Conjectures equivalent to the Borodin-Kostochka conjecture that a priori seem weaker

Daniel W. Cranston∗ Landon Rabern†

March 23, 2012

Abstract

Borodin and Kostochka conjectured that every graph $G$ with maximum degree $\Delta \geq 9$ satisfies $\chi \leq \max\{\omega, \Delta - 1\}$. We carry out an in-depth study of minimum counterexamples to the Borodin-Kostochka conjecture. Our main tool is the classification of graph joins $A \ast B$ with $|A| \geq 2$, $|B| \geq 2$ which are $f$-choosable, where $f(v) := d(v) - 1$ for each vertex $v$. Since such a join cannot be an induced subgraph of a vertex critical graph with $\chi = \Delta$, we have a wealth of structural information about minimum counterexamples to the Borodin-Kostochka conjecture.

Our main result is to prove that certain conjectures that are a priori weaker than the Borodin-Kostochka Conjecture are in fact equivalent to it. One such equivalent conjecture is the following: Any graph with $\chi \geq \Delta = 9$ contains $K_3 \ast E_6$ as a subgraph.

1 Introduction

1.1 A short history of the problem

The first non-trivial result about coloring graphs with around $\Delta$ colors is Brooks’ theorem from 1941.

Theorem 1.1 (Brooks [4]). Every graph with $\Delta \geq 3$ satisfies $\chi \leq \max\{\omega, \Delta\}$.

In 1977, Borodin and Kostochka conjectured that a similar result holds for $\Delta - 1$ colorings. Counterexamples exist showing that the $\Delta \geq 9$ condition is tight (see Figures 1, 2, 3 and 4).

Conjecture 1.2 (Borodin and Kostochka [3]). Every graph with $\Delta \geq 9$ satisfies $\chi \leq \max\{\omega, \Delta - 1\}$.

In the same paper they proved the following weakening. The proof is quite simple once you have a decomposition lemma of Lovász from the 1960’s [12].
Theorem 1.3 (Borodin and Kostochka [3]). Every graph satisfying $\chi \geq \Delta \geq 7$ contains a $K_{\lfloor \frac{\Delta+1}{2} \rfloor}$.

In the 1980’s, Kostochka proved the following using a complicated recoloring argument together with a technique for reducing $\Delta$ in a counterexample based on hitting every maximum clique with an independent set.

Theorem 1.4 (Kostochka [10]). Every graph satisfying $\chi \geq \Delta$ contains a $K_{\Delta-28}$.

Kostochka [10] proved the following result which shows that graphs having clique number sufficiently close to their maximum degree contain an independent set hitting every maximum clique. In [15] the second author improved the antecedent to $\omega \geq \frac{3}{4}(\Delta+1)$. Finally, King [9] made the result tight.

Lemma 1.5 (Kostochka [10]). If $G$ is a graph satisfying $\omega \geq \Delta + \frac{3}{2} - \sqrt{\Delta}$, then $G$ contains an independent set $I$ such that $\omega(G-I) < \omega(G)$.

Lemma 1.6 (Rabern [15]). If $G$ is a graph satisfying $\omega \geq \frac{3}{4}(\Delta+1)$, then $G$ contains an independent set $I$ such that $\omega(G-I) < \omega(G)$.

Lemma 1.7 (King [9]). If $G$ is a graph satisfying $\omega > \frac{3}{4}(\Delta+1)$, then $G$ contains an independent set $I$ such that $\omega(G-I) < \omega(G)$.

If $G$ is a vertex critical graph satisfying $\omega > \frac{3}{4}(\Delta+1)$ and we expand the independent set $I$ produced by Lemma 1.7 to a maximal independent set $M$ and remove $M$ from $G$, we see that $\Delta(G-M) \leq \Delta(G) - 1$, $\chi(G-M) = \chi(G) - 1$, and $\omega(G-M) = \omega(G) - 1$. Using this, the proof of many coloring results can be reduced to the case of the smallest $\Delta$ for which they work. In the case of graphs with $\chi = \Delta$, we get the following general result.

Definition 1. For $k, j \in \mathbb{N}$, let $C_{k,j}$ be the collection of all vertex critical graphs satisfying $\chi = \Delta = k$ and $\omega < k - j$. Put $C_k := C_{k,0}$. Note that $C_{k,j} \subseteq C_{k,i}$ for $j \geq i$.

For each $k \geq 9$, $C_k$ is precisely the set of counterexamples to the Borodin-Kostochka Conjecture with $\Delta = k$.

Lemma 1.8. Fix $k, j \in \mathbb{N}$ with $k \geq 3j + 6$. If $G \in C_{k,j}$, then there exists $H \in C_{k-1,j}$ such that $H \trianglelefteq G$.

Proof. Let $G \in C_{k,j}$. We first show that there exists a maximal independent set $M$ such that $\omega(G-M) < k - (j+1)$. If $\omega(G) < k - (j+1)$, then any maximal independent set will do for $M$. Otherwise, $\omega(G) = k - (j+1)$. Since $k \geq 3j + 6$, we have $\omega(G) = k - (j+1) > \frac{2}{3}(k+1) = \frac{2}{3}(\Delta(G) + 1)$. Thus by Lemma 1.7, we have an independent set $I$ such that $\omega(G-I) < \omega(G)$. Expand $I$ to a maximal independent set to get $M$.

Now $\chi(G-M) = k - 1 = \Delta(G-M)$, where the last equality follows from Brooks’ theorem and $\omega(G-M) < k - (j+1) \leq k - 1$. Since $\omega(G-M) < k - (j+1)$, for any $(k-1)$-critical induced subgraph $H \trianglelefteq G-M$ we have $H \in C_{k-1,j}$. \qed

As a consequence we get the following result of Kostochka that the Borodin-Kostochka conjecture can be reduced to the case when $\Delta = k = 9$. 

Lemma 1.9. Let \( \mathcal{H} \) be a hereditary graph property. For \( k \geq 5 \), if \( \mathcal{H} \cap C_k = \emptyset \), then \( \mathcal{H} \cap C_{k+1} = \emptyset \). In particular, to prove the Borodin-Kostochka conjecture it is enough to show that \( C_9 = \emptyset \).

A little while after Kostochka proved his bound, Mozhan [13] proved the following using a different technique.

**Theorem 1.10** (Mozhan [13]). Every graph satisfying \( \chi \geq \Delta \geq 10 \) contains a \( K_{\lceil \frac{2\Delta+1}{3} \rceil} \).

In his dissertation Mozhan improved on this result. We don’t know the method of proof as we were unable to obtain a copy of his dissertation. However, we suspect the method is a more complicated version of the proof of Theorem 1.10.

**Theorem 1.11** (Mozhan). Every graph satisfying \( \chi \geq \Delta \geq 31 \) contains a \( K_{\Delta-3} \).

In 1999, Reed used probabilistic methods to prove that the Borodin-Kostochka conjecture holds for graphs with very large maximum degree.

**Theorem 1.12** (Reed [16]). Every graph satisfying \( \chi \geq \Delta \geq 10^{14} \) contains a \( K_{\Delta} \).

A lemma from Reed’s proof of the above theorem is generally useful.

**Lemma 1.13** (Reed [16]). Let \( G \) be a vertex critical graph satisfying \( \chi = \Delta \geq 9 \) having the minimum number of vertices. If \( H \) is a \( K_{\Delta-1} \) in \( G \), then any vertex in \( G - H \) has at most 4 neighbors in \( H \). In particular, the \( K_{\Delta-1} \)'s in \( G \) are pairwise disjoint.

### 1.2 Our contribution

We carry out an in-depth study of minimum counterexamples to the Borodin-Kostochka conjecture. Our main tool is the classification, in Section 4, of graph joins \( A \ast B \) with \( |A| \geq 2, |B| \geq 2 \) which are \( f \)-choosable, where \( f(v) := d(v) - 1 \) for each vertex \( v \). Since such a join cannot be an induced subgraph of a vertex critical graph with \( \chi = \Delta \), we have a wealth of structural information about minimum counterexamples to the Borodin-Kostochka conjecture. In Section 2, we exploit this information and minimality to improve Reed’s Lemma 1.13 as follows (see Corollary 2.11).

**Lemma 1.14.** Let \( G \) be a vertex critical graph satisfying \( \chi = \Delta \geq 9 \) having the minimum number of vertices. If \( H \) is a \( K_{\Delta-1} \) in \( G \), then any vertex in \( G - H \) has at most 1 neighbor in \( H \).

Moreover, we lift the result out of the context of a minimum counterexample to the Borodin-Kostochka conjecture, to the more general context of graphs satisfying a certain criticality condition—we call such graphs mules. This allows us to prove meaningful results for values of \( \Delta \) less than 9.

Let \( K_t \) and \( E_t \) be the complete and edgeless graphs on \( t \) vertices, respectively. Since a graph containing \( K_{\Delta} \) as a subgraph also contains \( K_{t, \Delta-t} \) as a subgraph for any \( t \in [\Delta - 1] \), the Borodin-Kostochka conjecture implies the following conjecture. Our main result is that the two conjectures are equivalent.
Conjecture 1.15. Any graph with $\chi = \Delta \geq 9$ contains some $A_1 \ast A_2$ as an induced subgraph where $|A_1|, |A_2| \geq 3, |A_1| + |A_2| = \Delta$ and $A_i \neq K_1 + K_{|A_i|-1}$ for some $i \in [2]$.

In fact, using Kostochka’s reduction (Lemma 1.9) to the case $\Delta = 9$, the following conjecture is also equivalent.

Conjecture 1.16. Any graph with $\chi = \Delta = 9$ contains some $A_1 \ast A_2$ as an induced subgraph where $|A_1|, |A_2| \geq 3, |A_1| + |A_2| = 9$ and $A_i \neq K_1 + K_{|A_i|-1}$ for some $i \in [2]$.

As a special case, we get a couple more palatable equivalent conjectures (see Lemma 2.18 and the comment following it).

Conjecture 1.17. Any graph with $\chi = \Delta \geq 9$ contains $K_3 \ast E_{\Delta-3}$ as a subgraph.

Conjecture 1.18. Any graph with $\chi = \Delta = 9$ contains $K_3 \ast E_6$ as a subgraph.

The condition $A_i \neq K_1 + K_{|A_i|-1}$ is unnatural and by removing it we get a (possibly) weaker conjecture than the Borodin-Kostochka conjecture which has more aesthetic appeal.

Conjecture 1.19. Let $G$ be a graph with $\Delta(G) = k \geq 9$. If $K_{t,k-t} \not\subseteq G$ for all $3 \leq t \leq k-3$, then $G$ can be $(k-1)$-colored.

Conjecture 1.20. Conjecture 1.19 is equivalent to the Borodin-Kostochka conjecture.

Perhaps it would be easier to attack Conjecture 1.19 with $3 \leq t \leq k-3$ replaced by $2 \leq t \leq k-2$? We are unable to prove even this conjecture. Making this change and bringing $k$ down to 5 gives the following conjecture, which, if true, would imply the remaining two cases of Grünbaum’s girth problem for graphs with girth at least five.

Conjecture 1.21. Let $G$ be a graph with $\Delta(G) = k \geq 5$. If $K_{t,k-t} \not\subseteq G$ for all $2 \leq t \leq k-2$, then $G$ can be $(k-1)$-colored.

If $G$ is a graph with $\Delta(G) = k \geq 5$ and girth at least five, then it contains no $K_{t,k-t}$ for all $2 \leq t \leq k-2$ and hence Conjecture 1.21 would give a $(k-1)$-coloring. This conjecture would be tight since the Grünbaum graph and the Brinkmann graph are examples with $\chi = \Delta = 4$ and girth at least five.

Finally, we prove that the following conjecture is equivalent to the Borodin-Kostochka conjecture for graphs with independence number at most 6 (see Theorem 2.24).

Conjecture 1.22. Every graph satisfying $\chi = \Delta = 9$ and $\alpha \leq 6$ contains a $K_8$.

2 Mules

In this section we exclude more induced subgraphs in a minimum counterexample to the Borodin-Kostochka conjecture than we can exclude purely using list coloring properties. In fact, we lift these results out of the context of a minimum counterexample to graphs satisfying a certain criticality condition defined in terms of the following ordering.
Definition 2. If $G$ and $H$ are graphs, an \textit{epimorphism} is a graph homomorphism $f : G \rightarrow H$ such that $f(V(G)) = V(H)$. We indicate this with the arrow $\twoheadrightarrow$.

Definition 3. Let $G$ be a graph. A graph $A$ is called a \textit{child} of $G$ if $A \neq G$ and there exists $H \preceq G$ and an epimorphism $f : H \twoheadrightarrow A$.

Note that the child-of relation is a strict partial order on the set of (finite simple) graphs $\mathcal{G}$. We call this the \textit{child order} on $\mathcal{G}$ and denote it by `$\prec$'. By definition, if $H \prec G$ then $H \prec G$.

Lemma 2.1. The ordering $\prec$ is well-founded on $\mathcal{G}$; that is, every nonempty subset of $\mathcal{G}$ has a minimal element under $\prec$.

Proof. Let $\mathcal{T}$ be a nonempty subset of $\mathcal{G}$. Pick $G \in \mathcal{T}$ minimizing $|G|$ and then maximizing $\|G\|$. Since any child of $G$ must have fewer vertices or more edges (or both), we see that $G$ is minimal in $\mathcal{T}$ with respect to $\prec$. $\blacksquare$

Definition 4. Let $\mathcal{T}$ be a collection of graphs. A minimal graph in $\mathcal{T}$ under the child order is called a $\mathcal{T}$-\textit{mule}.

With the definition of mule we have captured the important properties (for coloring) of a counterexample first minimizing the number of vertices and then maximizing the number of edges. Viewing $\mathcal{T}$ as a set of counterexamples, we can add edges to or contract independent sets in induced subgraphs of a $\mathcal{T}$-mule and get a non-counterexample. We could do the same with a minimal counterexample, but with mules we have more minimal objects to work with. One striking consequence of this is that many of our proofs naturally construct multiple counterexamples to Borodin-Kostochka for small $\Delta$.

2.1 Excluding induced subgraphs in mules

Our main goal in this section is to prove Lemma 2.12, which says that (with only one exception) for $k \geq 7$, no $k$-mule contains $K_4 \ast E_{k-4}$ as a subgraph. This result immediately implies that the Borodin-Kostochka Conjecture is equivalent to Conjecture 2.13. This equivalence is a major step toward our main result. Our approach is based on Lemma 4.30, which implies that if $G$ is a counterexample to Lemma 2.12 then the vertices of the $E_{k-4}$ induce either $E_3$, a claw, a clique, or an almost complete graph. Our job in this section consists of showing that each of these four possibilities is, in fact, impossible. Ruling out the clique is easy. The cases of $E_3$ and the claw are handled in Lemma 2.8, and the case of an almost complete graph (which requires the most work) is handled by Corollary 2.11.

For $k \in \mathbb{N}$, by a $k$-\textit{mule} we mean a $C_k$-mule.

Lemma 2.2. Let $G$ be a $k$-mule with $k \geq 4$. If $A$ is a child of $G$ with $\Delta(A) \leq k$ then either

1. $A$ is $(k-1)$-colorable; or,
2. $A$ contains a $K_k$.
Proof. Let $A$ be a child of $G$ with $\Delta(A) \leq k$, $H \preceq G$ and $f : H \rightarrow A$ an epimorphism. Without loss of generality, $A$ is vertex critical. Suppose $A$ is not $(k-1)$-colorable. Then $\chi(A) \geq k \geq \Delta(A)$. Since $A \prec G$ and $G$ is a mule, $A \notin C_k$. Thus we have $\chi(A) \geq \Delta(A) \geq 3$, so Brooks’ theorem implies that $A = K_k$.  

Note that adding edges to a graph yields an epimorphism.

Lemma 2.3. Let $G$ be a $k$-mule with $k \geq 4$ and $H \preceq G$. Assume $x, y \in V(H)$, $xy \notin E(H)$ and both $d_H(x) \leq k-1$ and $d_H(y) \leq k-1$. If for every $(k-1)$-coloring $\pi$ of $H$ we have $\pi(x) = \pi(y)$, then $H$ contains $\{x, y\} \ast K_{k-2}$.

Proof. Suppose that for every $(k-1)$-coloring $\pi$ of $H$ we have $\pi(x) = \pi(y)$. Using the inclusion epimorphism $f_{xy} : H \rightarrow H + xy$ in Lemma 2.2 shows that either $H + xy$ is $(k-1)$-colorable or $H + xy$ contains a $K_k$. Since a $(k-1)$-coloring of $H + xy$ would induce a $(k-1)$-coloring of $H$ with $x$ and $y$ colored differently, we conclude that $H + xy$ contains a $K_k$. But then $H$ contains $\{x, y\} \ast K_{k-2}$ and the proof is complete.

We will often begin by coloring some subgraph $H$ of our graph $G$, and work to extend this partial coloring. More formally, let $G$ be a graph and $H \prec G$. For $t \geq \chi(H)$, let $\pi$ be a proper $t$-coloring of $H$. For each $x \in V(G-H)$, put $L_\pi(x) := \{1, \ldots, t\} - \bigcup_{y \in N(x) \cap V(H)} \pi(y)$. Then $\pi$ is completable to a $t$-coloring of $G$ iff $L_\pi$ admits a coloring of $G - H$. We will use this fact repeatedly in the proofs that follow. The following generalizes a lemma due to Reed [16], the proof is essentially the same.

Lemma 2.4. For $k \geq 6$, if a $k$-mule $G$ contains an induced $E_2 \ast K_{k-2}$, then $G$ contains an induced $E_3 \ast K_{k-2}$.

Proof. Suppose $G$ is a $k$-mule containing an induced $E_2 \ast K_{k-2}$, call it $F$. Let $x, y$ be the vertices of degree $k-2$ in $F$ and $C := \{w_1, \ldots, w_{k-2}\}$ the vertices of degree $k-1$ in $F$. Put $H := G - F$. Since $G$ is vertex critical, we may $k-1$ color $H$. Doing so leaves a list assignment $L$ on $F$ with $|L(z)| \geq d_F(z) - 1$ for each $z \in V(F)$. Now $|L(x)| + |L(y)| \geq d_F(x) + d_F(y) - 2 = 2k - 6 > k-1$ since $k \geq 6$. Hence we have $c \in L(x) \cap L(y)$. Coloring both $x$ and $y$ with $c$ leaves a list assignment $L'$ on $C$ with $|L'(w_i)| \geq k-3$ for each $1 \leq i \leq k-2$. Now, if $|L'(w_i)| \geq k - 2$ or $L'(w_i) \neq L'(w_j)$ for some $i, j$, then we can complete the partial $(k-1)$-coloring to all of $G$ using Hall’s Theorem. Hence we must have $d(w_i) = k$ and $L'(w_i) = L'(w_j)$ for all $i, j$. Let $N := \bigcup_{w \in C} N(w) \cap V(H)$ and note that $N$ is an independent set since it is contained in a single color class in every $(k-1)$-coloring of $H$. Also, each $w \in C$ has exactly one neighbor in $N$.

Proving that $|N| = 1$ will give the desired $E_3 \ast K_{k-2}$ in $G$. Thus, to reach a contradiction, suppose that $|N| \geq 2$.

We know that $H$ has no $(k-1)$-coloring in which two vertices of $N$ get different colors since then we could complete the partial coloring as above. Let $v_1, v_2 \in N$ be different. Since both $v_1$ and $v_2$ have a neighbor in $F$, we may apply Lemma 2.3 to conclude that $\{v_1, v_2\} \ast K_{v_1, v_2}$ is in $H$, where $K_{v_1, v_2}$ is a $K_{k-2}$.

First, suppose $|N| \geq 3$, say $N = \{v_1, v_2, v_3\}$. We have $z \in K_{v_1, v_2} \cap K_{v_1, v_3}$ for otherwise $d(v_1) \geq 2(k-2) > k$. Since $z$ already has $k$ neighbors among $K_{v_1, v_2} - \{z\}$ and $v_1, v_2, v_3$, we must have $K_{v_1, v_3} = K_{v_1, v_2}$. But then $\{v_1, v_2, v_3\} \ast K_{v_1, v_2}$ is our desired $E_3 \ast K_{k-2}$ in $G$.  

6
Hence we must have $|N| = 2$, say $N = \{v_1, v_2\}$. For $i \in [2]$, $v_i$ has $k - 2$ neighbors in $K_{v_1, v_2}$ and thus at most two neighbors in $C$. Hence $|C| \leq 4$. Thus we must have $k = 6$.

We may apply the same reasoning to $\{v_1, v_2\}^* K_{v_1, v_2}$ that we did to $F$ to get vertices $v_{2,1}, v_{2,2}$ such that $\{v_{2,1}, v_{2,2}\}^* K_{v_2,1, v_2,2}$ is in $G$. But then we may do it again with $\{v_{2,1}, v_{2,2}\}^* K_{v_2,1, v_2,2}$ and so on. Since $G$ is finite, at some point this process must terminate. But the only way to terminate is to come back around and use $x$ and $y$. This graph is 5-colorable since we may color all the $E_2$’s with the same color and then 4-color the remaining $K_4$ components. This final contradiction completes the proof.

\[ \square \]

\[ \text{Figure 1: The mule } M_{6,1}. \]

\[ \text{Figure 2: The mule } M_{7,1}. \]

**Lemma 2.5.** For $k \geq 6$, the only $k$-mules containing an induced $E_2 \ast K_{k-2}$ are $M_{6,1}$ and $M_{7,1}$.

**Proof.** Suppose we have a $k$-mule $G$ that contains an induced $E_2 \ast K_{k-2}$. Then by Lemma 2.4, $G$ contains an induced $E_3 \ast K_{k-2}$, call it $F$.

Let $x, y, z$ be the vertices of degree $k - 2$ in $F$ and let $C := \{w_1, \ldots, w_{k-2}\}$ be the vertices of degree $k$ in $F$. Put $H := G - C$. Since each of $x, y, z$ have degree at most 2 in $H$ and $G$ is a mule, the homomorphism from $H$ sending $x, y, z$ to the same vertex must produce a $K_k$. Thus we must have $k \leq 7$ and $H$ contains a $K_{k-1}$ (call it $D$) such that $V(D) \subseteq N(x) \cup N(y) \cup N(z)$. Put $A := G[V(F) \cup V(D)]$. Then $A$ is $k$-chromatic and as $G$ is a mule, we must have $G = A$. If $k = 7$, then $G = M_{7,1}$. Suppose $k = 6$ and $G \neq M_{6,1}$. Then one of $x, y, z$ has only one neighbor in $D$. By symmetry we may assume it is $x$. But we can add an edge from $x$ to a vertex in $D$ to form $M_{6,1}$ and hence $G$ has a proper child, which is impossible. \[ \square \]
Lemma 2.6. Let $G$ be a $k$-mule with $k \geq 6$ other than $M_{6,1}$ and $M_{7,1}$ and let $H \subset G$. If $x,y \in V(H)$ and both $d_H(x) \leq k - 1$ and $d_H(y) \leq k - 1$, then there exists a $(k - 1)$-coloring $\pi$ of $H$ such that $\pi(x) \neq \pi(y)$.

Proof. Suppose $x,y \in V(H)$ and both $d_H(x) \leq k - 1$ and $d_H(y) \leq k - 1$. First, if $xy \in E(H)$ then any $(k - 1)$-coloring of $H$ will do. Otherwise, if for every $(k - 1)$-coloring $\pi$ of $H$ we have $\pi(x) = \pi(y)$, then by Lemma 2.3, $H$ contains $\{x,y\} \ast K_{k-2}$. The lemma follows since this is impossible by Lemma 2.5.

Lemma 2.7. Let $G$ be a $k$-mule with $k \geq 6$ other than $M_{6,1}$ and $M_{7,1}$ and let $F \subset G$. Put $C := \{v \in V(F) \mid d(v) - d_F(v) \leq 1\}$. At least one of the following holds:

- $G - F$ has a $(k - 1)$-coloring $\pi$ such that for some $x,y \notin C$ we have $L_\pi(x) \neq L_\pi(y)$; or,
- $G - F$ has a $(k - 1)$-coloring $\pi$ such that for some $x \in C$ we have $|L_\pi(x)| = k - 1$; or,
- there exists $z \in V(G - F)$ such that $C \subseteq N(z)$.

Proof. Put $H := G - F$. Suppose that for every $(k - 1)$-coloring $\pi$ of $H$ we have $L_\pi(x) = L_\pi(y)$ for every $x,y \in C$. By assumption, the vertices in $C$ have at most one neighbor in $H$. If some $v \in C$ has no neighbors in $H$, then for any $(k - 1)$-coloring $\pi$ of $H$ we have $|L_\pi(v)| = k - 1$. Thus we may assume that every $v \in C$ has exactly one neighbor in $H$.

Let $N := \bigcup_{w \in C} N(w) \cap V(H)$. Suppose $|N| \geq 2$. Pick different $z_1, z_2 \in N$. Then, by Lemma 2.6, there is a $(k - 1)$-coloring $\pi$ of $H$ for which $\pi(z_1) \neq \pi(z_2)$. But then $\pi(x) \neq L_\pi(y)$ for some $x,y \in C$ giving a contradiction. Hence $N = \{z\}$ and thus $C \subseteq N(z)$.

By Lemma 4.24, no graph in $C_k$ contains an induced $E_3 \ast K_{k-3}$ for $k \geq 9$. For mules, we can improve this as follows.

Lemma 2.8. For $k \geq 7$, the only $k$-mule containing an induced $E_3 \ast K_{k-3}$ is $M_{7,1}$.

Proof. Suppose the lemma is false and let $G$ be a $k$-mule, other than $M_{7,1}$, containing such an induced subgraph $F$. Let $z_1, z_2, z_3 \in F$ be the vertices with degree $k - 3$ in $F$ and $C$ the rest of the vertices in $F$ (all of degree $k - 1$ in $F$). Put $H := G - F$.

First suppose there is not a vertex $x \in V(H)$ which is adjacent to all of $C$. Let $\pi$ be a $(k - 1)$-coloring of $H$ guaranteed by Lemma 2.7 and put $L := L_\pi$. Since $|L(z_1)| + |L(z_2)| + |L(z_3)| \geq 6k - 3 > k - 1$ we have $1 \leq i < j \leq 3$ such that $L(z_i) \cap L(z_j) \neq \emptyset$. Without loss of generality, $i = 1$ and $j = 2$. Pick $c \in L(z_1) \cap L(z_2)$ and color both $z_1$ and $z_2$ with $c$. Let $L'$ be the resulting list assignment on $F - \{z_1, z_2\}$. Now $|L'(z_3)| \geq k - 4$ and $|L'(v)| \geq k - 3$ for each $v \in C$. By our choice of $\pi$, either two of the lists in $C$ differ or for some $v \in C$ we have $|L'(v)| \geq k - 2$. In either case, we can complete the $(k - 1)$-coloring to all of $G$ by Hall’s Theorem.

Hence we must have $x \in V(H)$ which is adjacent to all of $C$. Thus $G$ contains the induced subgraph $K_{k-3} \ast G[z_1, z_2, z_3, x]$. Therefore $k = 7$ and $x$ is adjacent to each of $z_1, z_2, z_3$ by Lemma 4.30. Hence $G$ contains the induced subgraph $K_5 \ast E_3$ contradicting Lemma 2.5.

Lemma 2.9. For $k \geq 7$, no $k$-mule contains an induced $\overline{P}_3 \ast K_{k-3}$. 

8
Proof. Suppose the lemma is false and let \( G \) be a \( k \)-mule containing such an induced subgraph \( F \). Note that \( M_{7,1} \) has no induced \( P_3 \ast K_{k-3} \), so \( G \neq M_{7,1} \). Let \( z \in V(F) \) be the vertex with degree \( k - 3 \) in \( F \), \( v_1, v_2 \in F \) the vertices of degree \( k - 2 \) in \( F \) and the rest of the vertices in \( F \) (all of degree \( k - 1 \) in \( F \)). Put \( H := G - F \).

First suppose there is not a vertex \( x \in V(H) \) which is adjacent to all of \( C \). Let \( \pi \) be a \((k - 1)\)-coloring of \( H \) guaranteed by Lemma 2.7 and put \( L := L_{\pi} \). Then, we have \(|L(z)| \geq k - 4\) and \(|L(v_1)| \geq k - 3\). Since \( k \geq 7 \), \(|L(z)| + |L(v_1)| \geq 2k - 7 > k - 1\). Hence, by Lemma 4.7 we may color \( z \) and \( v_1 \) the same. Let \( L' \) be the resulting list assignment on \( F - \{z, v_1\} \). Now \(|L'(v_2)| \geq k - 4\) and \(|L'(v)| \geq k - 3\) for each \( v \in C \). By our choice of \( \pi \), either two of the lists in \( C \) differ or for some \( v \in C \) we have \(|L'(v)| \geq k - 2\). In either case, we can complete the \((k - 1)\)-coloring to all of \( G \) by Hall’s Theorem.

Hence we must have \( x \in V(H) \) which is adjacent to all of \( C \). Thus \( G \) contains the induced subgraph \( K_4 \ast G[z, v_1, v_2, x] \). By Lemma 4.30, \( G[z, v_1, v_2, x] \) must be almost complete and hence \( x \) must be adjacent to both \( v_1 \) and \( v_2 \). But then \( G[v_1, v_2, x] \ast C \) is a \( K_k \) in \( G \), giving a contradiction.

Reed proved that for \( k \geq 9 \), a vertex outside a \((k - 1)\)-clique \( H \) in a \( k \)-mule can have at most 4 neighbors in \( H \). We improve this to at most one neighbor.

![Figure 3: The mule \( M_{7,2} \).](image)

Lemma 2.10. For \( k \geq 7 \) and \( r \geq 2 \), no \( k \)-mule except \( M_{7,1} \) and \( M_{7,2} \) contains an induced \( K_r \ast (K_1 + K_{k-(r-1)}) \).

Proof. Suppose the lemma is false and let \( G \) be a \( k \)-mule, other than \( M_{7,1} \) and \( M_{7,2} \), containing such an induced subgraph \( F \) with \( r \) maximal. By Lemma 2.5 and Lemma 2.9 the lemma holds for \( r \geq k - 3 \). So we have \( r \leq k - 4 \). Now, let \( z \in V(F) \) be the vertex with degree \( r \) in \( F \), \( v_1, v_2, \ldots, v_{k-(r-1)} \in V(F) \) the vertices of degree \( k - 2 \) in \( F \) and the rest of the vertices in \( F \) (all of degree \( k - 1 \) in \( F \)). Put \( H := G - F \).

Let \( Z_1 := \{za \mid a \in N(v_1) \cap V(H) \} \). Consider the graph \( D := H + z + Z_1 \). Since \( v_1 \) has at most two neighbors in \( H \), \(|Z_1| \leq 2\) and thus to form \( D \) from \( H + z \), we added \( E(A) \) where \( A \in \{K_1, K_2, P_3\} \). Since \(|V| \geq 2 \), \( \Delta(D) \leq k \). Hence Lemma 2.2 shows that \( H + z \) contains a \( K_k - E(A) \) or \( \chi(D) \leq k - 1 \). Suppose \( \chi(D) \geq k \). If \( A = K_1 \), \( A = K_2 \), or \( A = P_3 \), then we have a contradiction by the fact that \( \omega(G) < k \), Lemma 2.5 and Lemma 2.9 respectively. Thus we must have \( \chi(D) \leq k - 1 \), which gives a \((k - 1)\)-coloring of \( H + z \) in which \( z \) receives a color \( c \) which is not received by any of the neighbors of \( v_1 \) in \( H \). Thus \( c \) remains in the list of \( v_1 \) and we may color \( v_1 \) with \( c \). After doing so, each vertex in \( C \) has a list of size at least \( k - 3 \) and \( v_i \) for \( i > 1 \) has a list of size at least \( k - 4 \). If any pair of vertices in
C had different lists, then we could complete the partial coloring by Hall’s Theorem. Let $N := \bigcup_{w \in C} N(w) \cap V(H)$ and note that $N$ is an independent set since it is contained in a single color class in the $(k-1)$-coloring of $H$ just constructed.

Suppose $|N| \geq 2$. Pick $a_1, a_2 \in N$. Consider the graph $D := H + z + Z_1 + a_1a_2$. Plainly, $\Delta(D) \leq k$. To form $D$ from $H+z$ we added $E(A)$, where $A \subseteq \{K_1, K_2, P_3, K_3, P_4, K_2 + P_3\}$. Hence Lemma 2.2 shows that $H + z$ contains a $K_k - E(A)$ or $\chi(D) \leq k - 1$. If $\chi(D) \geq k$, then we have a contradiction since $A = K_1$, $A = K_2$, and $A = P_3$ are impossible as above.

To show that $A = K_3$, $A = P_4$, and $A = K_2 + P_3$ are impossible, we apply Lemma 2.8 (this is where we use the fact that $G \neq M_{7,1}$), Lemma 4.32 (since $K_1 - E(P_4) = P_4 * K_{t-4}$), and Lemma 4.27 respectively.

Thus we must have $\chi(D) \leq k - 1$, which gives a $(k-1)$-coloring of $H + z$ in which $a_1$ and $a_2$ are in different color classes and $z$ receives a color not received by any neighbor of $v_1$ in $H$. As above we can complete this partial coloring to all of $G$ by first coloring $z$ and $v_1$ the same and then using Hall’s Theorem.

Hence there is a vertex $x \in V(H)$ which is adjacent to all of $C$. Note that $x$ is not adjacent to any of $v_1, v_2, \ldots, v_{k-(r+1)}$ by the maximality of $r$. Let $Z_2 := \{xa \mid a \in N(v_2) \cap V(H)\}$. Consider the graph $D := H + z + Z_1 + Z_2$. As above, both $Z_1$ and $Z_2$ have cardinality at most 2. Since $|C| \geq 2$, both $x$ and $z$ have degree at most $k$ in $D$. Since both $xa$ and $za$ were added only if $a$ was a neighbor of both $v_1$ and $v_2$, all the neighbors of $v_1$ in $H$ have degree at most $k$ in $D$. Similarly for $v_2$’s neighbors. Hence $\Delta(D) \leq k$. To form $D$ from $H + z$ we added $E(A)$ where $A \subseteq \{K_1, K_2, P_3, K_3, P_4, K_2 + P_3, 2K_2, P_3, 2P_3, C_4\}$. Hence Lemma 2.2 shows that $H + z$ contains a $K_k - E(A)$ or $\chi(D) \leq k - 1$.

Suppose $\chi(D) \geq k$. Then $A = K_1$, $A = K_2$, $A = P_3$, $A = K_3$, $A = P_4$, and $A = K_2 + P_3$ are impossible as above. Applying Lemma 4.27 shows that $A = 2K_2$, $A = P_3$, and $A = 2P_3$ are impossible. Thus we must have $A = C_4$. If $k \geq 8$, then Lemma 4.23 gives a contradiction. Hence we must have $k = 7$. Since $H + z$ contains an induced $K_3 * 2K_2$, we must have $N(v_1) \cap V(H) = N(v_2) \cap V(H)$, say $N(v_1) \cap V(H) = \{w_1, w_2\}$. Moreover, $xz \in E(G)$, $w_1w_2 \in E(G)$ and there are no edges between $\{w_1, w_2\}$ and $\{x, z\}$ in $G$.

Put $Q := \{v_1, \ldots, v_{k-(r+1)}\}$. Then for $v \in Q$, by the same argument as above, we must have $N(v) \cap V(H) = \{w_1, w_2\}$. Hence $Q$ is joined to $\{w_1, w_2\}$, $C$ is joined to $Q$, and $\{x, z\}$ and both $\{x, z\}$ and $\{w_1, w_2\}$ are joined to the same $K_3$ in $H$. We must have $r = 3$ for otherwise one of $x, z, w_1, w_2$ has degree larger than 7. Thus we have an $M_{7,2}$ in $G$ and therefore $G$ is $M_{7,2}$, a contradiction.

Thus we must have $\chi(D) \leq k - 1$, which gives a $(k-1)$-coloring of $H + z$ in which $z$ receives a color $c_1$ which is not received by any of the neighbors of $v_1$ in $H$ and $x$ receives a color $c_2$ which is not received by any of the neighbors of $v_2$ in $H$. Thus $c_1$ is in $v_1$’s list and $c_2$ is in $v_2$’s list. Note that if $x$ and $z$ are adjacent then $c_1 \neq c_2$. Hence, we can 2-color $G[x, z, v_1, v_2]$ from the lists. This leaves $k - 3$ vertices. The vertices in $C$ have lists of size at least $k - 3$ and the rest have lists of size at least $k - 5$. Since the union of any $k - 4$ of the lists contains one list of size $k - 3$, we can complete the partial coloring by Hall’s Theorem.

**Corollary 2.11.** For $k \geq 7$, if $H$ is a $(k-1)$-clique in a $k$-mule $G$ other than $M_{7,1}$ and $M_{7,2}$, then any vertex in $G - H$ has at most one neighbor in $H$.

**Proof.** Let $v \notin H$ be adjacent to $r$ vertices in $H$. Now $G[H \cup \{v\}] = K_r * (K_1 + K_{k-(r+1)})$. If $r \geq 2$, then $G[H \cup \{v\}]$ is forbidden by Lemma 2.10.
**Lemma 2.12.** For \( k \geq 7 \), no \( k \)-mule except \( M_{7,1} \) contains \( K_4 \ast E_{k-4} \) as a subgraph.

**Proof.** Let \( G \) be a \( k \)-mule other than \( M_{7,1} \) and suppose \( G \) contains an induced \( K_4 \ast D \) where \( |D| = k - 4 \). Then \( G \) is not \( M_{7,2} \). By Lemma 4.30, \( D \) is \( E_3 \), a claw, a clique, or almost complete. If \( D \) is a clique then \( G \) contains \( K_k \), a contradiction. Now Corollary 2.11 shows that \( D \) being almost complete is impossible. Finally, Lemma 2.8 shows that \( D \) cannot be \( E_3 \) or a claw. This contradiction completes the proof. \( \square \)

Since \( K_4 \ast E_{\Delta-4} \subseteq K_\Delta \), Lemma 2.12 shows that the following conjecture is equivalent to the Borodin-Kostochka conjecture.

**Conjecture 2.13.** Any graph with \( \chi \geq \Delta \geq 9 \) contains \( K_4 \ast E_{\Delta-4} \) as a subgraph.

**Lemma 2.14.** Let \( G \) be a \( k \)-mule with \( k \geq 8 \). Let \( A \) and \( B \) be graphs with \( 4 \leq |A| \leq k - 4 \) and \( |B| = k - |A| \) such that \( A \ast B \subseteq G \). Then \( A = K_1 + K_{|A|-1} \) and \( B = K_1 + K_{|B|-1} \).

**Proof.** Note that \( |B| \geq 4 \). By Lemma 4.49, \( A \ast B \) is almost complete, \( K_5 \ast E_3 \) or our desired conclusion holds. The first and second cases are impossible by Corollary 2.11 and Lemma 2.8. \( \square \)

This shows that the following conjecture is a natural weakening of Borodin-Kostochka.

**Conjecture 2.15.** Let \( G \) be a graph with \( \Delta(G) = k \geq 9 \). If \( K_{t,k-t} \not\subseteq G \) for all \( 4 \leq t \leq k-4 \), then \( G \) can be \((k-1)\)-colored.

In the next section we create the tools needed to reduce the 4 in these lemmata to 3.

### 2.2 Tooling up

For an independent set \( I \) in a graph \( G \), we write \( G_{[I]} \) for the graph formed by collapsing \( I \) to a single vertex and discarding duplicate edges. We write \([I]\) for the resulting vertex in the new graph. If more than one independent set \( I_1, I_2, \ldots, I_m \) are collapsed in succession we indicate the resulting graph by \( G_{[I_1][I_2] \cdots [I_m]} \).

**Lemma 2.16.** Let \( G \) be a \( k \)-mule other than \( M_{7,1} \) and \( M_{7,2} \) with \( k \geq 7 \) and \( H \triangleleft G \). If \( x, y \in V(H) \), \( xy \not\in E(H) \) and \( |N_H(x) \cup N_H(y)| \leq k \), then there exists a \((k-1)\)-coloring \( \pi \) of \( H \) such that \( \pi(x) = \pi(y) \).

**Proof.** Suppose \( x, y \in V(H) \), \( xy \not\in E(H) \) and \( |N_H(x) \cup N_H(y)| \leq k \). Let \( H' := H_{[x,y]} \). Then \( H' \sim H \) via the natural epimorphism \( f: H \twoheadrightarrow H' \). By applying Lemma 2.2 we either get the desired \((k-1)\)-coloring \( \pi \) of \( H \) or a \( K_{k-1} \) in \( H \) with \( V(K_{k-1}) \subseteq N(x) \cup N(y) \). But \( k-1 \geq 6 \), so one of \( x \) or \( y \) has at least three neighbors in \( K_{k-1} \) violating Corollary 2.11. \( \square \)

**Lemma 2.17.** Let \( G \) be a \( k \)-mule other than \( M_{7,1} \) and \( M_{7,2} \) with \( k \geq 7 \) and \( H \triangleleft G \). Suppose there are disjoint nonadjacent pairs \( \{x_1, y_1\}, \{x_2, y_2\} \subseteq V(H) \) with \( d_H(x_1), d_H(y_1) \leq k-1 \) and \( |N_H(x_2) \cup N_H(y_2)| \leq k \). Then there exists a \((k-1)\)-coloring \( \pi \) of \( H \) such that \( \pi(x_1) \neq \pi(y_1) \) and \( \pi(x_2) = \pi(y_2) \).
Proof. Put \( H' := \frac{H}{[x_2,y_2]} + x_1y_1 \). Then \( H' \prec H \) via the natural epimorphism \( f : H \to H' \). Suppose the desired \((k-1)\)-coloring \( \pi \) of \( H \) doesn’t exist. Apply Lemma 2.2 to get a \( K_k \) in \( H' \). Put \( z := [x_2,y_2] \). By Lemma 2.5 the \( K_k \) must contain \( z \) and by Lemma 2.10 the \( K_k \) must contain \( x_1y_1 \); hence the \( K_k \) contains \( x_1, y_1, z \). Thus \( H \) contains an induced subgraph \( A := \{x_1, y_1\} \ast K_{k-3} \) where \( V(A) \subseteq N_H(x_2) \cup N_H(y_2) \). Then \( x_2 \) and \( y_2 \) each have at most two neighbors in the \( K_{k-3} \) by Lemma 2.12 and Lemma 4.34. Thus \( k = 7 \) and both \( x_2 \) and \( y_2 \) have exactly two neighbors in the \( K_4 \). One of \( x_2 \) or \( y_2 \) has at least one neighbor in \( \{x_1, y_1\} \), so by symmetry we may assume that \( x_2 \) is adjacent to \( x_1 \). But then \( \{x_2\} \cup V(A) \) induces either a \( K_2 \ast \text{antichair} \) (if \( x_2 \not\leftrightarrow y_1 \)) or a graph containing \( K_2 \ast C_4 \) (if \( x_2 \leftrightarrow y_1 \)), and both are impossible by Lemma 4.50.  

2.3 Using our new tools

\[ \text{Figure 4: The mule } M_8. \]

Lemma 2.18. For \( k \geq 7 \), the only \( k \)-mules containing \( K_3 \ast E_{k-3} \) as a subgraph are \( M_{7,1} \), \( M_{7,2} \) and \( M_8 \).

Proof. Suppose not and let \( G \) be a \( k \)-mule other than \( M_{7,1} \), \( M_{7,2} \) and \( M_8 \) containing \( F := C \ast B \) as an induced subgraph where \( C = K_3 \) and \( B \) is an arbitrary graph with \(|B| = k - 3\). By Lemma 4.34 \( B \) is: \( E_3 \ast K_{|B|-3} \), almost complete, \( K_1 + K_{|B|-t} \), \( K_1 + K_1 + K_{|B|-t-1} \), or \( E_3 + K_{|B|-3} \). The first two options are impossible by Lemma 2.12.

First, suppose there is no \( z \in V(G - F) \) with \( C \subseteq N(z) \). Let \( \pi \) be the \((k-1)\)-coloring of \( G - F \) guaranteed by Lemma 2.7. Put \( L := L_\pi \). Let \( I \) be a maximal independent set in \( B \). If there are \( x, y \in I \) and \( c \in L(x) \cap L(y) \), then we may color \( x \) and \( y \) with \( c \) and then greedily complete the coloring to the rest of \( F \) giving a contradiction. Thus we must have
Therefore $|I| \leq 2$ and hence $B$ is $K_t + K_{|B|-t}$. Put $N := \bigcup_{w \in C} N(w) \cap V(G - F)$. Then $|N| \geq 2$ by assumption. Pick $x_1, y_1 \in N$ and nonadjacent $x_2, y_2 \in V(B)$ and put $H := G[V(G - F) \cup \{x_2, y_2\}]$. Plainly, the conditions of Lemma 2.17 are satisfied and hence we have a $(k - 1)$-coloring $\gamma$ of $H$ such that $\gamma(x_1) \neq \gamma(y_1)$ and $\gamma(x_2) = \gamma(y_2)$. But then we can greedily complete this coloring to all of $G$, a contradiction.

Thus we have $z \in V(G - F)$ with $C \subseteq N(z)$. Put $B' := G[V(B) \cup \{z\}]$ and $F' := G[V(F) \cup \{z\}]$. As above, using Lemma 4.34 and Lemma 2.12, we see that $B'$ is $K_t + K_{|B'|-t}$, $K_1 + K_t + K_{|B'|-t-1}$ or $E_3 + K_{|B'|-3}$.

Suppose $B'$ is $E_3 + K_{|B'|-3}$, say the $E_3$ is $\{z_1, z_2, z_3\}$. Since $k \geq 7$, we have $w_1, w_2 \in V(B') - \{z_1, z_2, z_3\}$. Then $d_{F'}(z_3) + d_{F'}(w_1) = k$ and hence we may apply Lemma 2.16 to get a $(k - 1)$-coloring $\zeta$ of $G - F'$ such that there is some $c \in L_\zeta(z_3) \cap L_\zeta(w_1)$. Now $|L_\zeta(z_1)| + |L_\zeta(z_2)| + |L_\zeta(w_2)| \geq 2 + 2 + k - 1 = k$ and hence there is a color $c_1$ that is in at least two of $L_\zeta(z_1)$, $L_\zeta(z_2)$ and $L_\zeta(w_2)$. If $c_1 = c$, then $c$ appears on an independent set of size 3 in $B'$ and we may color this set with $c$ and greedily complete the coloring. Otherwise, $B'$ contains two disjoint nonadjacent pairs which we can color with different colors and again complete the coloring greedily, a contradiction.

Now suppose $B'$ is $K_1 + K_t + K_{|B'|-t-1}$. By Lemma 2.10 we must have $2 \leq t \leq |B'|-3$. Let $x$ be the vertex in the $K_1$, $w_1, w_2 \in V(K_t)$ and $z_1, z_2 \in V(K_{|B'|-t-1})$. Then $d_{F'}(w_1) + d_{F'}(z_1) = k+1$ and hence we may apply Lemma 2.16 to get a $(k - 1)$-coloring $\zeta$ of $G - F'$ such that there is some $c \in L_\zeta(w_1) \cap L_\zeta(z_1)$. Now $|L_\zeta(x)| + |L_\zeta(w_2)| + |L_\zeta(z_2)| \geq 2 + k - 1 = k + 1$ and hence there is at least two colors $c_1, c_2$ that are each in at least two of $L_\zeta(x)$, $L_\zeta(w_2)$ and $L_\zeta(z_2)$. If $c_1 \neq c$ or $c_2 \neq c$, then $B'$ contains two disjoint nonadjacent pairs which we can color with different colors and then complete the coloring greedily. Otherwise $c$ appears on an independent set of size 3 in $B'$ and we may color this set with $c$ and greedily complete the coloring, a contradiction.

Therefore $B'$ must be $K_t + K_{|B'|-t}$. By Lemma 2.10 we must have $3 \leq t \leq |B'|-3$. Thus $k \geq 8$. Let $X$ and $Y$ be the two cliques covering $B'$. Let $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Put $H := G[V(G - F') \cup \{x_1, x_2, y_1, y_2\}]$ and $H' := G[V(G - F') \cup \{x_1, y_1\} \cup \{x_2, y_2\}]$. For $i \in [2]$, $d_{F'}(x_i) + d_{F'}(y_i) = k + 2$ and thus $\Delta(H') \leq k$. If $\chi(H') \leq k - 1$, then we have a $(k - 1)$-coloring of $H$ which can be greedily completed to all of $G$, a contradiction. Hence, by Lemma 2.2 $H'$ contains
$K_k$. Hence $H - \{x_1, y_1, x_2, y_2\}$ contains a $K_{k-2}$, call it $A$, such that $V(A) \subseteq N(x_i) \cup N(y_i)$ for $i \in [2]$. Since $d_{F^*}(x_i) + d_{F^*}(y_i) = k + 2$, we see that $N_H(x_i) \cap N_H(y_i) = \emptyset$ for $i \in [2]$. But we can play the same game with the pairs $\{x_1, y_2\}$ and $\{x_2, y_1\}$. We conclude that $N(x_1) \cap V(A) = N(x_2) \cap V(A)$ and $N(y_1) \cap V(A) = N(y_2) \cap V(A)$. In fact we can extend this equality to all of $X$ and $Y$. Put $Q := N(x_1) \cap V(A)$ and $P := N(y_1) \cap V(A)$. Then we conclude that $X$ is joined to $Q$ and $Y$ is joined to $P$. Moreover, we already know that $X$ and $Y$ are joined to the same $K_3$. The edges in these joins exhaust the degrees of all the vertices, hence $G$ is a 5-cycle with vertices blown up to cliques. If $k = 8$, then $|X| = |Y| = 3$ and thus $|Q| = |P| = 3$, but then $G = M_8$, a contradiction. So $k \geq 9$, but now $G$ is a line graph of a multigraph, so this is impossible by the Borodin-Kostochka conjecture for line graphs proved in [14].

Since $K_3 \ast E_{\Delta - 3} \subseteq K_\Delta$, Lemma 2.18 shows that Conjecture 1.17 is equivalent to the Borodin-Kostochka conjecture.

**Lemma 2.19.** Let $G$ be a $k$-mule with $k \geq 7$ other than $M_{7,1}$, $M_{7,2}$ and $M_8$. Let $A$ and $B$ be graphs with $3 \leq |A| \leq k - 3$ and $|B| = k - |A|$ such that $A \ast B \leq G$. Then $A = K_1 + K_{|A|-1}$ and $B = K_1 + K_{|B|-1}$.

**Proof.** Suppose the lemma is false and let $A \ast B \leq G$ be a counterexample.

First suppose $|A|, |B| \geq 4$. Then, by Lemma 4.49, $A \ast B$ is almost complete or $K_5 \ast E_3$.

The first and second cases are impossible by Corollary 2.11 and Lemma 2.8 respectively.

Thus we may assume $|A| = 3$. By Lemma 2.18, $A \in \{E_3, P_3, K_1 + K_2\}$. If $A = E_3$, then $B$ is complete by Lemma 4.44 but this is impossible by Lemma 2.8. If $A = P_3$, then $B$ is complete by Lemma 1.25 but this is impossible by Lemma 2.5. Hence $A = K_1 + K_2$. By Lemma 4.48, $B$ is complete or $K_1 + K_{|B|-1}$. The former is impossible by Lemma 2.9 and the latter by supposition.

Lemma 2.19 proves our main result, that Conjecture 1.15 is equivalent to the Borodin-Kostochka conjecture.

### 2.4 The low vertex subgraph of a mule

In this section we show that if a mule is not regular, then the subgraph of non-maximum-degree vertices is severely restricted. For a vertex critical graph $G$ we write $\mathcal{L}(G)$ for the subgraph induced on the vertices of degree $\chi(G) - 1$ in $G$ and $\mathcal{H}(G)$ for the subgraph induced on the rest of the vertices. We call $v \in V(G)$ low if $v \in V(\mathcal{L}(G))$ and high otherwise.

**Lemma 2.20.** For $k \geq 6$, no $k$-mule contains an induced $E_2 \ast K_{k-2}$ with some vertex low.

**Proof.** Since $M_{6,1}$ and $M_{7,1}$ contain no such induced subgraph, the lemma follows from Lemma 2.5.

**Lemma 2.21.** If $G$ is a $k$-mule with $k \geq 6$, then $\mathcal{L}(G)$ is complete.

**Proof.** Let $G$ be a $k$-mule with $k \geq 6$ and suppose $G$ has nonadjacent low vertices $x$ and $y$. Then $G + xy \not\sim G$ and hence, by Lemma 2.2, $G + xy$ contains a $K_k$. But then $G$ contains an $E_2 \ast K_{k-2}$ with some vertex low, contradicting Lemma 2.20. Hence $\mathcal{L}(G)$ is complete.
Lemma 2.22. If $G$ is a $k$-mule with $k \geq 6$ other than $M_{6,1}$ and $M_{7,1}$, then $|\mathcal{L}(G)| \leq k - 2$.

Proof. Let $G$ be a $k$-mule with $k \geq 6$ other than $M_{6,1}$ and $M_{7,1}$. By Lemma 2.21 $\mathcal{L}(G)$ is complete and hence $|\mathcal{L}(G)| \leq k - 1$. Suppose $|\mathcal{L}(G)| = k - 1$. Since $G$ doesn’t contain $K_k$, no high $z$ is adjacent to all of $\mathcal{L}(G)$. Hence, by Lemma 2.7 there is a $(k - 1)$-coloring of $\mathcal{H}(G)$ that we can complete to all of $G$ using Hall’s Theorem. This contradiction completes the proof. □

Lemma 2.23. Let $G$ be a $k$-mule with $k \geq 6$. If a high $x \in V(G)$ has at least three low neighbors, then $x$ is adjacent to all low vertices in $G$.

Proof. Assume the lemma is false. Let $x$ be a high degree vertex with at least three neighbors in $V(\mathcal{L}(G))$. If $|V(\mathcal{L}(G))| = 3$, then the claim holds. So assume that $|V(\mathcal{L}(G))| \geq 4$ and choose $y \in V(\mathcal{L}(G)) \setminus N(x)$. Let $A = V(\mathcal{L}(G)) \cap N(x)$. By Lemma 2.21 $\mathcal{L}(G)$ is complete. Thus, $G[\{x, y\} \cup A] = E_2 \ast K_{|A|}$. Since $L(v) = d(v)$ for all $v \in (A \cup \{y\})$, Lemma 4.54 implies that $E_2 \ast K_{|A|}$ cannot appear in $G$. This contradiction implies the lemma. □

2.5 Restrictions on the independence number

The Borodin-Kostochka conjecture has been proven for graphs with independence number at most two [1]. Here we prove that if we wish to prove the Borodin-Kostochka conjecture for graphs with independence number at most $a$ for any $a \leq 6$, it suffices to construct a $K_{\Delta - 1}$.

For $a \geq 2$, let $\mathcal{C}_a^k$ be those $G \in \mathcal{C}_k$ with $\alpha(G) \leq a$. By a $(k, a)$-mule we mean a $\mathcal{C}_a^k$-mule. Note that if $G \in \mathcal{C}_a^k$ and for some $H \in \mathcal{C}_k$ we have $H \prec G$, then $H \in \mathcal{C}_a^k$ as well. Therefore any $(k, a)$-mule is also a $k$-mule.

Theorem 2.24. For $k \geq 7$ and $2 \leq a \leq k - 3$, no $(k, a)$-mule except $M_{7,1}$ contains a $K_{k-1}$.

Proof. Suppose otherwise and let $G$ be such a $(k, a)$-mule containing a $K_{k-1}$, call it $H$. By Corollary 2.11 each vertex in $G - H$ has at most one neighbor in $H$. Let $\pi$ be a $(k - 1)$-coloring of $G - H$. Then $|L_\pi(v)| \geq k - 3$ for all $v \in V(H)$. Since $H$ cannot be colored from $L_\pi$, applying Hall’s Theorem shows that either $|\text{Pot}(L_\pi)| \leq k - 2$ or there is some $x \in V(H)$ such that $|\text{Pot}_{H - x}(L_\pi)| \leq k - 3$. In the former case, $\pi$ must have some color class to which each vertex of $H$ is adjacent and hence $\alpha(G) \geq k - 1$, a contradiction. In the latter case, $\pi$ must have two color classes to which each vertex of $H - x$ is adjacent and hence $G$ has two disjoint independent sets of size $k - 2$. Again we have a contradiction since $\alpha(G) \geq k - 2$. □

It follows that Conjecture 1.22 is equivalent to the Borodin-Kostochka conjecture for graphs with independence number at most 6.

3 Connectivity of complements

As a basic application of our list coloring lemmas, we prove that for $k \geq 5$ any $G \in \mathcal{C}_k$ has maximally connected complement.
Lemma 3.1. Fix $k \geq 5$. If $G \in C_k$ and $A \ast B \triangleleft G$ for graphs $A$ and $B$ with $1 \leq |A| \leq |B|$, then $|A \ast B| \leq \Delta(G) + 1$.

Proof. Let $G \in C_k$ and $A \ast B \triangleleft G$ for graphs $A$ and $B$ with $1 \leq |A| \leq |B|$. Assume $|A \ast B| > \Delta(G) + 1$. To avoid a vertex with degree larger than $\Delta(G)$, we must have $\Delta(A) \leq |A| - 2$ and $\Delta(B) \leq |B| - 2$. In particular, both $A$ and $B$ are incomplete, so $2 \leq |A| \leq |B|$ and both $A$ and $B$ contain an induced $E_2$. Hence, by Lemma 4.27 both $A$ and $B$ are the disjoint union of complete subgraphs and at most one $P_3$.

First, assume $|A| = 2$, say $A = \{x_1, x_2\}$. Since $|B| \geq \Delta(G)$, we conclude that $N(x_1) = N(x_2)$. Thus $x_1$ and $x_2$ are nonadjacent twins in a vertex critical graph which is impossible.

Thus we may assume that $|A| \geq 3$. If $A$ contained an induced $P_3$, then $G$ would have an induced $E_2 \ast (K_1 \ast B)$. For $K_1 \ast B$ to be the disjoint union of complete subgraphs and at most one $P_3$, $B$ must either be $E_2$ or complete, both of which are impossible. Hence $A$ is a disjoint union of at least two complete subgraphs. The same goes for $B$.

Assume that $A$ is edgeless. Then, by Lemma 4.44, $B$ must be $E_3$ or $P_3$. Hence $\Delta(G) + 1 < |A| + |B| = 6$, giving the contradiction $\Delta(G) \leq 4$.

Since $A$ is the disjoint union of at least two complete subgraphs and contains an edge, it contains $P_3$. By Lemma 4.48, $B$ must be either $E_3$ or the disjoint union of a vertex and a complete subgraph. As above, $B = E_3$ is impossible. In particular $B$ contains $P_3$, and using Lemma 4.48 again, we conclude that $A$ is the disjoint union of a vertex and a complete subgraph giving the final contradiction $\omega(G) \geq \omega(A \ast B) \geq \omega(A) + \omega(B) \geq |A| + |B| - 2 \geq \Delta(G)$.

Lemma 3.2. Fix $k \geq 5$. If $G \in C_k$, then $\overline{G}$ is maximally connected; that is, $\kappa(\overline{G}) = \delta(\overline{G})$.

Proof. Let $G \in C_k$ and let $S$ be a cutset in $\overline{G}$ with $|S| = \kappa(\overline{G})$. To get a contradiction, assume that $|S| < \delta(\overline{G}) = |G| - (\Delta(G) + 1)$. Since $\overline{G} - S$ is disconnected, $G - S = A \ast B$ for some graphs $A$ and $B$ with $1 \leq |A| \leq |B|$. We have $|A| + |B| = |\overline{G} - S| = |G| - |S| > |G| - (|G| - (\Delta(G) + 1)) = \Delta(G) + 1$. But then Lemma 3.1 gives a contradiction.

4 List coloring lemmas

In this section we use list-coloring lemmas to forbid a large class of graphs from appearing as subgraphs of mules. In each case, we assume that such a graph $H \triangleleft G$ appears as an induced subgraph of a mule $G$. By the minimality of $G$, we can color $G \setminus H$ with $\Delta - 1$ colors. If $H$ can be colored regardless of which colors are forbidden by its colored neighbors in $G \setminus H$, then we can clearly extend this coloring to all of $G$. We use the term $d_1$-choosable to describe such a graph $H$.

We characterize all graphs $A \ast B$ with $|A| \geq 2$, $|B| \geq 2$ that are not $d_1$-choosable. The characterization is somewhat lengthy, so we split it into a number of lemmas. For the case $|A| \geq 4$, $|B| \geq 4$, see Lemma 4.49. When $|A| = 3$, we consider the four cases $A = E_3$ (Lemma 4.44), $A = P_3$ (Lemma 4.48), $A = P_3$ (Lemma 4.31), and $A = K_3$ (Lemma 4.34). When $|A| = 2$, we consider the case $A = E_2$ in Lemma 4.27 and the case $A = K_2$ in Lemma 4.52. Finally, in Lemma 4.58, we characterize all triangle-free graphs $B$ such that $K_1 \ast B$ is not $d_1$-choosable.
Let $G$ be a graph. A list assignment to the vertices of $G$ is a function from $V(G)$ to the finite subsets of $N$. A list assignment $L$ to $G$ is good if $G$ has a coloring $c$ where $c(v) \in L(v)$ for each $v \in V(G)$. It is bad otherwise. We call the collection of all colors that appear in $L$, the pot of $L$. That is $\text{Pot}(L) := \bigcup_{v \in V(G)} L(v)$. For a subgraph $H$ of $G$ we write $\text{Pot}_H(L) := \bigcup_{v \in V(H)} L(v)$. For $S \subseteq \text{Pot}(L)$, let $G_S$ be the graph $G[[v \in V(G) \mid L(v) \cap S \neq \emptyset]]$. We also write $G_c$ for $G_{\{c\}}$. We let $\mathcal{B}(L)$ be the bipartite graph that has parts $V(G)$ and $\text{Pot}(L)$ and an edge from $v \in V(G)$ to $c \in \text{Pot}(L)$ iff $c \in L(v)$. For $f : V(G) \to N$, an $f$-assignment on $G$ is an assignment $L$ of lists to the vertices of $G$ such that $|L(v)| = f(v)$ for each $v \in V(G)$. We say that $G$ is $f$-choosable if every $f$-assignment on $G$ is good.

### 4.1 Shrinking the pot

In this section we prove a lemma about bad list assignments with minimum pot size. Some form of this lemma has appeared independently in at least two places we know of—Kierstead [8] and Reed and Sudakov [17]. We will use this lemma repeatedly in the arguments that follow.

Given a graph $G$ and $f : V(G) \to N$, we have a partial order on the $f$-assignments to $G$ given by $L < L'$ if $|\text{Pot}(L)| < |\text{Pot}(L')|$. When we talk of minimal $f$-assignments, we mean minimal with respect to this partial order.

**Lemma 4.1.** Let $G$ be a graph and $f : V(G) \to N$. Assume $G$ is not $f$-choosable and let $L$ be a minimal bad $f$-assignment. Assume $L(v) \neq \text{Pot}(L)$ for each $v \in V(G)$. Then, for each nonempty $S \subseteq \text{Pot}(L)$, any coloring of $G_S$ from $L$ uses some color not in $S$.

**Proof.** Suppose not and let $\emptyset \neq S \subseteq \text{Pot}(L)$ be such that $G_S$ has a coloring $\phi$ from $L$ using only colors in $S$. For $v \in V(G)$, let $h(v)$ be the smallest element of $\text{Pot}(L) - L(v)$ (this is well defined by assumption). Pick some $c \in S$ and construct a new list assignment $L'$ as follows.

$$
L'(v) = \begin{cases} 
L(v) & \text{if } v \in V(G) - V(G_S) \\
L(v) & \text{if } v \in V(G_S) \text{ and } c \notin L(v) \\
(L(v) - \{c\}) \cup \{h(v)\} & \text{if } v \in V(G_S) \text{ and } c \in L(v)
\end{cases}
$$

Note that $L'$ is an $f$-assignment and $\text{Pot}(L') = \text{Pot}(L) - \{c\}$. Thus, by minimality of $L$, we can properly color $G$ from $L'$. In particular, we have a coloring of $V(G) - V(G_S)$ from $L$ using no color from $S$. We can complete this to a coloring of $G$ from $L$ using $\phi$. This contradicts the fact that $L$ is bad. \qed

**Definition 5.** A bipartite graph with parts $A$ and $B$ has positive surplus (with respect to $A$) if $|N(X)| > |X|$ for all $\emptyset \neq X \subseteq A$.

**Lemma 4.2.** Let $G$ be a graph and $f : V(G) \to N$. Assume $G$ is not $f$-choosable and let $L$ be a minimal bad $f$-assignment. Assume $L(v) \neq \text{Pot}(L)$ for each $v \in V(G)$. Then $\mathcal{B}(L)$ has positive surplus (with respect to $\text{Pot}(L)$).

**Proof.** Suppose not and choose $\emptyset \neq X \subseteq \text{Pot}(L)$ such that $|N(X)| \leq |X|$ minimizing $|X|$. If $|X| = 1$, then $G_X$ can be colored from $X$ contradicting Lemma 4.1. Hence $|X| \geq 2$. 

17
By minimality of $|X|$, for any $Y \subset X$, $|N(Y)| \geq |Y| + 1$. Hence, for any $x \in X$, we have $|N(X)| \geq |N(X - \{x\})| \geq |X - \{x\}| + 1 = |X|$. Thus, by Hall’s Theorem, we have a matching of $X$ into $N(X)$, but $|N(X)| \leq |X|$ so this gives a coloring of $G_X$ from $X$ contradicting Lemma 4.1.

Our approach to coloring a graph (particularly a join) will often be to consider nonadjacent vertices $u$ and $v$ and show that their lists contain a common color. By the pigeonhole principle, this follows immediately when $|L(u)| + |L(v)| > |Pot(L)|$. We will use the following lemma frequently throughout the remainder of this paper.

**Small Pot Lemma.** Let $G$ be a graph and $f : V(G) \to \mathbb{N}$ with $f(v) < |G|$ for all $v \in V(G)$. If $G$ is not $f$-choosable, then $G$ has a minimal bad $f$-assignment $L$ such that $|Pot(L)| < |G|$.

**Proof.** Suppose not and let $L$ be a minimal bad $f$-assignment. For each $v \in V(G)$ we have $|L(v)| = f(v) < |G| \leq |Pot(L)|$ and hence $L(v) \neq Pot(L)$. Thus by Lemma 4.2 we have the contradiction $|G| \geq |N(Pot(L))| > |Pot(L)|$.

### 4.2 Degree choosability

**Definition 6.** Let $G$ be a graph and $r \in \mathbb{Z}$. Then $G$ is $d_r$-choosable if $G$ is $f$-choosable where $f(v) = d(v) - r$.

Note that a vertex critical graph with $\chi = \Delta + 1 - r$ contains no induced $d_r$-choosable subgraph. Since we are working to prove the Borodin-Kostochka Conjecture, we will focus on the case $r = 1$ and primarily study $d_1$-choosable graphs. For $r = 0$, we have the following well known generalization of Brooks’ Theorem (see [2], [6], [11], [5] and [7]).

**Definition 7.** A Gallai tree is a graph all of whose blocks are complete graphs or odd cycles.

**Classification of $d_0$-choosable graphs.** For any connected graph $G$, the following are equivalent.

- $G$ is $d_0$-choosable.
- $G$ is not a Gallai tree.
- $G$ contains an induced even cycle with at most one chord.

We give a couple lemmas about $d_0$-assignments that will be useful in our study of $d_1$-assignments. The following lemma was used in [11].

**Lemma 4.3.** Let $L$ be a bad $d_0$-assignment on a connected graph $G$ and $x \in V(G)$ a non-cutvertex. Then $L(x) \subseteq L(y)$ for each $y \in N(x)$.

**Proof.** Suppose otherwise that we have $c \in L(x) - L(y)$ for some $y \in N(x)$. Coloring $x$ with $c$ leaves at worst a $d_0$-assignment $L'$ on the connected $H := G - x$ where $|L'(y)| > d_H(y)$. But then we can complete the coloring, a contradiction.

**Lemma 4.4.** If $L$ is a bad $d_0$-assignment on a connected graph $G$, $|Pot(L)| < |G|$.  

18
Proof. Suppose that the lemma is false and choose a connected graph $G$ together with a bad $d_0$-assignment $L$ where $|\text{Pot}(L)| \geq |G|$ minimizing $|G|$. Plainly, $|G| \geq 2$. Let $x \in G$ be a noncutvertex (any end block has at least one). By Lemma 4.3, $L(x) \subseteq L(y)$ for each $y \in N(x)$. Thus coloring $x$ decreases the pot by at most one, giving a smaller counterexample. This contradiction completes the proof. \hfill \Box

We also need a few basic lemmas about how $d_r$-choosability behaves with respect to induced subgraphs.

**Lemma 4.5.** Fix $r \geq 0$. Let $G$ be a graph and $H \subseteq G$ a $d_r$-choosable subgraph. If $L$ is a $d_r$-assignment on $G$ and $G - H$ is properly colorable from $L$, then $G$ is properly colorable from $L$.

*Proof.* Color $G - H$ from $L$. Let $L'$ be the resulting list assignment on $H$. Since each $v \in V(H)$ must be adjacent to as many vertices as colors in $G - H$ we see that $L'$ is again a $d_r$-assignment. The lemma follows. \hfill \Box

**Lemma 4.6.** Fix $r \geq 0$. Let $G$ be a graph and $H \subseteq G$ a $d_r$-choosable subgraph. If there exists an ordering $v_1, \ldots, v_t$ of the vertices of $G - H$ such that $v_i$ has degree at least $r + 1$ in $G[V(H) \cup \bigcup_{1 \leq j \leq i - 1} v_j]$ for each $i$, then $G$ is $d_r$-choosable.

*Proof.* Let $L$ be a $d_r$-assignment on $G$. Go through $G - H$ in order $v_t, \ldots, v_1$ coloring $v_i$ with the smallest available color in $L(v_i)$. Since when we go to color $v_i$, it has at least $r + 1$ uncolored neighbors we succeed in coloring $G - H$. Now the lemma follows from Lemma 4.5. \hfill \Box

We will also use the following immediate consequence of the pigeonhole principle.

**Lemma 4.7.** If $S_1, \ldots, S_m$ are nonempty subsets of a finite set $T$ and $\sum_{i \geq 1} |S_i| > (m - 1)|T|$, then $\bigcap_{i \geq 1} S_i \neq \emptyset$.

### 4.3 Handling joins

The main result of this section is Lemma 4.14, which plays a key role in our classification of bad graphs $A * B$. Specifically, Lemma 4.14 is essential to the proof of Lemma 4.23, which considers the case when $|A| \geq 4$ and $B$ is arbitrary.

**Lemma 4.8.** Fix $r \geq 0$. Let $A$ be a graph with $|A| \geq r + 1$ and $B$ a nonempty graph. If $A * B$ is $d_r$-choosable, then $A * C$ is $d_r$-choosable for any graph $C$ with $B \subseteq C$.

*Proof.* Assume $A * B$ is $d_r$-choosable and let $C$ be a graph with $B \subseteq C$. Put $H = C - B$. For each $v \in V(H)$, $|L(v)| \geq d(v) - r \geq d_H(v) + r + 1 - r = d_H(v) + 1$. Thus we may color $H$ from its lists. By Lemma 4.5, we can complete the coloring to all of $A * C$. \hfill \Box

**Lemma 4.9.** Fix $r \geq 0$. Let $A$ be a graph with $|A| \geq r$ and $B$ a nonempty graph. If $A * B$ is $d_r$-choosable, then $A * C$ is $d_r$-choosable for any connected graph $C$ with $B \subseteq C$. 

19
Proof. Assume \( A \ast B \) is \( d_r \)-choosable and let \( C \) be a connected graph with \( B \subseteq C \). Put \( H = C - B \). For each \( v \in H \), \( |L(v)| \geq d(v) - r \geq d_H(v) + r - r = d_H(v) \). Since \( C \) is connected, each component of \( H \) has a vertex \( v \) that hits a vertex in \( B \) and hence has \( |L(v)| \geq d_H(v) + 1 \). Thus we may color \( H \) from its lists. By Lemma 4.5, we can complete the coloring to all of \( A \ast C \).

\[ \square \]

**Lemma 4.10.** Fix \( r \geq 0 \). Let \( G \) be a \( d_{r-1} \)-choosable graph with at least \( 2r + 2 \) vertices. Then \( E_2 \ast G \) is \( d_r \)-choosable.

Proof. Let \( x, y \) be the vertices in the \( E_2 \). Suppose \( E_2 \ast G \) is not \( d_r \)-choosable. Then by the Small Pot Lemma, we have a \( d_r \)-assignment \( L \) with \( |Pot(L)| < 2 + |G| \). Now \( |L(x)| + |L(y)| \geq d(x) + d(y) - 2r \geq 2 |G| - 2r \geq 2 + |G| > |Pot(L)| \), since \( |G| \geq 2r + 2 \). Thus we can use a single common color on \( x \) and \( y \), leaving a \( d_{r-1} \)-assignment on \( G \). We may now complete the coloring, giving a contradiction.

Since every graph is \( d_{-1} \)-choosable we get immediately.

**Corollary 4.11.** For \( r \geq 0 \), both \( E_2^{r+2} \) and \( E_2^{r+1} \ast K_2 \) are \( d_r \)-choosable.

**Lemma 4.12.** Fix \( r \geq 0 \). Let \( A \) be a graph with \( |A| \geq 3r + 2 \) and \( B \) an arbitrary graph. If \( A \ast B \) is not \( d_r \)-choosable, then \( \omega(B) \geq |B| - 2r \).

Proof. Suppose \( G := A \ast B \) is not \( d_r \)-choosable and let \( L \) be a minimal bad \( d_r \)-assignment. Then, by the Small Pot Lemma, \( |Pot(L)| \leq |G| - 1 \). Let \( g : S \rightarrow Pot_S(L) \) be a partial coloring of \( B \) from \( L \) maximizing \( |S| - |im(g)| \) and then minimizing \( |S| \). Color \( S \) using \( g \) and let \( L' \) be the resulting list assignment.

Put \( H := G - S \) and \( C := B - S \). First suppose that \( |S| - |im(g)| \geq r + 1 \). For each \( v \in C \) we have \( |L'(v)| \geq d_C(v) - r + 3r + 2 > d_C(v) \), so we can complete \( g \) to \( C \). This leaves each \( v \in V(A) \) with a list of size at least \( d_A(v) - r + |S| - |im(g)| > d_A(v) \). Hence, we can complete the coloring to all of \( G \). Thus \( L \) is not bad after all, giving a contradiction.

So instead we assume that \( |S| - |im(g)| \leq r \). By the minimality condition on \( |S| \) we see that \( g \) has no singleton color classes. In particular, \( |S| \geq 2 |im(g)| \). By combining this inequality with \( |S| - |im(g)| \leq r \), we get \( |S| \leq 2r \). Since \( |C| = |B| - |S| \geq |B| - 2r \), the conclusion will follow if we can show that \( C \) is complete.

By definition, \( |Pot(L')| = |Pot(L)| - |im(g)| \). By the maximality condition on \( g \), every pair of nonadjacent vertices in \( C \) must have disjoint lists under \( L' \) (otherwise we could use a common color on nonadjacent vertices in \( C \) and increase \( |S| - |im(g)| \)). Let \( I \) be a maximal independent set in \( C \). To reach a contradiction, we assume that \( |I| \geq 2 \). Then for all the
elements of $I$ to have disjoint lists, we must have
\[
\sum_{v \in I} |L'(v)| \leq |Pot(L')|
\]
\[
\sum_{v \in I} (d_H(v) - r) \leq |Pot(L')|
\]
\[
\sum_{v \in I} (|A| + d_C(v) - r) \leq |Pot(L')|
\]
\[
(|A| - r) |I| + \sum_{v \in I} d_C(v) \leq |Pot(L')|
\]
\[
(|A| - r) |I| + |C| - |I| \leq |Pot(L')|
\]
\[
(|A| - r - 1) |I| + |B| - |S| \leq |A| + |B| - 1 - |im(g)|
\]
\[
(|A| - r - 1) \leq |A| - 1 + |S| - |im(g)|
\]
\[
2(|A| - r - 1) \leq |A| - 1 + |S| - |im(g)|
\]
\[
|A| - 2r - 1 \leq |S| - |im(g)|
\]
\[
r + 1 \leq |S| - |im(g)|.
\]

This final inequality contradicts our assumption that $|S| - |im(g)| \leq r$. Hence $|I| \leq 1$; that is, $C$ is complete.

**Lemma 4.13.** Fix $r \geq 1$. Let $A$ be a connected graph and $B$ an arbitrary graph such that $A*B$ is not $d_r$-choosable. Let $L$ be a minimal bad $d_r$-assignment on $A*B$. If $B$ is colorable from $L$ using at most $|B| - r$ colors, then $|Pot(L)| \leq |A| + |B| - 2$.

**Proof.** By the Small Pot Lemma, $|Pot(L)| \leq |A| + |B| - 1$, so to get a contradiction suppose that $|Pot(L)| = |A| + |B| - 1$ and that $B$ is colorable from $L$ using at most $|B| - r$ colors. If $|Pot_A(L)| \geq |Pot(L)| + 1 - r$, then coloring $B$ with at most $|B| - r$ colors leaves at worst a $d_0$-assignment $L'$ on $A$ with $|Pot(L')| \geq |A|$. Hence the coloring can be completed to $A$ by Lemma 4.4, a contradiction.

Thus we may assume that $|Pot_A(L)| \leq |Pot(L)| - r$. Put $S := Pot(L) - Pot_A(L)$. Let $\pi$ be a coloring of $B$ from $L$ using at most $|B| - r$ colors, say $\pi$ uses colors $C$. Then $|C| = |B| - r$ and $S \cap C = \emptyset$ for otherwise coloring $B$ leaves at worst a $d_{r-1}$-assignment on $A$. Also, $\pi^{-1}(c) \not\subseteq V(G_S)$ for any $c \in C$ since otherwise we could recolor $\pi^{-1}(c)$ with colors from $S$ to get at worst a $d_{r-1}$-assignment on $A$. In particular, $|G_S| \leq \sum_{c \in C} (|\pi^{-1}(c)| - 1) = |B| - |C| = r \leq |S|$. But this inequality contradicts Lemma 4.2.

We now use Lemma 4.13 to strengthen Lemma 4.12.

**Lemma 4.14.** Fix $r \geq 1$. Let $A$ be a connected graph with $|A| \geq 3r + 1$ and $B$ an arbitrary graph. If $A*B$ is not $d_r$-choosable, then $\omega(B) \geq |B| - 2r$.

**Proof.** Suppose $G := A*B$ is not $d_r$-choosable and let $L$ be a minimal bad $d_r$-assignment. Then, by the Small Pot Lemma, $|Pot(L)| \leq |G| - 1$. Let $g: S \rightarrow Pot_S(L)$ be a partial coloring of $B$ from $L$ maximizing $|S| - |im(g)|$ and then minimizing $|S|$. Color $S$ using $g$ and let $L'$ be the resulting list assignment.
Put $C := B - S$. Running through the argument in Lemma 4.12 with $3r + 1$ in place of $3r + 2$ shows that we must have $|S| - |im(g)| = r$. But then completing $g$ to $C$ gives a coloring of $B$ from $L$ using at most $|B| - r$ colors. Thus, by Lemma 4.13, $|Pot(L)| \leq |G| - 2$. Now running through the argument in Lemma 4.12 again completes the proof.

4.4 The $r = 1$ case

4.4.1 Some preliminary tools

The Small Pot Lemma says that if $A \ast B$ is not $d_1$-choosable, then $A \ast B$ has a bad $d_1$-assignment $L$ such that $|Pot(L)| \leq |A| + |B| - 1$. In this section, we study conditions under which $|Pot(L)| \leq |A| + |B| - 2$. We also prove a key lemma for coloring graphs of the form $K_1 \ast B$. In the following section, our results here help us to find nonadjacent vertices with a common color.

Lemma 4.15. Let $A$ be a graph with $|A| \geq 2$, $B$ an arbitrary graph and $L$ a $d_1$-assignment on $A \ast B$. If $B$ has an independent set $I$ such that $(|A| - 1)|I| + |E_B(I)| > |Pot(L)|$, then $B$ can be colored from $L$ using at most $|B| - 1$ colors.

Proof. Suppose that $B$ has an independent set $I$ such that $(|A| - 1)|I| + |E(I)| > |Pot(L)|$. Now

$$\sum_{v \in I} |L(v)| = \sum_{v \in I} (d(v) - 1) = (|A| - 1)|I| + \sum_{v \in I} d_B(v) = (|A| - 1)|I| + |E_B(I)| > |Pot(L)|.$$ 

Hence we have distinct $x, y \in I$ with a common color $c$ in their lists. So we color $x$ and $y$ with $c$. Since $|A| \geq 2$, this leaves at worst a $d_1$-assignment on the rest of $B$. Completing the coloring to the rest of $B$ gives the desired coloring of $B$ from $L$ using at most $|B| - 1$ colors.

Lemma 4.16. Let $G$ be a graph and $I$ a maximal independent set in $G$. Then $|E(I)| \geq |G| - |I|$. If $I$ is maximum and $|E(I)| = |G| - |I|$, then $G$ is the disjoint union of $|I|$ complete graphs.

Proof. Each vertex in $G - I$ is adjacent to at least one vertex in $I$. Hence $|E(I)| \geq |G| - |I|$. Now assume $I$ is maximum and $|E(I)| = |G| - |I|$. Then $N(x) \cap N(y) = \emptyset$ for every distinct pair $x, y \in I$. Also, $N(x)$ must be a clique for each $x \in I$, since otherwise we could swap $x$ out for a pair of nonadjacent neighbors and get a larger independent set. Since we can swap $x$ with any of its neighbors to get another maximum independent set, we see that $G$ has components $\{G[v] \cup N(v) \mid v \in I\}$. 

Lemma 4.17. Let $A$ be a connected graph with $|A| \geq 4$ and $B$ an incomplete graph. If $A \ast B$ is not $d_1$-choosable, then $A \ast B$ has a minimal bad $d_1$-assignment $L$ such that $|Pot(L)| \leq |A| + |B| - 2$.

Proof. Suppose $A \ast B$ is not $d_1$-choosable and let $L$ be a minimal bad $d_1$-assignment on $A \ast B$. Then, by the Small Pot Lemma, $|Pot(L)| \leq |A| + |B| - 1$. Let $I$ be a maximum independent
set in $B$. Since $B$ is incomplete, $|I| = \alpha(B) \geq 2$. By Lemma 4.16 $|E_B(I)| \geq |B| - |I| = |B| - \alpha(B)$. As $|A| \geq 4$ we have $(|A| - 1)|I| + |E_B(I)| \geq (|A| - 1)\alpha(B) + |B| - \alpha(B) \geq (|A| - 2)\alpha(B) + |B| \geq 2|A| - 4 + |B| > |A| + |B| - 1 \geq |\text{Pot}(L)|$. Hence by Lemma 4.15 $B$ can be colored from $L$ using at most $|B| - 1$ colors. But then we are done by Lemma 4.13.

**Lemma 4.18.** Let $A$ be a connected graph with $|A| = 3$ and $B$ a graph that is not the disjoint union of at most two complete subgraphs. If $A \ast B$ is not $d_1$-choosable, then $A \ast B$ has a minimal bad $d_1$-assignment $L$ such that $|\text{Pot}(L)| \leq |B| + 1$.

**Proof.** Suppose $A \ast B$ is not $d_1$-choosable and let $L$ be a minimal bad $d_1$-assignment on $A \ast B$. Then, by the Small Pot Lemma, $|\text{Pot}(L)| \leq |B| + 2$.

Let $I$ be a maximum independent set in $B$. Since $B$ is not the disjoint union of at most two complete subgraphs, Lemma 4.16 implies that either $|E(I)| > |B| - |I|$ or $|I| \geq 3$. In the first case, $2|I| + |E(I)| > 2|I| + |B| - |I| \geq 2 + |B| \geq |\text{Pot}(L)|$. In the second case, $2|I| + |E(I)| \geq 2|I| + |B| - |I| \geq 3 + |B| > |\text{Pot}(L)|$.

Thus by Lemma 4.15 $B$ can be colored from $L$ using at most $|B| - 1$ colors. But then we are done by Lemma 4.13.

**Lemma 4.19.** Let $B$ be a graph containing an induced claw, $C_4$, $K_5^-$, $P_5$, bull, or $2P_3$. If $K_2 \ast B$ is not $d_1$-choosable, then $K_2 \ast B$ has a minimal bad $d_1$-assignment $L$ such that $|\text{Pot}(L)| \leq |B|$.

**Proof.** Suppose $K_2 \ast B$ is not $d_1$-choosable and let $L$ be a minimal bad $d_1$-assignment on $K_2 \ast B$. Then, by the Small Pot Lemma, $|\text{Pot}(L)| \leq |B| + 1$.

Let $H$ be an induced claw, $C_4$, $K_5^-$, $P_5$, bull or $2P_3$ in $B$ and $M$ a maximum independent set in $H$. Expand $M$ to a maximal independent set $I$ in $B$. We can easily verify that in each case $|E_H(M)| \geq |H| - |M| + 2$, which implies that $|E_B(I)| \geq |B| - |I| + 2$. Hence we have $(|K_2| - 1)|I| + |E_B(I)| \geq (|K_2| - 2)|I| + |B| + 2 = |B| + 2 > |\text{Pot}(L)|$. Now by Lemma 4.15 $B$ can be colored from $L$ using at most $|B| - 1$ colors. But then we are done by Lemma 4.13.

In the case that $A = K_1$, we might not be able to finish an arbitrary precoloring of $B$ from $L$ to all of $B$ as we did above. However, if there is a precoloring that has our desired properties, then there is a coloring of $B$ from the lists maintaining these properties. The following lemma makes this precise.

**Lemma 4.20.** Let $A$ and $B$ be graphs such that $G := A \ast B$ is $d_0$-choosable, but is not $d_1$-choosable; let $L$ be a bad $d_1$-assignment on $G$. Then

1. for any independent set $I \subseteq V(B)$ with $|I| = 3$, we have $\bigcap_{v \in I} L(v) = \emptyset$;

2. for disjoint nonadjacent pairs $\{x_1, y_1\}$ and $\{x_2, y_2\}$ at least one of the following holds

   (a) $L(x_1) \cap L(y_1) = \emptyset$;

   (b) $L(x_2) \cap L(y_2) = \emptyset$;

   (c) $|L(x_1) \cap L(y_1)| = 1$ and $L(x_1) \cap L(y_1) = L(x_2) \cap L(y_2)$.
Proof. For both (1) and (2) we prove the contrapositive.

(1) Suppose that $B$ has an independent set $I$ of size 3 such that there exists a color $c$ that appears in the list of each vertex in $I$; let $I = \{v_1, v_2, v_3\}$. Since $G$ is $d_0$-choosable, $G$ has an $L$-coloring. We will modify this coloring to get an $L$-coloring that uses $c$ on at least three vertices.

For each $v_i$ in $I$, if $v_i$ does not have a neighbor with color $c$, we recolor $v_i$ with $c$. If $c$ now appears three or more times in our current coloring, then we are done. Assume that $c$ appears on either a single vertex $w_1$ or on two vertices $w_1$ and $w_2$.

If both $w_1$ and $w_2$ have two neighbors in $I$, then we uncolor $w_1$ and $w_2$ and use color $c$ on all vertices of $I$. Otherwise, there exists a single vertex, say $w_1$, with at least two neighbors in $I$ for which $w_1$ is their only neighbor with color $c_1$. Uncolor $w_1$ and now use color $c$ on all of its neighbors in $I_1$ that no longer have a neighbor with color $c_1$. Since each uncolored vertex has at least two neighbors with color $c$, we can extend the coloring to all of $B$. Now since color $c$ is used 3 or more times on $B$, at most $|G| - 2$ colors are used on $G$, so we can extend the coloring to $A$.

(2) Suppose that $B$ has two disjoint independent sets $I_1$ and $I_2$ each of size 2 and there exist distinct colors $c_1$ and $c_2$ such that (for each $i \in \{1, 2\}$) color $c_i$ appears in the lists of both vertices of $I_i$. Since $G$ is $d_0$-choosable, $B$ has an $L$-coloring. We will show that $B$ has an $L$-coloring in which colors $c_1$ and $c_2$ each appear twice (or one appears at least three times). We will modify our coloring using recoloring arguments similar to that above, although we may need to recolor repeatedly. (If at any point our coloring of $B$ uses a single color three or more times, then we can stop, since we will be able to extend this coloring to $A$.)

If $c_1$ does not appear in our coloring, then we recolor some vertex of $I_1$ with $c_1$. Suppose that color $c_1$ appears only once in our coloring, say on vertex $u$. Either we can recolor some vertex in $I_1$ with $c_1$ or else both vertices in $I_1$ are adjacent to $u$. In this case, we uncolor $u$ and use $c_1$ on both vertices of $I_1$. Now we have some color available for $u$. Thus, we may assume that our coloring uses $c_1$ on exactly two vertices. If neither of these vertices with $c_1$ are in $I_2$, then we can use the same recoloring trick for color $c_2$. Neither vertex with $c_1$ will get recolored, so afterwards both colors $c_1$ and $c_2$ will appear on two vertices (and we’ll be able to extend the coloring to $A$).

Suppose instead that both vertices with color $c_1$ are in $I_2$. If neither vertex in $I_2$ is adjacent to a vertex where color $c_2$ is used, then we can recolor both of them with $c_2$. Next we can again apply the recoloring trick for color $c_1$. Since the vertices in $I_2$ with color $c_2$ will not get recolored, this will yield the desired coloring that uses each of $c_1$ and $c_2$ twice. So suppose that $c_1$ is used on both vertices in $I_2$ and $c_2$ is used on a vertex adjacent to at least one vertex in $I_2$. Since we may assume that $c_2$ appears on only one vertex, when we use the recoloring trick for $c_2$, we will color at least one vertex of $I_2$ with $c_2$. Thus, we may assume (up to symmetry of $I_1$ and $I_2$) that color $c_1$ appears on two vertices and that exactly one of them is in $I_2$; we may also assume that color $c_2$ appears on exactly one vertex.

We will show that after applying the recoloring trick at most three times we will get a coloring of $B$ that uses $c_1$ on two vertices and uses $c_2$ on two vertices. We call a vertex $v \in I_i$ miscolored if it is colored with color $c_{3-i}$. We will see that each time we apply the recoloring trick, either we increase the total number of vertices colored with $c_1$ and $c_2$ or else we decrease the number of miscolored vertices. Since we begin with at most two miscolored vertices, after applying the recoloring trick at most three times, our coloring will use colors
Lemma 4.23. Let $d$ be not 1.

Definition 8. Each of $d$ from Lemma 4.25 and Corollary 4.30. The remainder of the section considers the case when $|d| \geq 2$. When $|d| = 0$, we conclude that we’ve reduced the number of miscolored vertices. We now apply the recoloring trick for $d$. Again, we are done unless we’ve recolored a miscolored vertex. So assume that we did. Since we have no remaining miscolored vertices, when we now apply the recoloring trick for $d$, we get a coloring that uses each of $c_1$ and $c_2$ twice. Thus, we can extend the coloring of $B$ to $A$.

A simple variation of the (1) case in the above together with Lemma 4.13 gives the following pot-shrinking lemma for $K_1 \ast H$.

Lemma 4.21. Let $H$ be a $d_0$-choosable graph such that $G := K_1 \ast H$ is not $d_1$-choosable and $L$ a minimal bad $d_1$-assignment on $G$. If some nonadjacent pair in $H$ have intersecting lists, then $|\Pot(L)| \leq |H| - 1$.

Lemma 4.22. Let $A$ be a connected graph, let $G = A \ast B$, and suppose that either $B$ is $d_0$-choosable or $|A| \geq 2$. (1) Let $L$ be a $d_1$-assignment to $G$. If $B$ contains disjoint independent sets $I_1$ and $I_2$ such that $\sum_{v \in I_1} (d(v) - 1) \geq |\Pot(L)| + 1$ and $\sum_{v \in I_2} (d(v) - 1) \geq |\Pot(L)| + 2$, then $A \ast B$ has an $L$-coloring. (2) In particular, if $B$ contains disjoint independent sets $I_1$ and $I_2$ such that $\sum_{v \in I_1} (d(v) - 1) \geq |G| - 1$ and $\sum_{v \in I_2} (d(v) - 1) \geq |G|$, then $A \ast B$ is $d_1$-choosable.

Proof. Let $L$ be a bad $d_1$-list assignment. We prove (1) and (2) simultaneously. By the Small Pot Lemma, $|\Pot(L)| < |G|$. Thus, since $\sum_{v \in I_2} (d(v) - 1) > |\Pot(L)|$, we see that some color $\alpha$ appears on nonadjacent vertices in $I_2$. Either $B$ is $d_0$-choosable or $|A| \geq 2$, so using either Lemma 4.21 or Lemma 4.13 we get that $|\Pot(L)| = |G| - 2$, so $|G| - 1 \geq |\Pot(L)| + 1$.

Since $\sum_{v \in I_1} (d(v) - 1) \geq |\Pot(L)| + 1$, we see that two vertices of $I_1$ have a common color $\beta$. If $\beta$ appears 3 times in $I_2$, then we are done by Lemma 4.20. Otherwise, we use $\beta$ on the vertices of $I_1$ where it appears. After deleting $\beta$ from the lists of $I_2$, we can find a common color on two vertices of $I_2$. Again we are done, by Lemma 4.20.

4.4.2 A classification

In this section we classify the $d_1$-choosable graphs of the form $A \ast B$ where $|A| \geq 2$ and $|B| \geq 2$. When $|A| \geq 4$ and $A$ is connected (or similarly for $B$), the characterizations follows from Lemma 4.25 and Corollary 4.30. The remainder of the section considers the case when each of $A$ and $B$ is small and/or disconnected.

Definition 8. A graph $G$ is almost complete if $\omega(G) \geq |G| - 1$.

Lemma 4.23. Let $A$ be a connected graph with $|A| \geq 4$ and $B$ an arbitrary graph. If $A \ast B$ is not $d_1$-choosable, then $B$ is $E_3 \ast K_{|B|-3}$ or almost complete.
Proof. Suppose $A \ast B$ is not $d_1$-choosable and $B$ is neither $E_3 \ast K_{|B|-3}$ nor almost complete. Then, by Lemma 4.14 we have $\omega(B) = |B| - 2$.

Let $L$ be a minimal bad $d_1$-assignment on $A \ast B$. Then, by Lemma 4.17 $|\text{Pot}(L)| \leq |A| + |B| - 2$. Choose $x_1, x_2 \in V(B)$ so that $B - \{x_1, x_2\}$ is complete. Since $B$ is not $E_3 \ast K_{|B|-3}$ we have $x_1', x_2' \in V(B)$ such that $\{x_1, x_1'\}$ and $\{x_2, x_2'\}$ are disjoint pairs of nonadjacent vertices. We have $|L(x_1)| + |L(x_1')| \geq d(x_1) + d(x_1') - 2 \geq 2|A| + d_B(x_1) + |B| - 5$.

First suppose $d_B(x_1) > 0$ for some $i \in \{1, 2\}$. Without loss of generality, suppose $i = 1$. Then $|L(x_1)| + |L(x_1')| \geq |\text{Pot}(L)| + 2$ and $|L(x_2)| + |L(x_2')| \geq |\text{Pot}(L)| + 1$. Hence we have different colors $c_1, c_2$ such that $c_1 \in L(x_1) \cap L(x_1')$ and $c_2 \in L(x_2) \cap L(x_2')$. Coloring the pairs with these colors leaves a list assignment which is easily completable to all of $A \ast B$.

Hence we must have $d_B(x_1) = d_B(x_2) = 0$. But then $|L(x_1)| + |L(x_1')| \geq |\text{Pot}(L)| + 1$ for each $i \in \{1, 2\}$ and thus both $L(x_1) \cap L(x_1')$ and $L(x_2) \cap L(x_2')$ are nonempty. If they have different colors in common, we can finish as above. If they have the same color $c$ in common, then coloring $x_1, x_2$ and $x_1'$ with $c$ leaves a list assignment which is easily completable to all of $A \ast B$. \qed

Lemma 4.24. Let $A$ be a connected graph with $|A| \geq 6$ and $B$ an arbitrary graph. If $A \ast B$ is not $d_1$-choosable, then $B$ is almost complete.

Proof. Suppose $A \ast B$ is not $d_1$-choosable. By Lemma 4.23, $B$ is $E_3 \ast K_{|B|-3}$ or almost complete. Suppose that $B$ is $E_3 \ast K_{|B|-3}$ and let $x_1, x_2, x_3$ be the vertices in the $E_3$.

Let $L$ be a minimal bad $d_1$-assignment on $A \ast B$. Then, by Lemma 4.17 $|\text{Pot}(L)| \leq |A| + |B| - 2$. We have $\sum_{i=1}^{3} |L(x_i)| \geq \sum_{i=1}^{3} (d(x_i) - 1) = 3(|A| + |B| - 4)$. Since $|B| \geq 3$ we have $|A| + |B| \geq 9$ and hence $3(|A| + |B| - 4) > 2(|A| + |B| - 2) \geq 2|\text{Pot}(L)|$. Thus, by Lemma 4.7 we have $c \in \bigcap_{i=1}^{3} L(x_i)$. Coloring $x_1, x_2$ and $x_3$ with $c$ leaves a list assignment which is easily completable to the rest of $A \ast B$. This is a contradiction. Hence $B$ is almost complete. \qed

When $A$ is incomplete we can do much better.

Lemma 4.25. Let $A$ be a connected incomplete graph with $|A| \geq 4$ and $B$ an arbitrary graph. If $A \ast B$ is not $d_1$-choosable, then $B$ is complete.

Proof. By Lemma 4.8 it will suffice to show that $A \ast E_2$ is $d_1$-choosable. Suppose not and let $L$ be a minimal bad $d_1$-assignment on $A \ast E_2$. Then, by Lemma 4.17 $|\text{Pot}(L)| \leq |A|$. Let $x_1$ and $x_2$ be the vertices in the $E_2$. Then $|L(x_1)| + |L(x_2)| \geq d(x_1) + d(x_2) - 2 = 2|A| - 2 \geq |\text{Pot}(L)| + 2$. Hence we have different $c_1, c_2 \in L(x_1) \cap L(x_2)$.

First, suppose there exists $y \in V(A)$ such that $\{c_1, c_2\} \not\subseteq L(y)$. Without loss of generality, assume $c_1 \not\in L(y)$. Then coloring $x_1$ and $x_2$ with $c_1$ leaves a list assignment $L'$ on $A$ where $|L'(v)| \geq d_A(v)$ for all $v \in V(A)$ and $|L'(y)| > d_A(y)$. Hence the coloring can be completed, a contradiction.

Hence $\{c_1, c_2\} \subseteq L(v)$ for all $v \in V(A)$. If $\alpha(A) \geq 3$, then coloring a maximum independent set all with $c_1$ leaves an easily completable list assignment. Also, if $A$ contains two disjoint pairs of nonadjacent vertices, by coloring one with $c_1$ and one with $c_2$ we get another easily completable list assignment. Hence $A$ is almost complete.

Let $z \in V(A)$ such that $A - z$ is complete. Since $A$ is incomplete, we have $w \in V(A - z)$ nonadjacent to $z$. Also, as $A$ is connected we have $w' \in V(A - z)$ adjacent to $z$. Color
Lemma 4.29. \( K_t \) is \( d_1 \)-choosable if and only if \( t \geq 6 \).

Proof. If \( t \geq 6 \), then \( K_t \) is \( d_1 \)-choosable. To see this, let \( A = \{ 1, 2, \ldots, t \} \) and consider the list assignment on \( V(K_t) \) given by \( L(v) = [d(v) - 1] \) for each \( v \in V(K_t) \). Color each vertex \( v \) with \( \text{number of neighbors of } v \) in \( L(v) \). Then \( K_t \) is \( d_1 \)-choosable.

For the other direction, let \( A = \{ 1, 2, \ldots, t \} \) and consider the list assignment on \( V(K_t) \) given by \( L(v) = [d(v) - 1] \) for each \( v \in V(K_t) \). Color each vertex \( v \) with \( \text{number of neighbors of } v \) in \( L(v) \). Then \( K_t \) is \( d_1 \)-choosable.

Remark. Let \( A = \{ 1, 2, \ldots, t \} \) and consider the list assignment on \( V(K_t) \) given by \( L(v) = [d(v) - 1] \) for each \( v \in V(K_t) \). Color each vertex \( v \) with \( \text{number of neighbors of } v \) in \( L(v) \). Then \( K_t \) is \( d_1 \)-choosable.
Figure 5: A bad $d_1$-assignment on $K_5 \ast E_3$.

**Corollary 4.30.** For $t \geq 4$, $K_t \ast B$ is not $d_1$-choosable iff $B$ is almost complete; or $t = 4$ and $B$ is $E_3$ or a claw; or $t = 5$ and $B$ is $E_3$.

**Lemma 4.31.** $P_3 \ast B$ is not $d_1$-choosable iff $B$ is $E_2$ or complete.

**Proof.** Moving the center of $P_3$ to the other side of the join and applying Lemma 4.27 proves the lemma.

**Lemma 4.32.** $K_3 \ast P_4$ is $d_1$-choosable.

**Proof.** Suppose otherwise. Denote the vertices of the $P_4$ as $y_1, y_2, y_3, y_4$, in order. Note that $|L(y_1)| + |L(y_3)| = 4 + 5 \geq |G| + 1$ and $|L(y_2)| + |L(y_4)| = 5 + 4 \geq |G| + 1$. Now we apply (2) of Lemma 4.22 with $I_1 = \{y_1, y_3\}$ and $I_2 = \{y_2, y_4\}$.

Figure 6: The antipaw.

**Lemma 4.33.** $K_3 \ast \text{antipaw}$ is $d_1$-choosable.

**Proof.** Suppose not. We use the labeling of the antipaw given in Figure 6. Since the antipaw is not a disjoint union of at most two complete graphs, Lemma 4.18 gives us a minimal bad $d_1$-assignment $L$ on $K_3 \ast \text{antipaw}$ with $|Pot(L)| \leq 5$. Note that $|L(y_1)| + |L(y_4)| \geq 6$ and $|L(y_2)| + |L(y_3)| \geq 6$. Hence, by Lemma 4.20 $|L(y_1) \cap L(y_4)| = 1$ and $L(y_1) \cap L(y_4) = L(y_2) \cap L(y_3)$. But then we have $c \in L(y_2) \cap L(y_3) \cap L(y_4)$ and after coloring $y_2, y_3$, and $y_4$ with $c$ we can complete the coloring, getting a contradiction.

28
Lemma 4.34. $K_3 \ast B$ is not $d_1$-choosable iff $B$ is almost complete, $K_t + K_{|B|-t}$, $K_1 + K_t + K_{|B|-t-1}$, $E_3 + K_{|B|-3}$, or $|B| \leq 5$ and $B = E_3 \ast K_{|B|-3}$.

Proof. Let $K_3 \ast B$ be a graph that is not $d_1$-choosable and let $B$ be none of the specified graphs. Lemma 4.18 gives us a minimal bad $d_1$-assignment $L$ on $K_3 \ast B$ with $|Pot(L)| \leq |B| + 1$. Furthermore, the proof of Lemma 4.18 shows that we can color $B$ with at most $|B| - 1$ colors. In particular we have nonadjacent $x, y \in V(B)$ and $c \in L(x) \cap L(y)$. Coloring $x$ and $y$ with $c$ leaves a list assignment $L'$ on $D := B - \{x, y\}$. If $c \in L'(z)$ for some $z \in V(D)$, then $\{x, y, z\}$ is independent and we can color $z$ with $c$ and complete the coloring to get a contradiction. Hence $Pot(L') = Pot(L) - \{c\}$.

Suppose, for a contradiction, that $D$ is not the disjoint union of at most two complete subgraphs. If $\alpha(D) \geq 3$, let $J$ be a maximum independent set in $D$ and set $\gamma := 0$. Otherwise $D$ contains an induced $P_3$ $abc$ and we let $J \subseteq V(D)$ be a maximal independent set containing $\{a, c\}$ and set $\gamma := 1$. Lemma 4.16 implies that $\sum_{v \in J} d_D(v) \geq |D| - |J| + \gamma$. Since $L$ is bad, we must have

\[
\sum_{v \in J} |L'(v)| \leq |Pot(L')| \leq |Pot(L)| \leq |B| + 1 \leq |B| + |B| - 1 = 2|J| + |\sum_{v \in J} d_D(v)| \leq |B|
\]

\[
2|J| + \sum_{v \in J} d_D(v) \leq |B|
\]

\[
2|J| + |D| - |J| + \gamma \leq |B|
\]

\[
|J| + |D| + \gamma \leq |B|
\]

\[
|J| + |B| - 2 + \gamma \leq |B|.
\]

Hence $|J| \leq 2 - \gamma$, a contradiction. Therefore $D$ is indeed the disjoint union of at most two complete subgraphs. (Additionally, if $D$ is not complete then $v \in V(D)$ is not adjacent to both $x$ and $y$ since then we would get the same contradictory degree sum as in the case when $\gamma = 1$.) We now consider the case that $D$ is a complete graph and the case that $D$ is the disjoint union of two complete graphs.

First, suppose $D$ is a complete graph. Plainly, $|D| \geq 2$. Put $X := N(x) \cap V(D)$ and $Y := N(y) \cap V(D)$. Suppose $X - Y \neq \emptyset$ and pick $z \in X - Y$. We have $|L(y)| + |L(z)| \geq d(y) + d(z) - 2 = d_B(y) + d_B(z) + 4 \geq 0 + |B| - 2 + 4 = |B| + 2 > |Pot(L)|$. By repeating the argument given above for $B - \{x, y\}$, we see that $B - \{y, z\}$ is also the disjoint union of at most two complete subgraphs. In particular, $x$ is adjacent to all or none of $D - z$. If all, then $B$ is almost complete, if none then $B$ contains an induced $P_3$ or antipaw, and both possibilities give contradictions by Lemmas 4.32 and 4.33. Hence $X - Y = \emptyset$. Similarly, $Y - X = \emptyset$, so $X = Y$. Since $B$ is not $E_2 + K_{|B|-2}$, $|X| > 0$. If $X = V(D)$, then $B$ is almost complete. If $|V(D) - X| \geq 2$, then pick $w_1, w_2 \in V(D) - X$. Now by considering degrees, we see that $L(x) \cap L(w_1)$ and $L(y) \cap L(w_2)$ are both nonempty. Now we can color $x, y, w_1, w_2$ using only 2 colors, and then complete the coloring. Hence, we must have $|V(D) - X| = 1$, so let $\{w\} = V(D) - X$. Now $x$ and $y$ are joined to $D - w$ and hence $B$ is $E_3 \ast K_{|B|-3}$, a contradiction.
Thus $D$ must instead be the disjoint union of two complete subgraphs $D_1$ and $D_2$. For each $i \in [2]$, put $X_i := N(x) \cap V(D_i)$ and $Y_i := N(y) \cap V(D_i)$. From our parenthetical remark above, we know that $X_i \cap Y_i = \emptyset$. Suppose we have $z_1 \in V(D_1)$ and $z_2 \in V(D_2)$ such that $L(z_1) \cap L(z_2) \neq \emptyset$. Then, by Lemma 4.20, $L(z_1) \cap L(z_2) = L(x) \cap L(y)$. Since no independent set of size three can have a color in common, the edges $z_1x$ and $z_2y$ or $z_1y$ and $z_2x$ must be present. Using the same argument as for $B \setminus \{x,y\}$, we see that $B \setminus \{z_1,z_2\}$ is the disjoint union of at most two complete subgraphs. So each of $x$ and $y$ is adjacent to all or none of each of $V(D_1 - z_1)$ and $V(D_2 - z_2)$. Thus, by symmetry, we may assume that $V(D_1 - z_1) \subseteq X_1$ and $V(D_2 - z_2) \subseteq Y_2$. If $|D_1| = |D_2| = 1$, then $B$ is the disjoint union of two cliques, a contradiction. Hence we may assume that $|D_1| \geq 2$.

Pick $w \in V(D_1 - z_1)$. If $x$ is not adjacent to $z_1$, then $xwz_1$ is an induced $P_3$ in $B$. Since $X_1 \cap Y_1 = \emptyset$, this $P_3$ together with $y$ either induces a $P_4$ or an antipaw, contradicting Lemmas 4.32 and 4.33. Hence $X_1 = V(D_1)$. Similarly, if $|D_2| \geq 2$, then $Y_2 = V(D_2)$ and $B$ is the disjoint union of two complete subgraphs, a contradiction. Hence $D_2 = \{z_2\}$. But $z_2$ must be adjacent to $y$, so $B$ is again the disjoint union of two cliques, a contradiction.

Thus for every $z_1 \in V(D_1)$ and $z_2 \in V(D_2)$ we have $L(z_1) \cap L(z_2) = \emptyset$. Suppose there exist $z_1 \in V(D_1)$ and $z_2 \in V(D_2)$ such that $z_1$ and $z_2$ are each adjacent to at least one of $x$ and $y$. Then $|L(z_1)| + |L(z_2)| \geq d(z_1) + d(z_2) \geq 2d_B(z_1) + d_B(z_2) + 4 \geq |B| - 4 + 2 + 4 = |B| + 2 > |Pot(L)|$. Hence $L(z_1) \cap L(z_2) \neq \emptyset$, a contradiction.

Thus, by symmetry, we may assume that there are no edges between $D_1$ and $\{x,y\}$. Since no vertex in $D_2$ is adjacent to both $x$ and $y$, only one of $x$ or $y$ can have neighbors in $D_2$ lest $B$ contain an induced $P_4$ contradicting Lemma 4.32. Without loss of generality, we may assume that $y$ has no neighbors in $D_2$. Pick $w \in D_1$ and $z \in V(D_2)$.

Suppose that $|D_1| \geq 2$, $|D_2| \geq 2$, and there exists $t \in D_2$ such that $x$ and $t$ are nonadjacent. Now choose $u,v \in V(D_1)$ and $w \in V(D_2) \setminus \{t\}$. Now $\{v,w,y\}$ is independent and $|L(v)| + |L(w)| + |L(y)| \geq d(v) + d(w) + d(y) - 3 \geq d_B(v) + d_B(w) + d_B(y) + 6 \geq |B| \geq 2 > |Pot(L)|$. Hence either $L(v) \cap L(y) \neq \emptyset$ or $L(w) \cap L(y) \neq \emptyset$. Similarly, either $L(u) \cap L(x) \neq \emptyset$ or $L(t) \cap L(x) \neq \emptyset$. Thus, we can color 4 vertices using only 2 colors, and we can complete the coloring. So now either $|D_1| = 1$, $|D_2| = 1$, or $D_2 \subseteq N(x)$.

If $|D_2| = 1$, then either $B = K_1 + K_2 + K_{|B|-3}$ or else $B = E_3 + K_{|B|-3}$, both of which are forbidden. Similarly, if $|D_1| = 1$ and $x$ is adjacent to all or none of $D_2$, then $B = K_1 + K_1 + K_{|B|-2}$ or $E_3 + K_{|B|-3}$. Finally, if $x$ is adjacent to some, but not all of $D_2$, then $B$ contains an antipaw. By Lemma 4.33, this is a contradiction.

It remains to show that $K_3 * B$ is not $d_1$-choosable for any of the specified $B$’s. For $B$ almost complete, this follows from Lemma 4.28 and for $E_3 * K_{|B|-3}$, from Lemma 4.30. For all the rest of the options we will give a bad list assignment with lists $[|B| + 1]$ on the $K_3$. Suppose $K_1 + K_{|B|-t}$. On the $K_1$ the lists $[t + 1]$ and on the $K_{|B|-t}$ the lists $[|B| + 1] \setminus [t]$. Then any coloring of $K_3 * B$ from the lists must use three colors on the $K_3$ and hence at least one of the cliques loses at least two colors leaving it uncolorable. Now suppose $B = K_1 + K_t + K_{|B|-t-1}$. Use the list $\{1, |B| + 1\}$ on the $K_1$, the lists $[t + 1]$ on the $K_t$ and the lists $[|B| + 1] \setminus [t + 1]$ on the $K_{|B|-t-1}$. This list assignment is clearly bad on $K_3 * B$. Finally suppose $B = E_3 + K_{|B|-3}$. Give the three $K_1$’s the lists $\{1,2\}$, $\{1,3\}$, $\{2,3\}$ and the $K_{|B|-3}$ the list $[|B| + 1] \setminus [3]$. Again, this is clearly a bad list assignment on $K_3 * B$.

\[\square\]

Lemma 4.35. $K_2 * P_5$ is $d_1$-choosable.
**Proof.** Suppose otherwise. By Lemma 4.19, we have a minimal bad $d_1$-assignment $L$ on $P_5 \ast K_2$ with $\left| Pot(L) \right| \leq 5$. Let $y_1, y_2, y_3, y_4, y_5$ denote the vertices of the $P_5$ in order. Now $|L(y_2)| + |L(y_4)| \geq 6 \geq |Pot(L)| + 1$ and $|L(y_1)| + |L(y_3)| + |L(y_5)| \geq 7 \geq |Pot(L)| + 2$. So $\{y_2, y_4\}$ and $\{y_1, y_3, y_5\}$ satisfy the hypotheses of Lemma 4.22 giving a contradiction.

\[ \text{Figure 7: Labelings of the chair and the antichair.} \]

**Lemma 4.36.** $K_2 \ast \text{chair}$ is $d_1$-choosable.

**Proof.** Suppose otherwise. We use the labeling of the chair given in Figure 7a. Since the chair has an induced claw, Lemma 4.19 gives us a minimal bad $d_1$-assignment $L$ on $K_2 \ast \text{chair}$ with $\left| Pot(L) \right| \leq 5$. Now $|L(y_2)| + |L(y_5)| \geq 6 \geq |Pot(L)| + 1$ and $|L(y_1)| + |L(y_3)| + |L(y_4)| \geq 7 \geq |Pot(L)| + 2$. Then $\{y_2, y_5\}$ and $\{y_1, y_3, y_4\}$ satisfy the hypotheses of Lemma 4.22 giving a contradiction.

**Lemma 4.37.** $K_2 \ast \text{antichair}$ is $d_1$-choosable.

**Proof.** Suppose otherwise. We use the labeling of the antichair given in Figure 7b. Since the antichair has an induced $K_4^-$, Lemma 4.19 gives us a minimal bad $d_1$-assignment $L$ on $K_2 \ast \text{antichair}$ with $\left| Pot(L) \right| \leq 5$. We have $|L(y_2)| + |L(y_5)| \geq 7$ and hence $|L(y_2) \cap L(y_5)| \geq 2$. But then, by Lemma 4.20, we have the contradiction $|L(y_1)| + |L(y_3)| \leq 5$.

**Lemma 4.38.** $K_2 \ast C_5$ is $d_1$-choosable.

**Proof.** Suppose otherwise. By the Small Pot Lemma, we have a minimal bad $d_1$-assignment $L$ on $C_5 \ast K_2$ with $\left| Pot(L) \right| \leq 6$. Let $y_0, y_1, y_2, y_3, y_4$ denote in order the vertices of the $C_5$. Then for $0 \leq i < j \leq 4$ with $i - j \neq 1 \text{ (mod 5)}$ we have $|L(y_i)| + |L(y_j)| \geq d(y_i) + d(y_j) - 2 = 6$.

First suppose $\left| Pot(L) \right| \leq 5$. Then each nonadjacent pair has a color in common and by applying Lemma 4.20 multiple times we see that there must exist $c \in \bigcap_{0 \leq i \leq 4} L(y_i)$ and no nonadjacent pair can have a color other than $c$ in common. Put $S_1 = L(y_i) \setminus \{c\}$ and $T = Pot(L) \setminus \{c\}$. Then we must have $S_0 = T - S_3$, $S_1 = T - S_3 = T - S_4$ and $S_2 = T - S_4$. Hence $S_0 = S_1 = S_2$ contradicting $S_0 \cap S_2 = \emptyset$. Therefore we must have $\left| Pot(L) \right| = 6$. Thus for nonadjacent $y_i$ and $y_j$, $L(y_i) = Pot(L) - L(y_j)$. We have $L(y_0) = Pot(L) - L(y_3)$, $L(y_1) = Pot(L) - L(y_3)$, $L(y_2) = Pot(L) - L(y_4)$ and $L(y_3) = Pot(L) - L(y_4)$. Hence $L(y_0) = L(y_1) = L(y_2)$. Thus we may color $y_0$ and $y_2$ the same and complete this coloring to the rest of $B$ contradicting Lemma 4.13.

**Lemma 4.39.** $K_2 \ast 2P_3$ is $d_1$-choosable.
\textbf{Proof.} Suppose otherwise. Let $y_1, y_2, y_3$ and $y_4, y_5, y_6$ denote in order the vertices of the two $P_3$’s. Lemma 4.19 gives us a minimal bad $d_1$-assignment $L$ on $K_2 \ast 2P_3$ with $|\text{Pot}(L)| \leq 6$.

Since $|L(y_1)| + |L(y_3)| + |L(y_4)| + |L(y_6)| = 8 \geq |\text{Pot}(L)| + 2$, either three of these vertices share a common color, or else two pairs of them share distinct common colors. Thus, if $L(y_2) \cap L(y_5) \neq \emptyset$, then we can color $G$ by Lemma 4.20. Hence $L(y_2) \cap L(y_5) = \emptyset$.

By summing list sizes, we see that some pair among each of $\{y_1, y_3, y_5\}$ and $\{y_2, y_4, y_6\}$ must have a color in common. Since there are no edges between $\{y_1, y_3\}$ and $\{y_4, y_6\}$, if $L(y_1) \cap L(y_3) \neq \emptyset$ and $L(y_4) \cap L(y_6) \neq \emptyset$, then we get a contradiction. By symmetry, we may assume that the other two options are either $L(y_1) \cap L(y_3) \neq \emptyset$ and $L(y_2) \cap L(y_4) \neq \emptyset$ or else $L(y_1) \cap L(y_5) \neq \emptyset$ and $L(y_2) \cap L(y_4) \neq \emptyset$. In the former case, by Lemma 4.20 we must have $L(y_1) \cap L(y_3) \cap L(y_4) \neq \emptyset$, a contradiction. In the latter case, $L(y_1) \cap L(y_5) \neq L(y_2) \cap L(y_4)$ since $L(y_2) \cap L(y_5) = \emptyset$, contradicting Lemma 4.20.

\[\text{Note that if } L \text{ is a bad } d_1 \text{ assignment on } E_3 \ast B \text{ where the } E_3 \text{ is } \{x_1, x_2, x_3\}, \text{ then } L(x_1) \cap L(x_2) \cap L(x_3) = \emptyset.\]

\textbf{Lemma 4.40.} $E_3 \ast \text{anticlaw}$ is $d_1$-choosable.

\textbf{Proof.} Suppose otherwise. The Small Pot Lemma gives us a minimal bad $d_1$-assignment $L$ on $E_3 \ast \text{anticlaw}$ with $|\text{Pot}(L)| \leq 6$. Let the $E_3$ have vertices $x_1, x_2, x_3$, and let the anticlaw have vertices $y_1, y_2, y_3, y_4$, with $y_2, y_3, y_4$ mutually adjacent. Then $\sum_i |L(x_i)| = 9$ and hence there are three colors $c_1, c_2, c_3$ such that for each $t \in [3]$, $c_t \in L(x_t) \cap L(x_j)$ for some $1 \leq i < j \leq 3$.

Suppose there exists $i \in \{2, 3, 4\}$, say $i = 2$, such that $y_1$ and $y_2$ have a common color $c$. We use $c$ on $y_1$ and $y_2$, and let $L'(v) = L(v) - c$ for each uncolored $v$; note that $c$ must be
absent from some $x_i$, say $x_1$. Now since $|L'(x_2)| + |L'(x_3)| \geq 4$, we can color $x_2$ and $x_3$ such that at least two colors remain available on $y_3$. Finally, we greedily color $y_4, y_3, x_3$.

Otherwise, since $|\text{Pot}(L)| \leq 6$, we may assume that $L(y_1) = \{a, b\}$ and $L(y_2) = L(y_3) = L(y_4) = \{c, d, e, f\}$. Now we can color $x_1, x_2, x_3$ using only two colors, exactly one of which is in $\{a, b\}$. Finally, we greedily color $y_1, y_2, y_3, y_4$.

**Lemma 4.41.** $E_3 * 2K_2$ is $d_1$-choosable.

**Proof.** Suppose otherwise. The Small Pot Lemma gives us a minimal bad $d_1$-assignment $L$ on $E_3 * 2K_2$ with $|\text{Pot}(L)| \leq 6$. Let the $E_3$ have vertices $x_1, x_2, x_3$, and let the $2K_2$ have vertices $y_1$ adjacent to $y_2$ and $y_3$ adjacent to $y_4$. Then $\sum_i |L(x_i)| = 9$ and hence there are three colors $c_1, c_2, c_3$ such that for each $t \in [3], c_t \in L(x_i) \cap L(x_j)$ for some $1 \leq i < j \leq 3$. If all three $c_t$ appear on all four $y_i$, then we can 2-color the $2K_2$, and extend the coloring to the $E_3$. So we may assume instead without loss of generality that $c_1$ appears on $x_1$ and $x_2$, but not $y_1$. Now use $c_1$ on $x_1$ and $x_2$, then color greedily in the order $y_3, y_4, x_3, y_2, y_1$.

**Lemma 4.42.** $E_3 * E_4$ is $d_1$-choosable.

**Proof.** Suppose otherwise. Let the $E_3$ have vertices $x_1, x_2, x_3$ and let the $E_4$ have vertices $y_1, y_2, y_3, y_4$. If there exists $c \in \cap^3_{i=1} L(x_i)$, then we use $c$ on all $x_i$ and we can finish the coloring, so assume not. By the Small Pot Lemma, $|\text{Pot}(L)| \leq 6$, so there exist two $y_i$, say $y_1$ and $y_2$, with a common color $c$; use $c$ on $y_1$ and $y_2$. Now there exists some $x_i, y_3, x_3, y_4$ with $c \notin L(x_i)$. The 4-cycle induced by $x_1, x_2, y_3, y_4$ is 2-choosable; then we can extend the coloring to $x_3$.

**Lemma 4.43.** $E_3$ * antidiiamond is $d_1$-choosable.

**Proof.** Suppose otherwise. The Small Pot Lemma gives us a minimal bad $d_1$-assignment $L$ on $E_3 * \text{antidiiamond}$ with $|\text{Pot}(L)| \leq 6$. Let the $E_3$ have vertices $x_1, x_2, x_3$, and let the antidiiamond have vertices $y_1, y_2, y_3, y_4$ with $y_3$ adjacent to $y_4$. We can assume that $\cap^3_{i=1} L(x_i) = \emptyset$ (since otherwise we use a common color on the $x_i$ and then greedily complete the coloring). If $y_3$ or $y_4$ has a common color $c$ with $y_1$ or $y_2$, then we can use $c$ on those two vertices and proceed as in the case of $E_3 * E_4$, so assume not. Again $\sum_i |L(x_i)| = 9$ and hence there are three colors $c_1, c_2, c_3$ such that for each $t \in [3], c_t \in L(x_i) \cap L(x_j)$ for some $1 \leq i < j \leq 3$. So assume that $c_1$ appears on $x_1$ and $x_2$, and use it there. If $c_1$ appears on neither $y_1$ or $y_2$, then we greedily color in the order $y_3, y_4, x_3, y_1, y_2$. Otherwise $c_1$ appears on neither $y_3$ or $y_4$, so we greedily color in the order $y_1, y_2, x_3, y_3, y_4$.

**Lemma 4.44.** $E_3 * B$ is not $d_1$-choosable iff $B \in \{K_1, K_2, E_2, E_3, \overline{P}_3, K_3, K_4, K_5\}$.

**Proof.** Suppose we have $B$ such that $E_3 * B$ is not $d_1$-choosable. By Lemma 4.27, $B$ is the disjoint union of complete subgraphs and at most one $P_3$. If $B$ contained a $P_3$, then moving its middle vertex to the other side of the join would violate Lemma 4.25. By Lemma 4.42, $B$ has at most three components. By Lemma 4.43, if $B$ has three components, then $B = E_3$. By Lemma 4.41 and Lemma 4.40, if $B$ has two components then $B = E_2$ or $B = \overline{P}_3$. Otherwise $B$ is complete and Lemma 4.29 shows that $|B| \leq 5$. This proves the forward implication.

For the other direction, it is easy to verify that $E_3 * B$ is not $d_1$-choosable for the listed graphs. The cases $B \in \{K_1, K_2, E_2\}$ are nearly trivial. For $B = E_3$, we are simply recalling
Lemma 4.48. Finally, suppose that $B = \overline{P}_3$. Let $x_1, x_2, x_3$ denote the vertices of the $E_3$ and let $y_1, y_2, y_3$ denote the vertices of the $F_3$, where $y_2$ and $y_3$ are adjacent. Assign the lists $L(x_1) = \{1, 2\}$, $L(x_2) = \{1, 3\}$, $L(x_3) = \{2, 3\}$, $L(y_1) = \{1, 2\}$, and $L(y_2) = L(y_3) = \{1, 2, 3\}$. To color the $\overline{P}_3$, we clearly use at least two colors, but now some vertex of the $E_3$ has no remaining colors.

**Lemma 4.45.** $\overline{P}_3 \ast 2K_2$ is $d_1$-choosable.

**Proof.** Let $x_1, x_2, x_3$ be the vertices of $\overline{P}_3$, with $x_2$ adjacent to $x_3$, and let $y_1, y_2, y_3, y_4$ be the vertices of $2K_2$, with $y_1$ adjacent to $y_2$ and $y_3$ adjacent to $y_4$. By the Small Pot Lemma, $|Pot(L)| \leq 6$, so $x_2$ and $x_3$ have a common color $c_1$. If $c_1$ is absent from the list of some $y_i$, say $y_1$, then we can use $c_1$ on $x_1$ and $x_2$, then greedily color in the order $y_4, y_3, x_2, y_1$. Hence $c_1$ appears on all $y_i$. If $|Pot(L)| \leq 5$, then $x_1$ and $x_2$ have a second common color $c_2$. Since $c_1$ and $c_2$ must appear on all $y_i$, we can 2-color the $2K_2$, then greedily color $x_1, x_2$, and $x_3$. So we can conclude that $L(x_1) \cap L(x_2) = c_1$ and $L(x_1) \cap L(x_3) = c_1$. Similarly, we can 2-color the $2K_2$ if $y_1$ and $y_3$ have any common color other than $c_1$.

Now we use $c_1$ on $y_2$ and $y_4$, and let $L'(v) = L(v) - c_1$ for all uncolored $v$. Now $|Pot(L)| = |Pot(L)| - 1 = 5$. Let $S = \{x_1, x_2, x_3, y_1, y_3\}$. To show that we can finish the coloring, we use Hall’s Theorem. We only need to consider subsets $T \subset S$ of size 3 or 4. If $|T| = 3$, then either $\{y_1, y_3\} \subset T$, so $| \cup_{v \in T} L'(v) | \geq |L'(y_1)| + |L'(y_3)| \geq 4$, or else $T$ contains $x_2$ or $x_3$. Since $|L'(x_2)| = |L'(x_3)| = 3$, we are done. If $|T| = 4$, then either $\{y_1, y_3\} \subset T$ or $\{x_1, x_2\} \subset T$ or $\{x_1, x_3\} \subset T$. In each case $| \cup_{v \in T} L'(v) | \geq 4$.

**Lemma 4.46.** $\overline{P}_3 \ast \text{antidiamond}$ is $d_1$-choosable.

**Proof.** Let $x_1, x_2, x_3$ be the vertices of $\overline{P}_3$, with $x_2$ adjacent to $x_3$, and let $y_1, y_2, y_3, y_4$ be the vertices of the antidiamond, with $y_3$ adjacent to $y_4$. By the Small Pot Lemma, $|Pot(L)| \leq 6$, so $x_1$ and $x_2$ have a common color $c$. If $c$ is absent from $y_4$, then we use $c$ on $x_1$ and $x_2$, then greedily color $y_1, y_2, x_3, y_3, y_4$. Similarly, if $c$ is absent from $y_1$ and $y_2$, then we use $c$ on $x_1$ and $x_2$, then greedily color $y_3, y_4, x_3, y_2, y_1$. So $c$ must appear on $y_1$ (or $y_2$) and $y_3$, and we use it there. Let $L'(v) = L(v) - c$ for all uncolored vertices. Now if there exists $c_2 \in L'(y_2) \setminus L'(x_2)$, then we can use $c_2$ on $y_2$ and greedily color $x_1, y_4, x_3, x_2$. The same argument holds if there exists $c_2 \in L'(y_4) \setminus L'(x_2)$. Thus, we must have $(L'(y_2) \cup L'(y_4)) \subseteq L'(x_2)$, so $y_2$ and $y_4$ have a common color $c_2$. We use it on them and greedily color $x_1, x_2, x_3$.

**Lemma 4.47.** $\overline{P}_3 \ast E_4$ is $d_1$-choosable.

**Proof.** Let $x_1, x_2, x_3$ be the vertices of $\overline{P}_3$, with $x_2$ adjacent to $x_3$, and let $y_1, y_2, y_3, y_4$ be the vertices of $E_4$. If three of the $y_i$’s (say $y_1, y_2$, and $y_3$) have a common color $c$, then use $c$ on them, and now greedily color in the order $y_4, x_1, x_2, x_3$. By the Small Pot Lemma, $x_1$ and $x_2$ have a common color $c$, which we use on them. Now $c$ appears on at most two $y_i$, say $y_1$ and $y_2$, so we can greedily color in the order $y_1, y_2, x_3, y_3, y_4$.

**Lemma 4.48.** $\overline{P}_3 \ast B$ is not $d_1$-choosable iff $B$ is $E_3$, $K_{|B|}$, or $K_1 + K_{|B| - 1}$.

**Proof.** Since $\overline{P}_3$ contains an $E_2$, Lemma 4.27 shows that $B$ is the disjoint union of complete subgraphs and at most one $P_3$. If $B$ contained a $P_3$, then moving its middle vertex to the other side of the join would violate Lemma 4.25. By Lemma 4.45 at most one component of
B has more than one vertex. If B has more than two components, then Lemma 4.46 shows that B is independent and thus Lemma 4.47 shows that \( B = E_3 \). If B has two components then it is \( K_1 + K_{|B| - 1} \). Otherwise B is complete. This proves the forward implication.

The reverse implication is easily checked. For \( B = E_3 \), see Lemma 4.44. If \( B = K_{|B|} \), then G is almost complete. Suppose that \( B = K_{|B| - 1} \). Now \( \Delta(G) = \omega(G) = |B| + 1 \), so G is not \( d_1 \)-choosable.

Lemma 4.49. Let A and B be graphs with \( |A| \geq 4 \) and \( |B| \geq 4 \). The graph \( A \ast B \) is not \( d_1 \)-choosable iff \( A \ast B \) is almost complete, \( K_5 \ast E_3 \), or \( (K_1 + K_{|A| - 1}) \ast (K_1 + K_{|B| - 1}) \).

Proof. Suppose A and B are graphs with \( |A| \geq |B| \geq 4 \) such that \( A \ast B \) is not \( d_1 \)-choosable and not one of the specified graphs.

First suppose A is connected. If A is complete then by Corollary 4.30 \( |A| = 4 \) and B is a claw or B is almost complete. But this implies that \( G = K_5 \ast E_3 \) or G is almost complete. Hence A is incomplete. Now Lemma 4.44 shows that B is complete. By reversing the roles of A and B in this argument, we get a contradiction; so A is disconnected. The same argument shows that B is also disconnected.

Suppose \( \alpha(A) \geq 3 \). Then Lemma 4.44 shows that B is \( K_4 \) or \( K_5 \), both impossible as above. Thus \( \alpha(A) = 2 \) and hence A is the disjoint union of two complete graphs. The same goes for B. Now Lemma 4.48 shows that \( A = K_1 + K_{|A| - 1} \) and \( B = K_1 + K_{|B| - 1} \).

The reverse implication is easily checked. If \( A \ast B \) is almost complete, then clearly it is not \( d_1 \)-choosable. For \( A \ast B = K_5 \ast E_3 \), see Figure 5. So suppose that \( A \ast B = (K_1 + K_{|A| - 1}) \ast (K_1 + K_{|B| - 1}) \). Now \( \Delta(A \ast B) = \omega(A \ast B) = |A| + |B| - 2 \), so \( A \ast B \) is not \( d_1 \)-choosable.

4.4.3 Joins with \( K_2 \)

Definition 9. The net is formed by adding one edge incident to each vertex of \( K_3 \). The bowtie is formed by identifying one vertex in each of two copies of \( K_3 \). The M is formed from the bowtie by adding an edge incident to a vertex of degree 2.

Lemma 4.50. The graph \( K_2 \ast A \) is \( d_1 \)-choosable for all \( A \in \{2P_3, C_4, C_5, P_5, \text{chair, antichair, } K_1 \ast \text{antipaw, } K_1 \ast P_4, \text{net, } M \} \)

Proof. For eight of these ten choices of A, we have already proved that \( K_2 \ast A \) is \( d_1 \)-choosable. Specifically, we have proved this for \( 2P_3 \) (Lemma 4.39), \( C_5 \) (Lemma 4.38), \( P_5 \) (Lemma 4.35), chair (Lemma 4.36), antichair (Lemma 4.37), \( K_1 \ast \text{antipaw} \) (Lemma 4.33), \( K_1 \ast P_4 \) (Lemma 4.32), and \( C_4 \) (since \( C_4 = E_2^2 \), this is the case \( r = 1 \) in Corollary 4.11). Now we consider the remaining two cases: net and M.

Let \( G = K_2 \ast \text{net} \). Let \( x_1, x_2 \) denote the vertices of the \( K_2 \), let \( y_1, y_2, y_3 \) denote the degree-3 vertices in the net, and let \( z_1, z_2, z_3 \), denote the leaves of the net, with \( z_i \) adjacent to \( y_i \). We consider three cases. (1) If there exists \( c_1 \in \cap_{i=1}^3 L(z_i) \), then we first use \( c_1 \) on all three \( z_i \) and afterwards color \( y_1, y_2, y_3, x_1, x_2 \) greedily. (2) Suppose there exist \( y_i \) and \( z_j \), with \( i \neq j \), such that there exists \( c_1 \in L(y_i) \cap L(z_j) \); by symmetry we assume this is \( y_1 \) and \( z_2 \). We use \( c_1 \) on \( y_1 \) and \( z_2 \) and let \( L'(v) = L(v) - c_1 \) for each uncolored vertex v. Now we have \( |\text{Pot}(L')| < |G \setminus \{ y_1, z_2 \}| = 6 \). Since we have \( |L'(z_2)| + |L'(y_2)| + |L'(z_3)| \geq 1 + 3 + 2 = 6 \), we must have a common color \( c_2 \) (different from \( c_1 \)) on two of \( z_1, y_2, \) and \( z_3 \). We use this color
on these two vertices, then greedily color the remaining vertices of the net before coloring $x_1$ and $x_2$. (3) Observe that if $L(z_1)$ and $L(z_2)$ are disjoint, then (since $|\text{Pot}(L)| \leq 7$) either $L(z_1) \cap L(y_3) \neq \emptyset$ or $L(z_2) \cap L(y_3) \neq \emptyset$; in each case, we are in (2). Thus, if we are not in (1) or (2) above, then (again, since $|\text{Pot}(L)| \leq 7$) by symmetry we have $L(z_1) = \{a, b\}$, $L(z_2) = \{a, c\}$, $L(z_3) = \{b, c\}$, and $L(y_1) = L(y_2) = L(y_3) = \{d, e, f, g\}$. By symmetry, either $a \notin L(x_1)$ or $d \notin L(x_1)$. Thus, we use $a$ on $z_1$ and $z_2$ and we use $d$ on $y_3$. Now we greedily color $z_3$, $y_1$, $y_2$, $x_3$, $x_1$.

Let $G = K_2 \ast M$ and let $x_1$, $x_2$ denote the vertices of the $K_2$; for the $M$, let $y_1$ denote the 1-vertex, $y_2$ the 3-vertex, $y_3$ the 2-vertex adjacent to $y_2$, $y_4$ the 4-vertex, and $y_5$ and $y_6$ the remaining 2-vertices. By the Small Pot Lemma, $|\text{Pot}(L)| \leq 7$. Since $|L(y_1)| + |L(y_3)| + |L(y_6)| = 8$, two of them must have a common color $c$. If all three of $y_1$, $y_3$, $y_6$ have $c$, then we use $c$ on all three, and afterward we color greedily $y_2$, $y_4$, $y_5$, $x_1$, $x_2$. So now we consider three cases. (1) If $c$ appears in $L(y_3) \cap L(y_6)$, then we use $c$ on $y_3$ and $y_6$, and let $L'(v) = L(v) - c$ for each uncolored vertex $v$. By the Small Pot Lemma, $|\text{Pot}(L')| \leq 5$. Since $|L'(y_1)| + |L'(y_4)| \geq 2 + 4 > 5$, we have a common color $d$ (different from $c$) on $y_1$ and $y_4$. After we use $d$ on $y_1$ and $y_4$, we color greedily $y_2$, $y_5$, $x_1$, $x_2$. (2) If $c$ appears in $L(y_1) \cap L(y_3)$, then we use $c$ on $y_1$ and $y_3$ and let $L'(v) = L(v) - c$ for each uncolored vertex $v$. Again we have $|\text{Pot}(L')| \leq 5$ and $|L'(y_2)| + |L'(y_5)| \geq 3 + 3 > 5$. After using a common color on $y_2$ and $y_5$, we greedily color $y_4$, $y_6$, $x_1$, $x_2$.

(3) Now suppose that $c$ appears in $L(y_1) \cap L(y_6)$. If $c \in L(y_2)$, then we use $c$ on $y_2$ and $y_6$, and let $L'(v) = L(v) - c$ for each uncolored vertex $v$. Again we have $|\text{Pot}(L')| \leq 5$ and $|L'(y_1)| + |L'(y_3)| + |L'(y_5)| \geq 1 + 3 + 2$ (since $c \notin L(y_3)$). So again we use a common color on two of $y_1$, $y_3$, and $y_5$, then greedily color the remaining vertices of the $M$ before coloring $x_1$ and $x_2$. Suppose instead that $c \notin L(y_2)$. Now we use $c$ on $y_1$ and $y_6$, and then use a common color on $y_4$ and $y_5$ (since $|\text{Pot}(L')| \leq 5 < 6 = 4 + 2 \leq |L'(y_2)| + |L'(y_5)|$). Finally, we greedily color $y_3$, $y_4$, $x_1$, $x_2$.

**Lemma 4.51.** The graph $K_2 \ast (B + K_1)$ is not $d_1$-choosable iff $K_2 \ast B$ is not $d_1$-choosable.

**Proof.** Suppose $K_2 \ast B$ is not $d_1$-choosable and let $L$ be a bad list assignment (not using the colors in $[t]$). To form a list assignment for $K_2 \ast (B + K_1)$, we start with $L$, then assign $[t]$ to each vertex in the $K_1$ and add $[t]$ to the lists for the vertices in the $K_2$. Clearly $K_2 \ast (B + K_1)$ has no coloring from these lists.

Conversely, suppose $K_2 \ast B$ is $d_1$-choosable. Given a list assignment for $K_2 \ast (B + K_1)$, we greedily color the $K_1$; what remains is a list assignment for $K_2 \ast B$; thus, we can finish the coloring. 

Since $K_2 \ast 2P_3$ is $d_1$-choosable (Lemma 4.39) we see that any graph $B$ such that $K_2 \ast B$ is not $d_1$-choosable must have at most one incomplete component.

**Lemma 4.52.** If $K_2 \ast B$ is not $d_1$-choosable, then $B$ consists of a disjoint union of complete subgraphs, together with at most one incomplete component $H$. If $H$ has a dominating vertex $v$, then $K_2 \ast H = K_3 \ast (H - v)$, so by Lemma 4.34 we can completely describe $H$. Otherwise $H$ is formed either by adding an edge between two disjoint cliques or by adding a single pendant edge incident to each of two distinct vertices of a clique. Furthermore, all graphs formed in this way are not $d_1$-choosable.
Proof. Let $B$ be a graph such that $K_2 \ast B$ is not $d_1$-choosable, and let $H$ be the unique incomplete component of $B$. Suppose that $H$ does not contain a dominating vertex. We first show that $H$ is a tree of edge-disjoint cliques (clique tree), i.e., every cycle has an edge between every pair of its vertices. Since $K_2 \ast C_4$, $K_2 \ast C_5$, and $K_2 \ast P_5$ are $d_1$-choosable, we get that $H$ has no induced $C_4$, $C_5$, or $P_5$; thus $H$ is chordal. So if $H$ is not a clique tree, then $H$ contains an induced copy of $K_4^*$; call it $D$.

Let $w$ denote a vertex adjacent to $D$. Each vertex adjacent to $D$ can attach to the vertices of $D$ in 8 possible ways (up to isomorphism); it can attach to 0, 1, or 2 of the vertices of degree 2, and also to 0, 1, or 2 of the vertices of degree 3 (but it must attach to at least one vertex), thus $3 \times 3 - 1 = 8$ possibilities. Five of these possibilities yield a graph $J$ such that $K_2 \ast J$ is $d_1$-choosable (since $J$ contains an induced copy of either the antichain, $K_1 \ast$ antipaw, $K_1 \ast P_4$, or $C_4$). So we consider the other three possibilities (these are the three possibilities when $w$ is adjacent to both vertices of degree 3 in $D$).

If $D$ is not dominating, then some vertex $x$ is distance 2 from $D$, via $w$. In each case, the subgraph induced by $D$, $w$, and $x$ contains an induced $d_1$-choosable subgraph (in two cases this is a antichain, and in the third case it is $K_1 \ast$ antipaw). Hence, $D$ is dominating, and all of its neighbors are adjacent to both vertices of degree 3 in $D$. But now $H$ has two dominating vertices. This contradicts our assumption that $H$ has no dominating vertex. Hence, $H$ is a clique tree.

Since $H$ has no dominating vertex, it must contain an induced $P_4$, call it $P$. Since $H$ has neither a $P_4$ nor a “chair” as an induced subgraph, each vertex adjacent to $P$ must be adjacent to at least two vertices of $P$. Since $C_4$ and the antichain and $K_1 \ast P_4$ are all forbidden, each vertex adjacent to $P$ is adjacent to exactly two consecutive vertices of $P$. Since both $P_5$ and the net are forbidden, every vertex in $H$ is adjacent to $P$. Since $P_1 \ast$ antipaw is forbidden, every pair of vertices that are adjacent to the same two vertices of $P$ are also adjacent to each other. Finally, since $M$ is forbidden, $H$ must be formed in one of two ways. Either (a) begin with two disjoint cliques and add an edge between them, or else (b) begin with a clique and add exactly one edge incident to exactly two vertices of the clique. Furthermore, all graphs $H$ formed by either (a) or (b) are such that $K_2 \ast H$ is not $d_1$-choosable.

In (a), suppose that we begin with a $K_r$ and a $K_s$. We assign lists as follows: the $K_r$ gets $[r]$, the $K_s$ gets $\{r+1, \ldots, r+s\}$, the dominating vertices (on the other side of the join) get $[r+t]$; finally, the two endpoints of the additional edge also get $\alpha$ added to their lists. $K_2 \ast H$ is clearly not colorable from these lists, since all but one or $[r+t]$ must be used on $H$.

In (b), suppose that we begin with a $K_r$. We assign lists as follows: the $K_r$ gets $[r]$, the two degree 1 vertices get $\{r+1, r+2\}$, the dominating vertices (on the other side of the join) get $[r+2]$; finally, the two vertices in the $K_r$ that are endpoints of the pendant edges also get $r+1$ added to their lists. $K_2 \ast H$ is clearly not colorable from these lists, since all but one of $[r+2]$ must be used on $H$.

4.4.4 Mixed list assignments

Lemma 4.53. Let $A$ be a graph with $|A| \geq 4$. Let $L$ be a list assignment on $G := E_2 \ast A$ such that $|L(v)| \geq d(v) - 1$ for all $v \in V(G)$ and each component $D$ of $A$ has a vertex $v$ such that $|L(v)| \geq d(v)$. Then $L$ is good on $G$. 37
Proof. By the Small Pot Lemma, \(|\text{Pot}(L)| \leq |A| + 1\). Say the \(E_2\) has vertices \(\{x, y\}\). Then \(|L(x)| + |L(y)| \geq 2|A| - 2 > |A| + 1\) since \(|A| \geq 4\). Coloring \(x\) and \(y\) the same leaves at worst a \(d_0\) assignment \(L'\) on \(A\) where each component \(D\) has a vertex \(v\) with \(|L'(v)| > d_D(v)\). Hence we can complete the coloring.

Lemma 4.54. Let \(A\) be a graph with \(|A| \geq 3\). Let \(L\) be a list assignment on \(G := E_2 \ast A\) such that \(|L(v)| \geq d(v) - 1\) for all \(v \in V(G)\), \(|L(v)| \geq d(v)\) for some \(v\) in the \(E_2\) and each component \(D\) of \(A\) has a vertex \(v\) such that \(|L(v)| \geq d(v)\). Then \(L\) is good on \(G\).

Proof. By the Small Pot Lemma, \(|\text{Pot}(L)| \leq |A| + 1\). Say the \(E_2\) has vertices \(\{x, y\}\). Then \(|L(x)| + |L(y)| \geq 2|A| - 1 > |A| + 1\) since \(|A| \geq 3\). Coloring \(x\) and \(y\) the same leaves at worst a \(d_0\) assignment \(L'\) on \(A\) where each component \(D\) has a vertex \(v\) with \(|L'(v)| > d_D(v)\). Hence we can complete the coloring.

4.5 Joins with \(K_1\)

Let \(G\) be a \(d_0\)-choosable graph. If \(K_1 \ast G\) is not \(d_1\)-choosable, then we call \(G\) bad; otherwise we call \(G\) good. Adding a leaf to a graph does not change whether it is bad, so we focus on bad \(G\) such that \(\delta(G) \geq 2\). We will also restrict our attention to connected bad graphs.

In this section, we apply Lemma 4.20 to characterize all bad triangle-free graphs. An easy special case of this classification for triangle-free graphs is the following lemma. We frequently use the idea of an independent set with a common color, so we call an independent set of size \(k\) with a common color an independent \(k\)-set.

Lemma 4.55. If \(G\) is a connected bipartite graph with more edges than vertices, then \(K_1 \ast G\) is \(d_1\)-choosable.

Proof. Let \(A\) and \(B\) be the parts of \(G\). Let \(L\) be a minimal bad \(d_1\)-assignment for \(K_1 \ast G\). Since \(G\) has more edges than vertices, \(G\) has a cycle. Since \(G\) is also bipartite, \(G\) is \(d_0\)-choosable (by the classification of \(d_0\)-choosable graphs at the start of Section 4.2). By the Small Pot Lemma, \(\text{Pot}(L) \leq |G|\). Note that \(\sum_{v \in A} d(v) = |E(G)| > |V(G)| \geq |\text{Pot}(L)|\). Similarly \(\sum_{v \in B} d(v) > |\text{Pot}(L)|\). Now we apply Lemma 4.22 with \(I_1 = A\) and \(I_2 = B\). This proves the lemma.

Lemma 4.56. Let \(C\) be a collection of sets \(I_1, \ldots, I_k\), each of size 2. If for all \(i \neq j\), we have \(I_i \cap I_j \neq \emptyset\), then either there exists \(v \in \bigcap_{i=1}^k I_i\) or there exist \(v_1, v_2, v_3\) such that each \(I_i\) equals either \(\{v_1, v_2\}\) or \(\{v_1, v_3\}\) or \(\{v_2, v_3\}\).

Proof. Suppose that \(\bigcap_{i=1}^k I_i = \emptyset\). Consider distinct sets \(I_1\) and \(I_2\). Let \(I_1 = \{v_1, v_2\}\) and \(I_2 = \{v_1, v_3\}\). Since \(\bigcap_{i=1}^k I_i = \emptyset\), there exists \(I_3\) such that \(v_1 \notin I_3\). So we must have \(I_3 = \{v_2, v_3\}\). Now for all \(k \geq 4\), we must have \(|I_k \cap \{v_1, v_2, v_3\}| = 2\).

The core of a graph is its maximum subgraph with minimum degree at least 2. Alternatively, it’s the result if we repeatedly delete vertices of degree at most 1 for as long as possible.

Using Lemmas 4.20 and 4.56 we can prove the following classification.
Lemma 4.57. If a graph $G$ is bad, then $K_1 \ast G$ has a $d_1$-list assignment $L$ such that one of the following 5 conditions holds.

1. $L$ is a $d$-clique cover of $G$ of size at most $|G|$.
2. There exists $v \in V(G)$ such that $L$ is a $d$-clique cover of $G - v$ of size at most $|G| - 1$.
3. There exists a color $c$ such that the union of all independent 2-sets in $c$ induces $P_4$ and all other independent 2-sets are the end vertices of the $P_4$.
4. The union of all independent 2-sets is $E_3$ or $E_2$.
5. All independent 2-sets in $L$ are the same color.

Proof. Let $z$ denote the $K_1$. We consider the possible ways for a bad list assignment $L$ to satisfy Lemma 4.20. Clearly $L$ has no independent $k$-sets, for $k \geq 3$. If $L$ has no independent 2-sets, then Condition 1 holds. If all independent 2-sets in $L$ are the same color, then Condition 5 holds. If $L$ has only the same independent 2-set in multiple colors, then the 2-sets induce $E_2$, so Condition 4 holds. So instead $L$ must have distinct independent 2-sets in distinct colors.

Assume that additionally all independent 2-sets intersect in a common vertex $v$. If $|\text{Pot}_{G-v}(L)| \leq |G|-1$, then Condition 2 holds. So instead $|\text{Pot}_{G-v}(L)| \geq |G|$. So there exist some $w \in G - v$ and some color $c \in L(w)$ such that $c \notin L(z)$. By Lemma 4.20 $G$ has an $L$-coloring that uses $c$ on $w$ and uses some other common color on two vertices of $G - w$. Now we can extend the coloring to $z$.

Now suppose that no vertex $v$ lies in all independent 2-sets. If all independent 2-sets are distinct colors, then Lemma 4.56 implies that Condition 4 holds. Suppose we have two independent 2-sets $I_1 = \{v_1, v_2\}$ and $I_2 = \{v_1, v_3\}$ in the same color $c$. Since $L$ has no independent 3-set, $v_2$ is adjacent to $v_3$. Recall that $L$ has an independent 2-set $I_3$ of another color $c'$. If $v_1 \notin I_3$, then $I_3$ is disjoint from either $I_1$ or $I_2$, so we can finish the coloring, by (2) in Lemma 4.20. Hence $v_1 \in I_3$. So the only independent 2-sets not containing $v_1$ must be of color $c$, say $\{v_2, v_4\}$. Since $L$ has no independent 3-sets, we must have $v_1$ adjacent to $v_4$. Now we see that every independent 2-set in a color other than $c$ must be $\{v_1, v_2\}$. This implies that $v_2$ and $v_3$ must be adjacent. Now Condition 3 holds.

Finally, suppose that $L$ has two independent 2-sets $I_1 = \{v_1, v_2\}$ and $I_2 = \{v_3, v_4\}$ in a common color. If we are not in the case above, then $G[v_1, v_2, v_3, v_4] = C_4$. Now every independent 2-set $I_3$ of another color can intersect at most one of $I_1$ and $I_2$, so we can color the graph by (2) in Lemma 4.20.

The classification in Lemma 4.57 is somewhat unsatisfying, since it does not immediately yield a method to construct all bad graphs of a certain size. In Lemma 4.58, we give a more satisfying characterizations for triangle-free bad graphs.

Lemma 4.58. Let $G$ be $d_0$-choosable and triangle-free. The graph $K_1 \ast G$ is not $d_1$-choosable iff the core of each component of $G$ is an even cycle, except for at most one component which has a core that is either $\theta_{2,3,2l+1}$ (for some integer $l$) or is formed from a disjoint union of even paths by adding a vertex adjacent to all their endpoints.
Proof. Suppose that $H$ is $d_0$-choosable and triangle-free, but that $K_1 \ast H$ is not $d_1$-choosable. Let $z$ denote the vertex of the $K_1$. Since $H$ is $d_0$-choosable, no component of $H$ can be a Gallai tree. Hence each component contains an even cycle. If $H$ is a counterexample to the theorem, then some component $D$ of $H$ contains at least $|D| + 1$ edges.

First suppose that there exist two components $D_1$ and $D_2$ of $H$ with at least $|D_1| + 1$ and $|D_2| + 1$ edges, respectively. Let $G_1$ and $G_2$ be the cores of $D_1$ and $D_2$ and let $L$ be a bad $d_1$-assignment for $K_1 \ast (G_1 + G_2)$. (We are guaranteed this bad list assignment from our bad $d_1$-list assignment for $K_1 \ast H$.) By greedily coloring $G_2$, we can get a bad $d_1$-assignment $L_1$ for $K_1 \ast G_1$. By the Small Pot Lemma, we may assume that $|Pot(L_1)| \leq |G_1|$. Since $|E(G_1)| > |G_1|$ we get $\sum_{v \in G_1} |L(v)| = 2|E(G_1)| > 2|G_1| \geq 2|Pot(L_1)|$, so we have a color class $\alpha$ of size $3$ in $G_1$, and hence an independent set of size $2$ in $G_1$ with the common color $\alpha$. By reversing the roles of $G_1$ and $G_2$, we can find an independent set of size $2$ in $G_2$ with a common color $\beta$. Now we can apply (2) from Lemma 4.20. Thus, the core of all but at most one component $D$ of $H$ is an even cycle. Let $G$ be the core of $D$ and let $L$ be a bad $d_1$-list assignment for $K_1 \ast G$.

We first prove that every color appears in the list of at most $3$ vertices of $G$. Our plan is to either find an independent set of size $3$ with a color common to its lists or to find two disjoint independent sets of size $2$ with a distinct color common to the lists of each. Then we can apply (1) or (2) from Lemma 4.20.

Claim 1. We may assume that $|Pot_G(L)| = |G| - 1$.

Since $G$ is $d_0$-choosable, we know that $G$ has an $L$-coloring. If $|Pot_G(L)| < |L(z)| = |G| - 1$, then we can clearly extend the coloring to $z$. If there is at least one independent 2-set, then applying Lemma 4.21 gives $|Pot_G(L)| = |G| - 1$. Otherwise, since $G$ is triangle-free, $|G| = |E|$ and we are done.

Claim 2. No color appears on $6$ or more vertices of $G$. Suppose the contrary. Since $G$ is triangle-free (and since $R(3, 3) = 6$), $3$ of these vertices form an independent $3$-set. Now we can apply (1) from Lemma 4.20.

Claim 3. No color appears on $5$ vertices of $G$. Suppose that color $\alpha$ appears on exactly $5$ vertices of $G$. Note that $\sum_{v \in V} |L(v)| = \sum_{v \in V} d_G(v) = 2|E(G)| \geq 2(|G| + 1)$. By Claim 1, $|Pot(L)| = |G| - 1$. So by the Pigeonhole Principle (since $2(|G| + 1) = 2(|G| - 1) + 4$) there exists a color $\beta \neq \alpha$ such that $\beta$ appears on at least $3$ vertices in $G$. Since $G$ is triangle-free, there exists an independent 2-set $I$ with $\beta$ as a common color. We may assume that the subgraph $G_\alpha$ induced by color $\alpha$ has no independent 3-set. Since $G$ is triangle-free, $G_\alpha$ must be $C_5$. Now $G_\alpha$ has an independent 2-set that is disjoint from $I$. Thus we can apply (2) from Lemma 4.20.

Claim 4. No distinct colors $\alpha$ and $\beta$ each appear on $4$ vertices of $G$. Suppose that colors $\alpha$ and $\beta$ each appear on $4$ vertices of $G$ and let $G_\alpha$ and $G_\beta$ be the subgraphs induced by these colors. Since $G_\alpha$ is bipartite and has no independent 3-set, we can partition $V(G_\alpha)$ into two independent 2-sets $I_1$ and $I_2$. Similarly, we can partition $V(G_\beta)$ into independent 2-sets $J_1$ and $J_2$. Now we can finish by (2) from Lemma 4.20 unless each of $I_1$ and $I_2$ intersects each of $J_1$ and $J_2$. This implies that $V(G_\alpha) = V(G_\beta) = I_1 \cup I_2$. Thus, we can use $\alpha$ on $I_1$ and $\beta$ on $I_2$.

Claim 5. If $G$ has a color class $\alpha$ of size $4$ and a color class $\beta$ of size $3$, then $\alpha$ induces a $P_4$, $\beta$ induces a $P_3$, and the $P_3$ and $P_4$ together induce a $C_5$. Let $G_\alpha$ and $G_\beta$ be the subgraphs induced by $\alpha$ and $\beta$. Let $I$ be an independent 2-set in $G_\beta$. Since $G_\alpha$ is bipartite and has no
independent 3-set, $G_\alpha$ is a subgraph of $C_4$ with at least two edges. If $G_\alpha$ is $C_4$, then let $J_1$ and $J_2$ denote the disjoint independent 2-sets in $G_\alpha$. No independent set intersects both $J_1$ and $J_2$. Thus, we can apply (2) to $I$ and some $J_i$. Similarly, if $G_\alpha = 2K_2$, then $I$ is disjoint from some independent 2-set in $G_\alpha$. Hence, $G_\alpha = P_4$ and $I$ consists of the endpoints of the $P_4$. If $G_\beta$ is not $P_3$, then we have a second choice for $I$, which cannot also be the endpoints of $G_\alpha$. Thus, $G_\beta = P_3$. Since $G$ is triangle-free, we get that $G_{\alpha\cup\beta} = C_5$.

**Claim 6.** No color appears on 4 vertices of $G$. By Claim 4, suppose that exactly one color, $\alpha$, appears on 4 vertices of $G$. Let $c_i$ denote the number of colors that appear on exactly $i$ vertices. We have $2|E| = \sum_{v \in V} d(v) = \sum_{v \in V} |L(v)| = c_1 + 2c_2 + 3c_3 + 4(1)$. By Claim 1, we know that $|Pot(L)| = |G| - 1$, so $c_1 + c_2 + c_3 + 1 = |G| - 1$. Multiplying the second equation by 2 and subtracting it from the first gives $2(|E| - |G|) = c_3 - c_1$. Let $x$ and $y$ denote the endpoints of $G_\alpha$, as given by Claim 5. For each color class $\beta$ of size 3, we can apply Claim 5. Thus each color in a class of size 3 appears on both $x$ and $y$; so does color $\alpha$. Hence $d(x) \geq 1 + c_3$ and $d(y) \geq 1 + c_3$. Note that $2(|E| - |G|) = \sum_{v \in V} (d(v) - 2) \geq (d(x) - 2) + (d(y) - 2) \geq 2c_3 - 2$; the first inequality holds because $G$ is the core, and thus $\delta(G) \geq 2$. Combining this inequality with the equality above, we get $c_3 - c_1 \geq 2c_3 - 2$, which implies that $2 - c_1 \geq c_3$. Finally, this implies that $|E| - |G| \leq 1$.

If $|E| - |G| = 0$, then $G$ is simply a 5-cycle, which is a Gallai tree, which yields a contradiction. Hence $|E| - |G| = 1$, which implies that $c_3 = 2$ and $c_1 = 0$. Thus $d(x) = d(y) = 3$. Since $|E| = |G| + 1$, $\delta(G) = 2$, and $x$ and $y$ lie on a common cycle, $G$ must be a theta graph $\Theta_{2,3,k}$. If any color class of size 2 is an independent set $I$, then $I$ must be $\{x, y\}$ (since otherwise we could use a common color on $I$ and on two vertices of $G_\alpha$); however, this is impossible, since we have already accounted for all of $L(x)$ and $L(y)$. Thus, every color class of size 2 must induce a $K_2$. Now a simple parity argument shows that the final path of the theta graph has odd length, so $G$ is $\Theta_{2,3,2l+1}$.

**Claim 7.** No vertex $u$ is contained in every independent 2-set. Suppose instead that such a $u$ exists. By Claim 1, $|Pot(L)| = |G| - 1$. Since $\delta(G) \geq 2$, we get $\sum_{v \in (V - u)} d_G(v) \geq 2(|G| - 1)$. Since $u$ appears in every independent 2-set (and $G$ is triangle-free), each color appears at most twice on $G - u$. Since $|Pot_{G - u}(L)| \leq |Pot(L)| \leq |G| - 1$, every color in $|Pot_{G - u}(L)|$ appears exactly twice on $G - u$ and furthermore $d_{G - u}(v) = 2$ for all $v \in V(G) - u$. Hence, $G - u$ is a disjoint union of paths. Since each color class must induce a $K_2$ in $G - u$, we see that each path must be of odd length.

**Claim 8.** No such $G$ exists. If there exist independent 2-sets $I_1$ and $I_2$, both with common color $\alpha$, then by Claim 6, $I_1$ and $I_2$ intersect. Clearly, if independent 2-sets $I_1$ and $I_2$ have distinct common colors, then they must intersect (or we are done by (2) from Lemma 4.20). Thus, every pair of independent 2-sets intersects. Now by Claim 7 and Lemma 4.56, there exist vertices $v_1$, $v_2$, $v_3$ such that every independent 2-set is either $\{v_1, v_2\}$ or $\{v_1, v_3\}$ or $\{v_2, v_3\}$.

Now, similar to in Claim 6, we have $2|E| = c_1 + 2c_2 + 3c_3$ and $c_1 + c_2 + c_3 = |G| - 1$. Again $2(|E| - |G|) = c_3 - c_1 - 2$. Now we have $2(|E| - |G|) = \sum_{v \in G} (d(v) - 2) \geq (d(v_1) - 2) + (d(v_2) - 2) + (d(v_3) - 2) \geq 2c_3 - 6$. So $c_3 - c_1 - 2 \geq 2c_3 - 6$, which implies that $c_3 \leq 4$, and hence $|E| - |G| \leq 1$. If $|E| - |G| = 0$, then $G$ is a cycle, so we may assume that $|E| - |G| = 1$, which implies that $c_1 = 0$ and $c_3 = 4$.

Since $c_3 = 4$, there exist distinct colors $\alpha$, $\beta$, and $\gamma$, each of which appear on independent
2-sets; say $\alpha$ appears on $\{v_1, v_2\}$, and $\beta$ appears on $\{v_1, v_3\}$ and $\gamma$ appears on $\{v_2, v_3\}$. Let $w_1$, $w_2$, and $w_3$ denote the third vertices with color $\alpha$, $\beta$, and $\gamma$, respectively. If two (or all three) of the $w_i$ coincide, then that vertex $w$ has degree at least 3, so $\sum_{v \in V}(d(v) - 2) \geq 2c_3 - 5$. This implies that $c_3 \leq 3$, which contradicts our assumption that $c_3 = 4$. If instead $w_1$, $w_2$, and $w_3$ are distinct, then $\{v_1, v_2, v_3, w_1, w_2, w_3\}$ induces a 6-cycle. Since the only two vertices in $G$ of degree 3 lie on the 6-cycle, $G$ is a theta graph. Suppose, without loss of generality, that $v_1$ and $v_2$ each appear in three color classes of size 3 and that $v_3$ appears in two of them. Now we have fully accounted for the colors in $L(v_1)$, $L(v_2)$, $L(v_3)$, and $L(w_1)$. However, we still need another color in each of $L(w_2)$ and $L(w_3)$. Since $c_1 = 0$, this gives a contradiction. We have now completed one direction of the proof. Below, we give the other direction.

Form $G'$ from $G$ by adding a pendant edge. Observe that $K_1 \ast G'$ is $d_1$-choosable iff $K_1 \ast G$ is $d_1$-choosable. Any $d_1$-list assignment will give a single color to the degree 1 vertex, so $K_1 \ast G'$ has a coloring from its lists iff $K_1 \ast G$ has a coloring from the resulting $d_1$-list assignment. Thus, given a graph $H$ and its core $G$, the graph $K_1 \ast H$ is $d_1$-choosable iff $K_1 \ast G$ is $d_1$-choosable. To complete the proof, we need only provide list assignments to show that $K_1 \ast G$ is not $d_1$-choosable when $G$ is a disjoint union of 4+ -cycles together with at most component that is $\Theta_{2,3,2l+1}$ or is formed from a disjoint union of odd paths by adding a vertex adjacent to all their endpoints.

For a $k$-cycle, we assign to each edge a distinct color and assign to each vertex the colors on its two incident edges. Since each color can be used only once in a proper coloring, every coloring of the $k$-cycle uses all $k$ colors in its lists. Thus, if we let $L(z)$ contain $k - 1$ of those colors, then $K_1 \ast C_k$ has no $L$-coloring. Furthermore, for any graph $G$ and any integer $k$, the graph $K_1 \ast (G + C_k)$ is $d_1$-choosable iff $K_1 \ast G$ is $d_1$-choosable.

Suppose that $G$ is formed from disjoint even paths by adding a vertex $v$ adjacent to all of their endpoints. We partition each path into copies of $K_2$ and give the vertices in each $K_2$ the same list, say $\{\alpha_i, \beta_i\}$. We use disjoint lists on each $K_2$ and we assign an arbitrary list of colors to vertex $v$. Finally, let $L(z) = \bigcup \{\alpha_i, \beta_i\}$. Any proper coloring of $G - u$ will use all the colors in $\bigcup \{\alpha_i, \beta_i\}$. Thus, $z$ will have no color.

Finally, suppose that $G = \Theta_{2,3,2l+1}$. Let $v_1, v_2, v_3, v_4, v_5$ denote the vertices of the 5-cycle, where $d(v_1) = d(v_4) = 3$. Let $L(v_1) = L(v_4) = \{a, b, c\}$, $L(v_2) = L(v_3) = \{a, d\}$, and $L(v_5) = \{b, c\}$. Partition the $2l - 1$ path of $G \setminus \{v_1, \ldots, v_5\}$ into copies of $K_2$. As above, give the vertices in each copy of $K_2$ the same list $\{\alpha_i, \beta_i\}$; use disjoint lists on the $K_2$s. Now let $L(z) = \{a, b, c, d\} \cup (\bigcup \{\alpha_i, \beta_i\})$. Since each of $\{a, b, c, d\}$ must be used on $v_1, \ldots, v_5$, no color remains for $z$.
## Notation

<table>
<thead>
<tr>
<th>Symbology</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>G</td>
</tr>
<tr>
<td>$|G|$</td>
<td>the number of edges $G$ has</td>
</tr>
<tr>
<td>$G[S]$</td>
<td>the subgraph of $G$ induced on $S$</td>
</tr>
<tr>
<td>$E_G(X,Y)$</td>
<td>the edges in $G$ with one end in $X$ and the other in $Y$</td>
</tr>
<tr>
<td>$E_G(X)$</td>
<td>$E_G(X,V(G) - X)$</td>
</tr>
<tr>
<td>$\chi(G)$</td>
<td>the chromatic number of $G$</td>
</tr>
<tr>
<td>$\omega(G)$</td>
<td>the clique number of $G$</td>
</tr>
<tr>
<td>$\alpha(G)$</td>
<td>the independence number of $G$</td>
</tr>
<tr>
<td>$\Delta(G)$</td>
<td>the maximum degree of $G$</td>
</tr>
<tr>
<td>$\delta(G)$</td>
<td>the minimum degree of $G$</td>
</tr>
<tr>
<td>$\kappa(G)$</td>
<td>the vertex connectivity of $G$</td>
</tr>
<tr>
<td>$\overline{G}$</td>
<td>the complement of $G$</td>
</tr>
<tr>
<td>$A + B$</td>
<td>the disjoint union of graphs $A$ and $B$</td>
</tr>
<tr>
<td>$A \ast B$</td>
<td>the join of graphs $A$ and $B$ (that is, $\overline{A + B}$)</td>
</tr>
<tr>
<td>$kG$</td>
<td>$\underbrace{G + G + \cdots + G}_{k \text{ times}}$</td>
</tr>
<tr>
<td>$G^k$</td>
<td>$\underbrace{G \ast G \ast \cdots \ast G}_{k \text{ times}}$</td>
</tr>
<tr>
<td>$H \subseteq G$</td>
<td>$H$ is a subgraph of $G$</td>
</tr>
<tr>
<td>$H \subset G$</td>
<td>$H$ is a proper subgraph of $G$</td>
</tr>
<tr>
<td>$H \triangleleft G$</td>
<td>$H$ is an induced subgraph of $G$</td>
</tr>
<tr>
<td>$H \preceq G$</td>
<td>$H$ is a proper induced subgraph of $G$</td>
</tr>
<tr>
<td>$H \prec G$</td>
<td>$H$ is a child of $G$</td>
</tr>
<tr>
<td>$f : S \hookrightarrow T$</td>
<td>an injective function from $S$ to $T$</td>
</tr>
<tr>
<td>$f : S \rightarrow T$</td>
<td>a surjective function from $S$ to $T$</td>
</tr>
<tr>
<td>$X := Y$</td>
<td>$X$ is defined as $Y$</td>
</tr>
<tr>
<td>$K_k$</td>
<td>the complete graph on $k$ vertices</td>
</tr>
<tr>
<td>$E_k$</td>
<td>the edgeless graph on $k$ vertices (that is, $\overline{K_k}$)</td>
</tr>
<tr>
<td>$P_k$</td>
<td>the path on $k$ vertices</td>
</tr>
<tr>
<td>$C_k$</td>
<td>the cycle on $k$ vertices</td>
</tr>
<tr>
<td>$K_{a,b}$</td>
<td>the complete bipartite graph with parts of size $a$ and $b$ (that is, $E_a \ast E_b$)</td>
</tr>
<tr>
<td>$[n]$</td>
<td>${1, 2, \ldots, n}$</td>
</tr>
<tr>
<td>$\mathbb{N}$</td>
<td>the natural numbers (0, 1, 2, \ldots)</td>
</tr>
<tr>
<td>$\mathbb{R}$</td>
<td>the real numbers</td>
</tr>
</tbody>
</table>
References


