge 5$ [221], and $s(G) \le 14 - g$ when $g \in \{6, 7, 8\}$ (Kündgen and Timmons [163]) or $g = 9$ (Timmons [221]). The upper bounds in [163] hold also for the list version, and the arguments for $g \in \{7, 8\}$ use only the corresponding strict bounds on $\mathrm{mad}(G)$ (that is, $\frac{14}{5}$ and $\frac{8}{3}$).

This brings us to the application of Lemma 7.9. Our aim is not to study general results on $s(G)$ for dense graphs, but rather to use discharging to study what sparseness is need to reach $s(G) \le 4$. Timmons [221] proved that every graph with $\mathrm{mad} < \frac{7}{3}$ has an I, F-partition. Later Bu et al. [71] improved the result to $\mathrm{mad} < \frac{26}{11}$, which yields $s(13) = 4$. The general version of Lemma 7.9 was used in [71] to prove that $\mathrm{mad}(G) < \frac{18}{7}$ (and hence girth 9 for planar graphs) implies $s(G) \le 5$.

We prove the result of [221] here; it yields $s(14) = 4$. Recall that 2-vertices on threads incident to a vertex v are *weak 2-neighbors* of v.

Theorem 7.10 (Timmons [221])**.** *If* $\mathrm{mad}(G) < \frac{7}{3}$*, then* $V(G)$ *can be partitioned into sets* I *and* F *such that* I *is 2-independent and* $G[F]$ *is a forest.*

Proof. We may assume that no component is a cycle, because in such a component it suffices to put one vertex into I. Having forbidden 2-regular components, Lemma 2.6 with $t = 2$ implies that G has a 1^--vertex, a 3-thread, or a 3-vertex with at least five weak 2-neighbors. Here the discharging rule is the obvious one when 1^--vertices are forbidden: each 2-vertex takes $\frac{1}{6}$ from each end of the maximal thread containing it.

It therefore suffices to prove these configurations reducible for the existence of I, F-partitions. A 1^--vertex v can be added to the forest in any such partition of $G - v$.

Let $\langle v, w, x, y, z \rangle$ be a 3-thread in G; vertices w, x, y each have degree 2. Let I', F' be an I, F-partition of $G - \{w, x, y\}$. If v or z is in I', then add $\{w, x, y\}$ to F' to form an I, F-partition of G. Otherwise, add x to I' and $\{w, y\}$ to F', as in Figure 22.

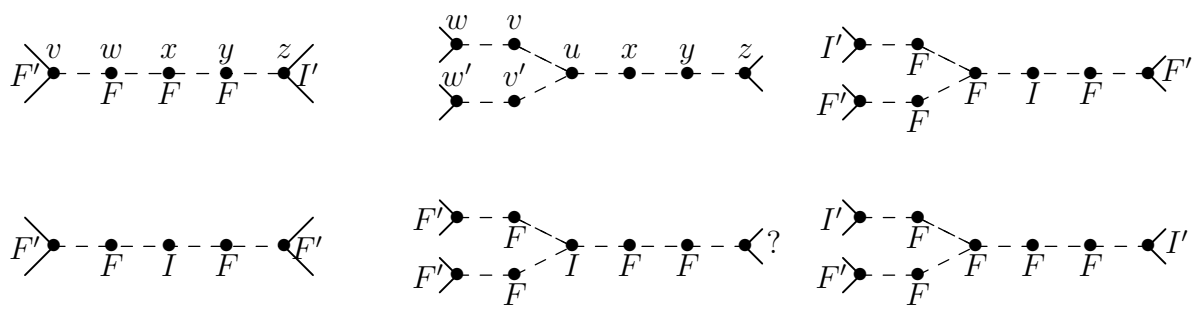

Figure 22: Reducibility cases for Theorem 7.10

Finally, consider a 3-vertex u with at least five weak 2-neighbors. If one of the threads has three 2-vertices, then G has a 3-thread. Otherwise, u has at least two incident 2-threads plus a 2-neighbor on the third incident thread. It suffices to consider a 2-thread $\langle u, x, y, z\rangle$ and two 1-threads $\langle u, v, w\rangle$ and $\langle u, v', w'\rangle$ incident to u (see Figure 22). Let $R = \{w, w'\}$ and $S = \{u, x, y, v, v\}$, and let $G' = G - S$. Let I', F' be an I, F-partition of G'. If $R \cap I' = \varnothing$, then add x to I' and the rest of S to F'. If $R \cap I' \neq \varnothing$ and $z \notin I'$, then add v to I' and the rest of S to F'. If $R \cap I' \neq \varnothing$ and $z \in I'$, then add all of S to F'. In each case, the resulting sets form an I, F-partition of G. $\square$

Reducibility of 3-vertices in this proof uses only four weak 2-neighbors, when they distribute as $(4, 0, 0)$ or $(3, 1, 0)$ or $(2, 1, 1)$ on the incident threads. This suggests using $\overline{d}(G) < \frac{12}{5}$; then 2-vertices taking $\frac{1}{5}$ from each end of their maximal incident threads would leave 2-vertices and 3-vertices both with charge at least $\frac{12}{5}$ when each 3-vertex has at most three weak 2-neighbors. The difficulty is that four weak 2-neighbors may be distributed as $(2, 2, 0)$, and this configuration is not reducible. Note that $\frac{7}{3} < \frac{26}{11} < \frac{12}{5}$. The strengthening of Theorem 7.10 to the hypothesis $\mathrm{mad}(G) < \frac{26}{11}$ [71] must address the situation where all 3-vertices with four weak 2-neighbors have this configuration. In fact, even the configuration consisting of two adjacent such 3-vertices is not reducible (Exercise 7.7).

It is not known whether the improvement of Theorem 7.10 to $\mathrm{mad}(G) < \frac{26}{11}$ is sharp. The smallest known value of $\mathrm{mad}(G)$ in graphs without an I, F-partition is $\frac{5}{2}$ (Exercise 7.8). The largest known girth of a planar graph without an I, F-partition is 7 [221].

Problem 7.11. Determine the largest b such that $\mathrm{mad}(G) < b$ guarantees an I, F-partition of G. Determine the smallest g such that all planar graphs with girth at least g have I, F-partitions. Do the answers differ from the values that guarantee $s(G) \leq 4$?

We mentioned that some of the results on star coloring of planar graphs extend to the list version of star coloring. In particular, Kündgen and Timmons [163] proved that $s_\ell(G) \leq 8$ when G is a planar graph with girth at least 6. This is a special case of the result of Chen, Raspaud, and Wang [81] that $s_\ell(G) \leq 8$ when $\mathrm{mad}(G) < 3$.

Another way to study star coloring of sparse graphs is to seek sharp bounds for fixed small values of the maximum degree. When $\Delta(G) = 2$, the upper bound $\Delta(G)^2 - \Delta(G) + 2$ by Albertson et al. [8] specializes to 4, which is sharp due to the 5-cycle. It specializes to 8 when $\Delta(G) = 3$, but Chen, Raspaud, and Wang [82] proved that $s(G) \le 6$ for all graphs with maximum degree 3. This also is sharp, by the 3-regular graph in Exercise 7.9.

Always $\mathrm{mad}(G) \le \Delta(G)$, so one can attempt to extend coloring results for graphs with $\Delta(G) \le D$ to the larger family with $\mathrm{mad}(G) \le D$, perhaps with some small additional number of colors. The results of [81] and [82] for $D = 3$ motivate such an effort. Unfortunately, no such result is possible for $D \ge 4$.

Proposition 7.12. *For $k \in \mathbb{N}$, there is a graph G such that* $\mathrm{mad}(G) < 4$ *and $s(G) > k$. In fact, there is also such a graph with $a(G) > k$.*

Proof. Let G_n be the graph obtained from the complete graph K_n by subdividing every edge once. Note that G_n has $n(n+1)/2$ vertices and $n(n-1)$ edges; thus $\overline{d}(G_n) = 4 - \frac{8}{n+1}$, and in fact $\mathrm{mad}(G_n) = 4 - \frac{8}{n+1}$.

If $n > k^2 + k$, then by the pigeonhole principle any k-coloring ϕ of G_n has some $k+2$ high-degree vertices with the same color. Now color each edge of a complete graph H on these vertices by the color of their common neighbor in ϕ. If two incident edges of H have the same color, then ϕ is not a star coloring of G_n. Thus $s(G_n) \le k$ requires $\chi'(H) \le k$. However, $\chi'(K_{k+2}) > k$, so $s(G_{k^2+k+1}) > k$.

The same idea gives a construction for $a(G) > k$ a bit later. With $n > 2k^2$, we obtain $2k+1$ high-degree vertices with the same color. Now ϕ fails to be an acyclic coloring of G_n if the edges of some cycle in H have the same color. A complete graph decomposes into k forests only if it has at most $2k$ vertices. Hence $a(G_{2k^2+1}) > k$. $\square$

The constructions in Proposition 7.12 suggest several questions.

Problem 7.13. For $3 < b < 4$, is it true that $s(G)$ or $a(G)$ (or their list versions) is bounded when $\mathrm{mad}(G) < b$? Given k, what is the minimum number of vertices in a graph G with $\mathrm{mad}(G) < 4$ and $s(G) > k$ (or $a(G) > k$)?

Exercise 7.7. Argue that the configuration consisting of two 3-vertices each incident to two 2-threads is not reducible for the existence of I, F-partitions.

Exercise 7.8. Construct an infinite family of graphs with maximum average degree $\frac{5}{2}$ that have no I, F-partition.

Exercise 7.9. (Fertin–Raspaud–Reed [111], Albertson et al. [8]) For each graph G in Figure 23, prove that $\chi_s(G) = 6$.

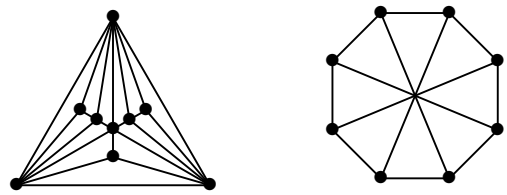

Figure 23: Graphs with star chromatic number 6

Exercise 7.10. (Bartnicki et al. [23]) Let G be a graph. A function $f\colon V(G) \to \{1, \ldots, k\}$ is an *additive k-coloring* of G if the function ϕ defined by $\phi(v) = \sum_{u \in N(v)} f(u)$ is a proper coloring of G. The *additive chromatic number* of a graph G, written $\chi_{\text{add}}(G)$, is the least k such that G has an additive k-coloring. From [23], every forest is additively 2-colorable (one can prove this inductively). Conclude that if G has an I, F-partition, then $\chi_{add}(G) \le 4$. (Comment: In fact, the result from [23] uses the Combinatorial Nullstellensatz to prove that forests are additively 2-choosable.)

7.3 Linear Coloring

Instead of restricting the diameter of bicolored forests in acyclic colorings, we now restrict their maximum degree. Introduced by Yuster [254], a *linear coloring* of a graph G is an acyclic coloring in which every bicolored forest has maximum degree at most 2. Let $\text{lc}(G)$ denote the minimum number of colors needed. Always $\text{lc}(G) \ge \lceil \frac{1}{2}\Delta(G) \rceil + 1$, because each color appears at most twice in $N(v)$ when $d(v) = \Delta(G)$, and v itself needs another color. For planar graphs with girth at least 7, 9, 11, or 13, Raspaud and Wang [194] proved that $\text{lc}(G) = \lceil \frac{1}{2}\Delta(G) \rceil + 1$ when the maximum degree is at least 13, 7, 5, or 3, respectively.

A graph G is *linearly k-choosable* if for every k-uniform list assignment L, there is a linear L-coloring of G. Esperet, Montassier, and Raspaud [107] introduced this concept, using the notation $\Lambda^l(G)$ for the least such k. For consistency with other notation in this paper, we use $\text{lc}_\ell(G)$. We will prove that $\text{lc}_\ell(G) \le \lceil \frac{1}{2}\Delta(G) \rceil + 4$ for every planar graph with girth at least 5, near the trivial lower bound for linear coloring. In [107] there are better bounds for larger girth. In fact, they are the usual corollaries of bounds in terms of $\text{mad}(G)$: if $\text{mad}(G) < b_j$, then G is linearly $(\lceil \frac{1}{2}\Delta(G) \rceil + j)$-choosable, where $b_3 = \frac{8}{3}$, $b_2 = \frac{5}{2}$, and $b_1 = \frac{16}{7}$, the latter requiring $\Delta(G) \ge 3$. See Exercises 7.11–7.13.

Theorem 7.14. *If G is planar with girth at least 5 and $\Delta(G) \le D$, then $\text{lc}_\ell(G) \le \lceil \frac{1}{2}D \rceil + 4$.*

Proof. Let G be a minimal counterexample, with a $(\lceil \frac{1}{2}D \rceil + 4)$-uniform list assignment L from which no linear coloring can be chosen. Every proper induced subgraph G' has a linear L-coloring ϕ. To extend ϕ successfully to a missing vertex w, we seek a color that does not appear on $N_G(w)$ and does not appear on two vertices having distance 2 from w in G.

By minimality, G is connected. If G has a 1-vertex v, then let $G' = G - v$. We must avoid at most $1 + \lfloor \frac{1}{2}(D-1) \rfloor$ colors. Since $|L(v)| = \lceil \frac{1}{2}D \rceil + 4$, a color is available for v.

We may therefore assume $\delta(G) \geq 2$. Since G also is planar with girth at least 5, Lemma 7.3 implies that G has a 2-vertex v with a 5^--neighbor u or a 5-face whose incident vertices are four 3-vertices and a 5^--vertex.

In the case $d_G(v) = 2$, let w be the other neighbor of v, and let $G' = G - v$. Since $d_G(u) \leq 5$, the number of forbidden colors is at most $2+\left\lfloor \frac{(D-1)+(5-1)}{2} \right\rfloor$, which equals $\lceil \frac{1}{2}D \rceil + 3$. Since $|L(v)| = \lceil \frac{1}{2}D \rceil + 4$, a color remains available for v.

Hence we may assume that G contains a 5-face whose incident vertices are 3-vertices v_1, v_2, v_3, v_4 in cyclic order plus a 5^--vertex w. Let x_2 and x_3 denote the remaining neighbors of v_2 and v_3, respectively, as labeled in Figure 20.

Let $G' = G - \{v_2, v_3\}$, and let ϕ be a linear L-coloring of G'. Each of the two missing vertices has two colored neighbors and at most $4 + (D-1)$ colored vertices at distance 2. Since $\lfloor (D+4-1)/2 \rfloor = \lceil \frac{1}{2}D \rceil + 1$, there are at most $\lceil \frac{1}{2}D \rceil + 3$ colors to avoid, so for each of v_2 and v_3 a color remains available.

However, each could be left with only the same single color available. To complete the proof, we show that v_2 or v_3 has at least two colors available; we extend to that vertex last. If $\phi(v_1) = \phi(x_2)$, then v_1 and x_2 together forbid only one color, and v_2 still has two available. Hence we may assume $\phi(v_1) \neq \phi(x_2)$. Similarly, we may also assume $\phi(v_4) \neq \phi(x_3)$.

Since $\phi(v_1) \neq \phi(x_2)$, coloring v_3 cannot give v_2 three neighbors with the same color. Also, we claim that coloring v_3 cannot create a 2-colored cycle. Since $\phi(v_4) \neq \phi(x_3)$, such a cycle must contain v_2 and one of $\{v_4, x_3\}$. By requiring that v_2 does not receive a color appearing on two or more vertices at distance 2 from v_2, we prevent such a 2-colored cycle.

Therefore, after extending the coloring to v_2 as before, it suffices for v_3 to avoid the colors appearing on $\{v_2, x_3, v_4\}$ or on at least two vertices of $N(v_4) \cup N(x_3)$. The number to avoid is at most $3 + \left\lfloor \frac{(D-1)+2}{2} \right\rfloor$, which equals $\lceil D/2 \rceil + 3$. Hence a color remains available for v_3. (That is, before coloring v_2, two colors were available at v_3.) $\square$

A similar but more detailed argument proves $\mathrm{lc}_\ell \leq \lceil \frac{1}{2}\Delta(G) \rceil + 3$ when G is planar with girth at least 5 and $\Delta(G) \geq 13$. To do this, we first refine Lemma 7.3 to show that if $\Delta(G) \geq 13$, then in (RC1) either a 2-vertex has a 4^--neighbor or it has a 5-neighbor and a 12^--neighbor. Furthermore, we can require in (RC2) that at most two outside neighbors of the four 3-vertices on the 5-face have high degree. These improvements use that faces only need $\frac{1}{3}$ from each incident 6^+-vertex v, not $\frac{d(v)-4}{d(v)}$. Thus 7^+-vertices have extra charge they can send to 2-neighbors and 3-neighbors. With more careful analysis, these new configurations are reducible even with only $\lceil \frac{1}{2}\Delta(G) \rceil + 3$ colors.

The problem is harder for general planar graphs, even in the non-list version. Li, Wang, and Raspaud [169] proved that if G is planar with $\Delta(G) \geq 52$, then $\mathrm{lc}(G) \leq (\lfloor .9\Delta(G) \rfloor + 5)$, dropping the coefficient below 1. Along the way, they proved $\mathrm{lc}(G) \leq \Delta(G) + 15$ for all planar G (Exercise 7.15). Finally, they asked whether there is a constant c such that $\mathrm{lc}(G) \leq \lceil \frac{1}{2}\Delta(G) \rceil + c$ for every planar graph G.

Beyond planar graphs, Yuster [254] studied $\mathrm{lc}(G)$ when $\Delta(G) = k$, proving that it is bounded by $O(k^{3/2})$ and constructing instances where $\Omega(k^{3/2})$ colors are needed. Contrast this with the acyclic chromatic number, where the upper bound is $O(k^{4/3})$ and examples are known needing $\Omega(k^{4/3}/(\log k)^{1/3})$ colors. Linear coloring requires more colors but is simpler to analyze in the sense that the upper and lower bounds on the extremal problem for fixed maximum degree differ only by a constant factor. In terms of the maximum degree, Li et al. [169] proved $\mathrm{lc}(G) \le \binom{\Delta(G)+1}{2}$ in general, plus $\mathrm{lc}(G) \le 8$ when $\Delta(G) = 4$ and $\mathrm{lc}(G) \le 14$ when $\Delta(G) = 5$. Also, Esperet et al. [107] proved that G is linearly 5-choosable when $\Delta(G) = 3$ and conjectured that $K_{3,3}$ is the only such graph that is not linearly 4-choosable.

We previously mentioned the arboricity problem, where the edge set rather than the vertex set is partitioned. To distinguish edge-partitioning from coloring problems, we let a *decomposition* of a graph G be a set of edge-disjoint subgraphs whose union is G. Decomposition into matchings, known as edge-coloring, is just proper coloring of the line graph. Decomposition into linear forests is not the same problem as linear coloring of the line graph, because vertices on a path in the line graph of G may correspond to the edges of a star in G. However, since a path in G does become a (shorter) path in the line graph, the *linear arboricity* of a graph (the minimum number of linear forests needed to decompose it) is always at least the linear chromatic number of its line graph. We use the common notation $\mathrm{la}(G)$ for the linear arboricity.

Trivially, $\mathrm{la}(G) \ge \lceil \Delta(G)/2 \rceil$, but equality when G is $2r$-regular would require each color class to be a spanning 2-regular subgraph and hence contain cycles. Akiyama, Exoo, and Harary [4, 5] conjectured that always $\mathrm{la}(G) \le \lceil (\Delta(G)+1)/2 \rceil$. This was proved for $\Delta(G) \in \{1, 2, 3, 4, 5, 6, 8, 10\}$ (together) in [4, 5, 104, 125]. For $\epsilon > 0$, Alon [9] proved that $\mathrm{la}(G) \le (\frac{1}{2} + \epsilon)\Delta(G)$ when $\Delta(G)$ is sufficiently large. When G is planar, the conjecture was proved for $\Delta(G) \ge 9$ by Wu [247] (presented below) and for $\Delta(G) = 7$ by Wu and Wu [248], so the proof is now complete for planar graphs.

In the proof, Borodin's extension of Kotzig's Theorem (Lemma 6.1) is very helpful: if a planar graph G with $\delta(G) \ge 3$ has no edge with weight at most 11, then it has a 3-alternating 4-cycle. Let a *linear t-decomposition* be a decomposition into t linear forests. We view the t linear forests as t colors in an edge-coloring.

Theorem 7.15 (Wu [247])**.** *If G is a planar graph with $\Delta(G) \ge 9$, then* $\mathrm{la}(G) \le \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$.

Proof. It suffices to show for each positive integer t with $t \ge 5$ that every planar graph G with $\Delta(G) < 2t$ has a linear t-decomposition.

For an edge uv with weight at most $2t+1$, consider a linear t-decomposition of $G - uv$. Since $d_{G-uv}(u) + d_{G-uv}(v) < 2t$, fewer than t colors appear twice at u or twice at v or once at each. Thus a color is available at uv to extend the linear decomposition to G.

Hence we may assume weight at least $2t+2$ for every edge. Since $\Delta(G) < 2t$, this yields $\delta(G) \ge 3$. Now Lemma 6.1 yields a 4-cycle $[u, x, v, y]$ in G with $d(u) = d(v) = 3$. Let

u' and v' be the remaining neighbors of u and v, respectively (see Figure 24). Note that $\{u, v\} = \{u', v'\}$ is forbidden, but $u' = v'$ is possible, requiring a similar analysis that we omit.

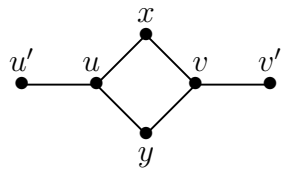

Figure 24: Reducible configuration for Theorem 7.15

The weight restrictions and $\Delta(G) \le 2t-1$ imply that each of x, y, u', v' has degree $2t-1$ in G. Hence a linear t-decomposition of $G - \{u, v\}$ has $2t-3$ colored edges at both x and y and $2t-2$ colored edges at both u' and v'. We conclude that at most one color is missing at each of these vertices; call a vertex "bad" if a color is missing. For a vertex z, let $C(z)$ and $C'(z)$ denote the sets of colors appearing at z exactly once and at most once, respectively.

Case 1: Neither x nor y is bad. Here $|C(x)| = |C(y)| = 3$. First color uu' from $C'(u')$, then ux from $C(x)$, and then uy from $C(y)$. Make the colors on ux and uy differ. If u' is bad, then the color on uu' may be used on ux or uy, but otherwise the three edges have distinct colors. Two colors from each of $C(x)$ and $C(y)$ remain, and we ensure these remaining pairs are not equal. Now we can choose distinct colors on the three edges at v.

Case 2: One of x and y is bad. We may assume by symmetry that x is bad; still $|C(y)| = 3$. Give ux the color missing at x, and give xv the one color in $C(x)$. Color vv' from $C'(v')$, different from the color on xv if v' is not bad. Now color vy from $C(y)$ avoiding the colors on xv and vv'. Now color uu' from $C'(u')$, and choose a color for uy from $C(y)$ avoiding that and the color on vy. Note that uu' or uy may have the same color as ux.

Case 3: x and y are both bad. If the one missing color at each of x and y is the same, then use it on ux and vy. Now use the color in $C(x)$ on xv and the color in $C(y)$ on yu. If u' is bad, then its missing color can be used on uu'; otherwise, color uu' from $C(u')$ to avoid the color on yu. The symmetric argument applies to color vv'.

If the missing colors at x and y are different, then use the color missing at x on ux and xv, and use the color missing at y on uy and yv. If a color is missing at u', use it on uu', and then use any color from $C'(v')$ on vv'. By symmetry, then, only $|C(u')| = |C(v')| = 2$ remains, and it suffices to give uu' and vv' distinct colors from these sets. $\square$

Also, Wu [247] proved that $\mathrm{la}(G) = \lceil \frac{1}{2}\Delta(G) \rceil$ when G is planar with $\Delta(G) \ge 13$, using a stronger form of a lemma by Borodin (see Exercise 7.17). Cygan, Kowalik, and Luzar [98] proved that the same conclusion holds in the larger class of planar G with $\Delta(G) \ge 10$, and they conjectured that $\Delta(G) \ge 6$ is sufficient.

Exercise 7.11. (Esperet–Montassier–Raspaud [107]) Prove that a 1-vertex, a 3-thread, and a 3-vertex incident to three 2-threads are all reducible for $\mathrm{lc}_\ell(G) \le \lceil \frac{1}{2}\Delta(G) \rceil + 1$ when $\Delta(G) \ge 3$. Conclude that $\mathrm{lc}_\ell(G) \le \lceil \frac{1}{2}\Delta(G) \rceil + 1$ when $\mathrm{mad}(G) < \frac{16}{7}$ and $\Delta(G) \ge 3$.

Exercise 7.12. ([107]) Prove that a 1-vertex, a 2-thread, and a 3-vertex with three 2-neighbors are all reducible for $\mathrm{lc}_\ell(G) \le \lceil \frac{1}{2}\Delta(G) \rceil + 2$. Conclude that $\mathrm{lc}_\ell(G) \le \lceil \frac{1}{2}\Delta(G) \rceil + 2$ when $\mathrm{mad}(G) < \frac{5}{2}$.

Exercise 7.13. ([107]) Prove that a 1-vertex, a 2-thread, and a 3-vertex with two 2-neighbors are all reducible for $\mathrm{lc}_\ell(G) \le \lceil \frac{1}{2}\Delta(G) \rceil + 3$. Conclude that $\mathrm{lc}_\ell(G) \le \lceil \frac{1}{2}\Delta(G) \rceil + 3$ when $\mathrm{mad}(G) < \frac{8}{3}$.

Exercise 7.14. (van den Heuvel–McGuinness [131]) Prove that every planar graph G with $\delta(G) \ge 3$ has a 5^--vertex v with at most two 12^+-neighbors such that v has a 7^--neighbor if $d(v) \in \{4, 5\}$, and v has an additional 6^--neighbor if $d(v) = 5$. (Hint: Extend the proof of Lemma 6.5 by allowing 5^--vertices to take some charge from their 11^--neighbors. Comment: This is the stronger version of Lemma 6.5 used in [131] to prove $\chi(G^2) \le 2\Delta(G) + 25$ when G is planar; see Exercise 6.13.)

Exercise 7.15. (Li–Wang–Raspaud [169]) The general bound $\mathrm{lc}(G) \le \binom{\Delta(G)+1}{2}$ in [169] yields $\mathrm{lc}(G) \le \Delta(G) + 15$ when $\Delta(G) \le 6$. Use Exercise 6.9 for $\Delta(G) \in \{7, 8\}$ and Exercise 7.14 for $\Delta(G) \ge 9$ to prove $\mathrm{lc}(G) \le \Delta(G) + 15$ for every planar graph G.

Exercise 7.16. (Esperet-Montassier-Raspaud [107]) Use Exercise 7.14 to prove $\mathrm{lc}_\ell(G) \le \Delta(G)+26$ for every planar graph G.

Exercise 7.17. (Wu [247]) Strengthen Lemma 5.3 to show that every planar graph contains an edge of weight at most 15 or a 2-alternating cycle such that some high-degree vertex on the cycle has an additional 2-neighbor outside the cycle. Conclude that $\mathrm{la}(G) = \lceil \frac{1}{2}\Delta(G) \rceil$ for every planar graph G with $\Delta(G) \ge 13$.

7.4 Improper Coloring

We close this section by studying a type of relaxed vertex coloring. When adjacency represents conflict, independent sets are "conflict-free", but one may not need such a strong restriction. There are many ways to weaken the requirements on subgraphs induced by color classes while still remaining rather sparse; one then obtains a coloring parameter by seeking the minimum number of such classes need to cover the vertex set.

Planar graphs are 4-colorable. Grötzsch's Theorem gives 3-colorability for the restricted class of triangle-free planar graphs. Alternatively, we study what relaxations of the restriction on color classes permit successful 3-colorings.

For example, one could require that the subgraph induced by a color class have no $(k+1)$-edge-connected subgraph. Setting $k = 0$ yields the ordinary chromatic number, and setting $k = 1$ yields the parameter called *vertex arboricity*. As observed by Stein, a consequence of Borodin's acyclic 5-colorability of planar graphs is that the vertices of a planar graph can be colored with three colors so that one class is independent and the other two induce forests.

Intermediate between independent sets and general forests are restricted forests with bounded degree or diameter. For example, one may ask what requirement on girth in planar graphs is needed to guarantee a 3-coloring of the vertices so that two colors induce matchings and the third induces a forest. It would also be natural to explore what bounds on vertex arboricity (or vertex linear arboricity) follow from $\mathrm{mad}(G) < b$.

In the remainder of this section, we simply bound the vertex degrees in induced subgraphs. The parameters measure how "improper" a color class is allowed to be.

Definition 7.16. A $(d_1, \ldots, d_t)$-*improper* (or "-defective" or "-relaxed") coloring of a graph G is a partition of $V(G)$ into sets $V_1, \ldots, V_t$ such that $\Delta(G[V_i]) \le d_i$ for $1 \le i \le t$. For simplicity, we abbreviate the term to $(d_1, \ldots, d_t)$-coloring and say that such a graph is $(d_1, \ldots, d_t)$-*colorable*. Without indicating the number of colors, a coloring is d-*improper* if each vertex has at most d neighbors of its own color. A graph is $t|d$-*choosable* if from any t-uniform list assignment a d-improper coloring can be chosen.

The Four Color Theorem states that every planar graph is $(0,0,0,0)$-colorable. When only three colors are used, we cannot guarantee $(0,0,0)$-colorings, but for $i,j,k \ge 0$ we can ask for classes of planar graphs guaranteed to be (i,j,k)-colorable or (i,j)-colorable. The questions seem even more natural in terms of $\mathrm{mad}(G)$, recalling the classical result of Lovász [172] stating that if $\sum_{i\in 1}^{t} d_i \ge D - t + 1$, then every graph with maximum degree D is $(d_1, \ldots, d_t)$-colorable.

Cowen, Cowen, and Woodall [87] proved that every planar graph is $(2,2,2)$-colorable. Eaton and Hull [103] and Škrekowski [211] extended this, independently proving that every planar graph is $3|2$-choosable, and Cushing and Kierstead [97] proved that they are $4|1$-choosable. Škrekowski [212] proved that triangle-free planar graphs are $3|1$-choosable. Toward Steinberg's Conjecture that planar graphs without cycles of lengths 4 or 5 are 3-colorable, it has been proved that such graphs are $(1,1,0)$-colorable [133, 250] and $(3,0,0)$-colorable [132] (strengthening [75]); also they are $3|1$-choosable [171] (see Exercise 7.23). Graphs in the more general family forbidding only adjacent triangles and 5-cycles are $(1,1,1)$-colorable [249]. Forbidding cycles of lengths 4 and 6 also yields $(3,0,0)$- and $(1,1,0)$-colorings [251]. These results generally are proved by discharging.

Cowen, Goddard, and Jesurum [88] proved results on $(d, \ldots, d)$-colorability of graphs on higher surfaces. They also showed that recognizing $(1,1,1)$-colorability or (d,d)-colorability is NP-complete, even for planar graphs. For lists of size 2, Škrekowski [213] proved that planar graphs with girth at least g are $2|d$-choosable for $(g,d) \in \{(9,1),(7,2),(6,3),(5,4)\}$.

Glebov and Zambalaeva [120] proved that planar graphs with girth at least 16 are $(0,1)$-colorable, which has been strengthened by recent work toward finding the best bounds on $\mathrm{mad}(G)$ to ensure $(d_1, \ldots, d_t)$-colorability. This is another way to use maximum average degree to refine our understanding of coloring, like the problems posed in Section 2. Bounding $\mathrm{mad}(G)$ is successful for (j,k)-colorability where bounding degeneracy and requiring large girth are not: Kostochka and Nešetřil [154] proved that for all $j,k,g \in \mathbb{N}$ there is a 2-degenerate graph of girth g that is not (j,k)-colorable.

Let $F(j,k) = \sup\{b\colon \mathrm{mad}(G) < b \text{ implies that } G \text{ is } (j,k)\text{-colorable}\}$. Note that $F(0,0) = 2$, since forests are 2-colorable and odd cycles are not. The result of [154] yields $F(j,k) < 4$

for all j and k, since $\mathrm{mad}(G) < 4$ when G is 2-degenerate. In [62], $F(j,k)$ was bounded below using a linear program.

Kurek and Ruciński [164] introduced the problem in terms of vertex Ramsey theory; their results imply $\frac{8}{3} \le F(1,1) \le \frac{14}{5}$, and they offered 400,000 (old) Polish zlotys for the exact value. Borodin, Kostochka, and Yancey [69] determined $F(1,1) = \frac{14}{5}$; their result does not extend to the list context. For the precise statement, let the *potential* of a set $A \subseteq V(G)$ be $a|A| - b||A||$, where $||A||$ denotes $|E(G[A])|$ and the coefficients a and b are chosen appropriately. They proved that if $7|A| - 10||A|| \ge 0$ for all $A \subseteq V(G)$, then G is $(1,1)$-colorable (see Exercise 7.20 for sharpness).

Havet and Sereni [128] proved $4 - \frac{4}{k+2} \le F(k,k) \le 4 - \frac{2k+4}{k^2+2k+2}$, even for the list version (for the upper bound, see Exercise 7.18). More generally, if $\mathrm{mad}(G) < t + \frac{tk}{t+k}$, then G is $t|k$-choosable (for $t \ge 2$). With Corrêa [86], they conjectured that for sufficiently large k, planar graphs with $\Delta(G) \le 2k+2$ are (k,k)-colorable (K_{2k+3} is not (k,k)-colorable).

Now consider $j < k$. Borodin and Ivanova [53] proved $F(0,1) \ge \frac{7}{3}$ (improving [120]); Borodin and Kostochka [66] determined $F(0,1) = \frac{12}{5}$. More precisely, G is $(0,1)$-colorable if $6|A| - 5||A|| \ge -2$ for all $A \subseteq V(G)$ (see Exercise 7.19 for sharpness). For triangle-free graphs, Kim, Kostochka, and Zhu [147] proved that G is $(0,1)$-colorable when $11|A| - 9||A|| \ge -4$ for all $A \subseteq V(G)$, which applies when $\mathrm{mad}(G) \le \frac{22}{9}$ and thus includes planar graphs with girth at least 11 (it is sharp even for girth 5; see Exercise 7.21). There are non-$(0,1)$-colorable planar graphs with girth 9, but it is unknown whether they exist with girth 10.

An earlier more general result (weaker for $k = 1$) is $3 - \frac{2}{k+2} \le F(0,k) \le 3 - \frac{1}{k+1}$ [61] (also $\frac{10k+22}{3k+9} \le F(1,k) \le \frac{14k}{4k+1}$ [63]). Borodin and Kostochka [66] proved $F(j,k) = 4 - \frac{2k+4}{(j+2)(k+1)}$ for $k \ge 2j+2$. More precisely, G is (j,k)-colorable if $(2k+2-\frac{k+2}{j+1})|A| - (k+1)||A|| > -1$ for all $A \subseteq V(G)$, and this is sharp. Their result does not cover $F(1,2)$, $F(1,3)$, and $F(2,2)$, and these remain unknown.

These results use discharging. To illustrate the method, and in particular the use of arbitrarily large reducible configurations, we prove the weaker bound $F(1,1) \ge \frac{8}{3}$, strengthened to the list context. This is the bound from [128] and [164]; our method is similar to [128] but less general. As usual, the discharging argument is simpler than the reducibility.

Lemma 7.17. *If $\delta(G) = 2$ and $\overline{d}(G) < \frac{8}{3}$, then G has two adjacent 2-vertices, or some component of the subgraph of G induced by 3-vertices is a tree whose incident edges all are incident also to 2-vertices in G.*

Proof. If G has no such configuration, then let each vertex have initial charge equal to its degree. Let H be the subgraph of G induced by the 3-vertices in G. Discharge as follows:

(R1) Each 2-vertex takes $\frac{1}{3}$ from each neighbor.
(R2) Each 3-vertex takes $\frac{1}{3}$ from each 4^+-neighbor.
(R3) After (R1) and (R2), the vertices in a component of H share their charge equally.

Since 2-vertices have no 2-neighbors, each 2-vertex ends happy. A 4^+-vertex of degree j loses at most 1/3 to each neighbor and hence keeps at least $\frac{2j}{3}$, which is at least $\frac{8}{3}$.

To study the 3-vertices, let S be the vertex set of a component of H (see Figure 25). Let F be the set of edges with endpoints in S whose other endpoints are 2-vertices in G. Since a spanning tree of $H[S]$ has $|S|-1$ edges, and $\sum_{v\in S} d_G(v) = 3|S|$, at most $|S|+2$ edges of G incident to S are not in this tree. Hence if $|F| \geq |S|+2$, then $H[S]$ is a tree, and all edges with one endpoint in S are incident to 2-vertices. This is the second specified configuration; in the discharging argument, we assume that this does not occur.

If $|F| \leq |S|$, then S has enough charge to send $\frac{1}{3}$ along each edge in F while retaining an average of $\frac{8}{3}$ for each vertex of S (which is then spread equally). Hence the vertices of S end happy. Note that this case is forced when $H[S]$ is not a tree.

If $|F| = |S|+1$, then $H[S]$ is a tree and some vertex $v \in S$ has a 4^+-neighbor u. Since v takes $\frac{1}{3}$ from u, again S has enough charge. □

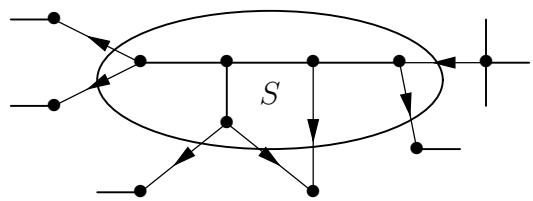

Figure 25: Discharging for Lemma 7.17

Theorem 7.18. *If* $\mathrm{mad}(G) < \frac{8}{3}$*, then* G *is* $2|1$*-choosable.*

Proof. Let G be a minimal counterexample, with a 2-uniform list assignment L from which no 1-improper coloring can be chosen. We may assume that G is connected. If v is a 1-vertex, then a 1-improper L-coloring of $G-v$ can be extended to G by giving v a color from $L(v)$ not used on its neighbor. Hence we may assume $\delta(G) \geq 2$. It now suffices to show that the configurations in Lemma 7.17 are reducible.

If G contains a 2-thread $\langle u', u, v, v' \rangle$, then let $G' = G - \{u, v\}$, and let ϕ be a 1-improper L-coloring of G'. Extend ϕ to G by giving u a color in $L(u) - \phi(u')$ and v a color in $L(v) - \phi(v')$; note that u and v may then have the same color.

Let H be the subgraph of G induced by the 3-vertices. It remains to prove that tree components of H are reducible when each external neighbor has degree 2. For this we prove a stronger claim. Say that a vertex in a partial coloring is

solo if it has a fixed single color not used on any neighbor,
dual if it is free to have either of two colors and each appears on at most one neighbor,
empty if it has no colored neighbor.

Colors on dual vertices will be chosen later.

Let S be the vertex set of a tree induced by 3-vertices in G (not necessarily a component of H; there may be outside 3-neighbors). Leaving S are $|S|+2$ edges. We prove a claim:

If the other endpoints of the edges leaving S are labeled solo/dual/empty from 1-improper L-colorings of $G-S$, then G has a 1-improper L-coloring.

We use induction on $|S|$. If $S=\{v\}$, then the colors in $L(v)$ cannot both be forced to appear on at least two neighbors of v outside S, since there are only three such neighbors. If a color $c \in L(v)$ is not forced to appear on $N(v)$, then fix the colors on dual neighbors to avoid c and used c on v to complete the extension. If c is forced to appear on one neighbor, then that vertex is solo and again c can be used on v to complete the extension.

For $|S|>1$, let v be an endpoint of a longest path in the tree, so v has two outside labeled neighbors. Cases on the left in Figure 26 show v having no empty neighbor; on the right v has at least one empty neighbor. In each case we label v solo (with color fixed) or dual (with color to be chosen later), and the induction hypothesis completes the extension.

Suppose first that v has no empty neighbor. If some color $c \in L(v)$ is not forced to appear on $N(v)$, then make the choices on any dual vertices in $N(v)$ to avoid c, use c on v, and label v solo. If both colors in $L(v)$ are forced to appear on $N(v)$, then the vertices in $N(v)$ must both be solo with those colors; label v dual, with color to be chosen later.

If v has one empty neighbor w, then color v with a color from $L(v)$ not used on its colored neighbor u (fixing the color of u if u is dual), color w with a color from $L(w)$ not used on v, and label both v and w solo. If both outside neighbors of v are empty, then color v from $L(v)$ arbitrarily, color the neighbors of v with other colors from their lists, and label all three vertices solo. This completes the proof of the claim.

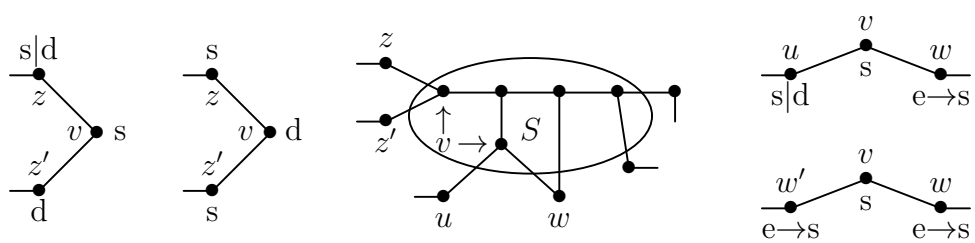

Figure 26: Induction for components of H in Theorem 7.18

To apply the claim, let S be the vertex set of a component of H. Consider a 1-improper L-coloring ϕ of $G-S$. Each vertex w outside S having a neighbor in S is a 2-vertex in G. If both neighbors of w lie in S, then label w empty. Otherwise, w has exactly one neighbor y in $G-S$, so we may assume $\phi(w) \in L(w)-\phi(y)$ and label w solo. Now the outside neighbors of S have been labeled, and by the technical claim ϕ can be extended to a 1-improper L-coloring of G. $\square$

The subtrees $G[S]$ in Theorem 7.18 are arbitrarily large reducible configurations, like the alternating cycles of Lemma 5.3 and i-alternating subgraphs in Lemma 4.13. Use of such configurations is sometimes called "global discharging". The authors of [61] expanded on this notion, introducing "soft components" (see Exercise 7.24) to describe large reducible configurations. They also introduced "feeding areas" to extend the notion of long-distance discharging (sending charge along arbitrarily long paths) used in [59] and in [52]. Additional applications of such ideas appear in [60, 67, 129, 230, 247]. In some sense, these techniques of global discharging can be viewed as the start of the "second generation" of the discharging method, expanding its use to more difficult problems.

Exercise 7.18. (Havet–Sereni [156]) The *book* B_r is $K_2 \oplus \overline{K}_r$, the join[1] of K_2 with r independent vertices (also obtained from $K_{2,r}$ by adding an edge joining the high-degree vertices). Form F_k from k disjoint copies of B_{k+1} by selecting one vertex of degree $k+2$ from each copy and merging them. Form the graph $G_{k,n}$ from $2n+1$ copies of F_k by adding a cycle through the vertices of degree $k(k+2)$ and subdividing all but one edge of the cycle ($G_{2,1}$ appears in Figure 27). Prove that $G_{k,n}$ is not (k,k)-colorable but $\mathrm{mad}(G) = \frac{(2n+1)(4k^2+6k+4)-2}{(2n+1)(k^2+2k+2)-1}$. Conclude that $F(k,k) < 4 - \frac{4}{k+2}$.

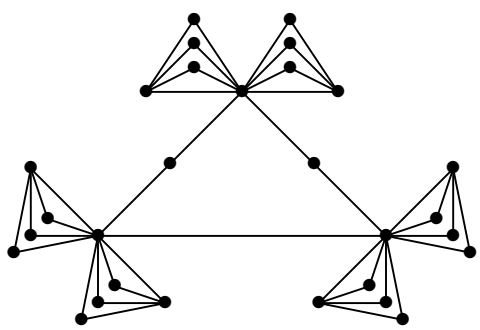

Figure 27: The non-$(2,2)$-colorable graph $G_{2,1}$ in Exercise 7.18

Exercise 7.19. (Borodin–Kostochka [66]) Prove that the graph on the left in Figure 28 is not $(0,1)$-colorable. Generalize to an infinite family of non-$(0,1)$-colorable graphs G such that $6|V(G)| - 5|E(G)| = -3$, thereby proving $F(0,1) \le \frac{12}{5}$.

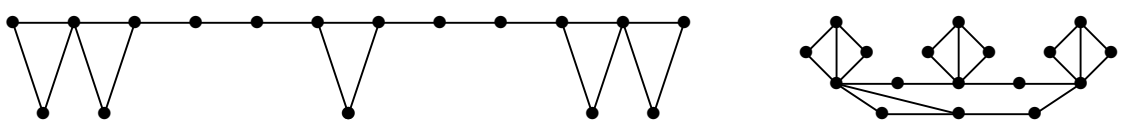

Figure 28: Non-$(0,1)$-colorable and non-$(1,1)$-colorable graphs

[1]The notation $\oplus$ for the join is consistent with the "Czech notation" of Nešetřil for products: the notation pictures the join of K_2 with K_2 and yet reflects that the size of the vertex set is additive under join.

Exercise 7.20. (Borodin–Kostochka–Yancey [69]) Prove that the graph on the right in Figure 28 is not $(1,1)$-colorable. Generalize to an infinite family of non-$(1,1)$-colorable graphs G such that $7|V(G)| - 5|E(G)| = -1$, thereby proving $F(1,1) \le \frac{14}{5}$.

Exercise 7.21. (Kim–Kostochka–Zhu [147]) Let H be the 10-vertex graph shown on the left in Figure 29. Let G be the graph in which three copies of H are attached to a 5-cycle by first merging the special vertex v with a vertex of the 5-cycle (as shown on the right) and then adding paths of length 3 joining the special vertices u and v in each copy. Prove that G is a non-$(0,1)$-colorable triangle-free graph satisfying $11|V(G)| - 9|E(G)| = -5$. Extend the construction to obtain an infinite family of graphs with these properties.

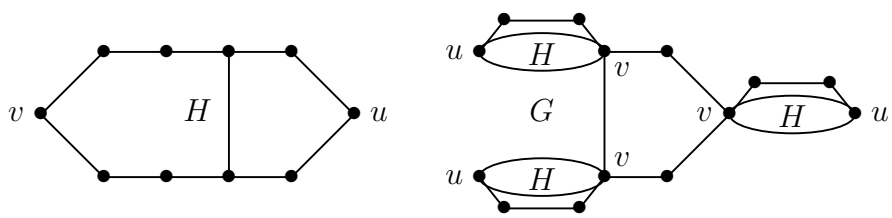

Figure 29: The non-$(0,1)$-colorable graph G in Exercise 7.21

Exercise 7.22. Conclude from Lemma 7.3 that every planar graph with girth at least 5 is $2|4$-choosable.

Exercise 7.23. (Lih et at. [171]) Prove that every planar graph with no 4-cycles or 5-cycles has a 2^--vertex, two adjacent 3-vertices, or a triangle whose vertices have degrees $3,4,4$. Use this to prove that every planar graph with no 4-cycles or 5-cycles is $3|1$-choosable. (Comment: [171] proves $3|1$-choosability for planar graphs with with 4-cycles or k-cycles, where $k \in \{5,6,7\}$, and [99] extends this to $k \in \{8,9\}$.)

Exercise 7.24. (Borodin et al. [61]) For $(1,k)$-colorability, 1^--vertices are reducible. This exercise develops a family of arbitrarily large reducible configurations; we may assume $\delta(G) \ge 2$.

Say that $(k+1)^-$-vertices are *small*, $(k+2)$-vertices are *medium*, and $(k+3)^+$-vertices are *large*. A small vertex is *weak* if it has exactly one non-small neighbor. An edge is *soft* if it joins a non-big vertex to a 2-vertex or a medium vertex to a weak vertex.

And induced subgraph H is a *soft component* if it contains no big vertex, contains every 2-vertex whose neighbors both lie in $V(H)$, and each edge xy with $x \in V(H)$ and $y \notin V(H)$ is soft and satisfies $d_G(x) \ge d_G(y)$. Prove that soft components are reducible for $(1,k)$-colorability.

8 Further Applications

Discharging has also been used in many other problems. We briefly describe several.

Graph decomposition. The basic definitions for graph decomposition appear in Section 7.3. Also, say that a graph G is *d-bounded* if $\Delta(G) \le d$. Motivated by an application to "game coloring number", He et al. [129] introduced the study of decomposing a graph

into a forest and a d-bounded subgraph. Many papers proved such results for planar graphs under girth restrictions and then for bounded $\mathrm{mad}(G)$. Montassier et al. [182] observed that subdividing each edge of a $(2d+2)$-regular graph yields a graph with average degree $2+\frac{2d}{d+2}$ having no decomposition into a forest and a d-bounded graph. They asked for the largest b such that $\mathrm{mad}(G) < b$ is sufficient, showing that $\frac{2d+4}{d+6+6/d}$ is sufficient. These results were proved by discharging, with [182] using pots of charge corresponding to special subgraphs.

Kim et al. [148] proved that the construction in [182] is sharp: $\mathrm{mad}(G) < 2+\frac{2d}{d+2}$ guarantees decomposition into a forest and a d-bounded graph. Although this implies the earlier results for planar graphs with large girth, it was not proved by discharging. Instead, a more detailed statement allowing differing bounds on vertex degrees in the d-bounded graph was proved inductively, using potential functions such as those defined in Section 7.4.

Montassier et al. [181] asked what restriction on $\mathrm{mad}(G)$ guarantees a decomposition into $k+1$ forests, with one required to be d-bounded? Recall from Theorem 6.6 that G decomposes into k forests if and only if $\lceil \mathrm{Arb}(G) \rceil \le k$, where $\mathrm{Arb}(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1}$. Intuitively, if the *fractional arboricity* $\mathrm{Arb}(G)$ just slightly exceeds k, then it should be possible to control the last forest. The *Nine Dragon Tree*[2] *Conjecture* [181] asserts that if $\mathrm{Arb}(G) \le k + \frac{d}{k+d+1}$, then G decomposes into $k+1$ forests with one being d-bounded.

The conjecture was proved when $k=1$ for $d \le 2$ in [181] and for $d \le 6$ in [148]. These proofs used discharging and potential functions. When $d > k$, [148] proved that $\mathrm{Arb}(G) \le k + \frac{d}{k+d+1}$ suffices for decomposability of G into k forests plus one d-bounded graph (not necessarily a forest). This hypothesis is a bit stronger than $\mathrm{mad}(G) < 2k + \frac{2d}{k+d+1}$. The result was first proved by discharging, but the inductive proof of a more detailed statement about potential functions is much shorter.

For $d \le k+1$, the stronger restriction $\mathrm{Arb}(G) \le k + \frac{d}{2k+2}$ implies the conclusion of the NDT Conjecture ([148]). Since $\frac{d}{2k+2} = \frac{d}{k+d+1}$ when $d = k+1$, the NDT Conjecture is thus true in that case. The technique here again was inductive.

The *Strong NDT Conjecture* asserts that under the same arboricity restriction, the last forest can also be required to have at most d edges in each component. Equivalent to the NDT Conjecture when $d=1$, this has been proved for $(k,d) = (1,2)$ [148].

When $k=3$, Balogh et al. [22] proved that every planar graph decomposes into three forests with one being 8-bounded (recall Exercise 6.11). They conjectured that 8 can be improved to 4, which Gonçalves [121] proved; this does not follow from [148]. Gonçalves [121] also proved that planar graphs with girth at least g decompose into two forests with one being d-bounded, for $(g,d) = (6,4)$ or $(g,d) = (7,2)$; these results do follow from [148].

Other graph properties. The subject of homomorphism, introduced in Section 2, includes the subject of coloring, but in full generality it is not as thoroughly developed. Nevertheless, there are analogous results. Dvořák, Škrekovski, and Valla [102] proved that

[2]The Nine Dragon Tree, a banyan tree atop a mountain in Kaohsiung, Taiwan, is far from acyclic.

every planar graph G having no odd cycle of length less than 9 is H-colorable, where H is the Petersen graph (that is, there is a homomorphism from G to the Petersen graph). The proof involves a quite length discharging argument. Note that being H-colorable is stronger than being 3-colorable when H is 3-colorable.

It remains unknown whether 9 can be improved to 7. It cannot be improved to 5; the graph obtained from K_4 by doubly subdividing the two edges of a matching is not H-colorable. This graph has average degree $\frac{5}{2}$; Chen and Raspaud [77] proved (again by a substantial discharging argument) that every triangle-free graph with $\mathrm{mad}(G) < \frac{5}{2}$ is H-colorable. As a weaker version of the conjecture that planar graphs with girth at least $4t$ have circular chromatic number at most $2 + \frac{1}{t}$, they conjectured that such graphs have fractional chromatic number at most $2 + \frac{1}{t}$, and the result of [102] confirms this for $t = 2$.

A set $S \subseteq V(G)$ is *dominating* in a graph G if every vertex in $V(G) \setminus S$ has a neighbor in S. The *domination number* $\gamma(G)$ is the size of the smallest dominating set in G. Reed [192] proved $\gamma(G) \leq \frac{3}{8}n$ for every n-vertex graph G with $\delta(G) \geq 3$. The proof assumes a counterexample G and covers G with vertex disjoint paths, subject to various minimality conditions. The desired dominating set now consists of roughly every third vertex on each path (with special attention to the endpoints).

Careful counting of the edges from the endpoints is needed. Discharging can provide a convenient language in which to present counting arguments that involve inequalities. Kostochka and Stodolsky [157] used this approach to strengthen Reed's argument, proving that 3-regular graphs have dominating sets using at most $\frac{4}{11}$ of their vertices. Beginning with charge 1 at each vertex of a dominating set and charge 0 elsewhere, they redistributed charge so that each vertex finished with charge at most $\frac{4}{11}$. By further developing this discharging approach, Kostochka and Stocker [155] reduced the ratio to $\frac{5}{14}$.

Discharging has also been used to prove non-local structural properties of graphs that are sparse by virture of being embedded in surfaces. For example, Sanders and Zhao [201] proved that every graph that embeds in a surface of Euler characteristic k has a spanning 2-connected subgraph with maximum degree at most $10 - 2k$. For $k \leq 10$, they improved this to the best possible $6 - 2k$. The proof is by discharging.

For a graph G, the *edge-deck* is the multiset $\{G - e\colon e \in E(G)\}$. A graph G is *edge-reconstructible* if G is uniquely determined by its edge-deck. In 1964, Harary conjectured that every graph with at least four edges is edge-reconstructible. Zhao [256, 257] applied discharging to the special case of Harary's Edge-Reconstruction Conjecture for graphs embedded on sufaces. Building on work of Fiorini and Lauri, he proved that if G is a triangulation of a surface with Euler characteristic k and $|V(G)| \geq -43k$, then G is edge-reconstructible.

A "planar cover" of a graph G is a homomorphism from a planar graph to G that maps vertex neighborhoods bijectively. A number of papers have applied discharging to Negami's Conjecture that a graph has a finite planar cover if and only if it embeds in the projective

plane; they are surveyed by Hliněný [134], who wrote many of them.

We also mention some classical graph-theoretic problems not involving coloring or embeddings on surfaces. A well-known conjecture by Thomassen [218] asserts that every 4-connected line graph is Hamiltonian. In this direction, Li and Yang [170] proved that every 3-connected, essentially 10-connected line graph is Hamiltonian-connected. Lai, Wang, and Zhan [167] then gave a proof of this result by discharging.

Belotserkovsky [31] used discharging to determine the minimum number of edges in a 2-connected graph with n vertices and diameter at most d, when n is sufficently large: it is $\lceil \frac{dn-2d-1}{d-1} \rceil$.

Let $\nu(G)$ denote the maximum number of pairwise edge-disjoint triangles in a graph G, and let $\tau(G)$ denote the minimum number of edges needed to hit all the triangles. Trivially, $\nu(G) \le \tau(G) \le 3\nu(G)$. Tuza [222] conjectured that always $\tau(G) \le 2\nu(G)$, which is sharp for a graph consisting of edge-disjoint copies of K_4. Tuza [223] proved the conjecture for planar graphs, and Krivelevich [160] proved it for all 4-colorable graphs. Extending Tuza's result in a different direction, Puleo [193] used discharging to prove the conjecture whenever $\mathrm{mad}(G) < 7$.

Topological graphs. Radoičić and Tóth [191] described various uses of discharging in combinatorial geometry, starting with a short, clever (non-inductive!) proof of Euler's Formula itself found by Thurston [220]. They gave an extensive summary of coloring topics on plane graphs before moving on to discrete geometry, where they proved a special case of a conjecture of Pach and Sharir [189]: If H is a bipartite graph, then there is a constant c_H such that every H-avoiding intersection graph of n convex sets in the plane has at most $c_H n$ edges, where a graph is H-avoiding if it does not contain H. Radoičić and Tóth used discharging to prove the conjecture for $H \in \{K_{2,3}, C_6\}$.

To motivate their proof, Radoičić and Tóth described the discharging proof by Ackerman and Tardos [2] about the maximum number of edges in quasi-planar graphs. A *topological graph* is a graph drawn in the plane; it is a *geometric graph* when the edges are drawn as line segments. A topological graph is k-quasi-planar if it does not have k pairwise crossing edges. Thus a 2-quasi-planar graph is a plane graph, and 3-quasi-planar graphs are simply called *quasi-planar*.

Capolyeas and Pach [74] conjectured for every fixed k that the maximum number of edges in a k-quasi-planar graph on n vertices is linear in n. Agarwal et al. [3] proved this for $k = 3$, and Ackerman [1] proved it for $k = 4$ (by discharging). The short discharging proof by Ackerman and Tardos [2] gave a much better bound on the coefficient for $k = 3$: there are at most $8n - 20$ edges. A construction gave a lower bound of $7n - O(\sqrt{n})$ edges. They used two rounds of discharging and noted that stopping after the first round still yields an upper bound of $10n - 20$.

The discharging ideas are unusual. First draw the graph G with the fewest crossings.

The fact that G is 3-quasi-planar implies that in the graph G' obtained by introducing new vertices for the crossings, the new vertices are all 4-vertices and every 3-face contains at least one original vertex. Assign charge $l(f) - 4 + v(f)$ to each face f of G', where $v(f)$ denotes the number of original vertices incident to f. The desire is for each face f to have charge at least $\frac{v(f)}{5}$, after which the second round proceeds. The only faces that do not have this much charge are triangles having only one original vertex u. Such a face f defines a channel between its boundary edges e and e' that are incident to u. Basically, f takes $\frac{1}{5}$ from the first face along this channel that is not a 4-face having no original vertices.

Ackerman and Tardos [2] also provided a very short discharging proof that bounds the number of edges of the multigraph G by $19n$, using balanced charging on G'. In a minimal counterexample, each vertex of G has degree at least 20. Every original vertex sends $\frac{4}{5}$ to each incident face. Every 3-face in G' that has only one original vertex takes $\frac{1}{5}$ in the same way as above. Since $\delta(G) \geq 20$, each vertex remains happy.

Sharir and Welzl [207] used discharging to prove that the number of triangulations (with straight edges) of a set of n points in the plane is at most 43^n. Sharir and Sheffer [206] refined this approach to improve the upper bound to 30^n. For a set S of n points in the plane, their methods also bound the number of planar straight-line graphs on S in various classes, including all planar graphs (207.84^n), spanning cycles ($O(68.67^n)$), spanning trees ($O(146.69^n)$), and forests ($O(164.17^n)$).

Given a planar n-point set S, their basic idea is to prove a lower bound on the average number of 3-vertices in a uniformly chosen triangulation of S. Their approach is complicated, so we just present a brief sketch; for more detail, see Sections 2 and 3 of [206]. They show that if the average number of 3-vertices in such a triangulation is at least cn for some positive constant c, then the total number of triangulations of S is at most c^{-n}. To prove a lower bound on the average number of 3-vertices, each 4^+-vertex sends total charge at least 1 to 3-vertices, and each 3-vertex receives charge at most $1/c$. Much of the work consists of proving an upper bound on the charge received by each 3-vertex. The innovation that allows this approach to work is sending charge *between triangulations*. Since some triangulations have no 3-vertices, each 4^+-vertex in such a triangulation sends charge to 3-vertices in "nearby" triangulations (under an appropriate edge-flipping metric).

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