Antimagic Labelings of Regular Bipartite Graphs

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Antimagic Labelings

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Pf. for $\Delta(G) = n - 1$. Let $d(v) = n - 1$. Label $G - v$ arbitrarily. Label the final star in order of partial sum.

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Thm. [Alon et al. 2004] Every complete partite graph other than K_2 is antimagic.

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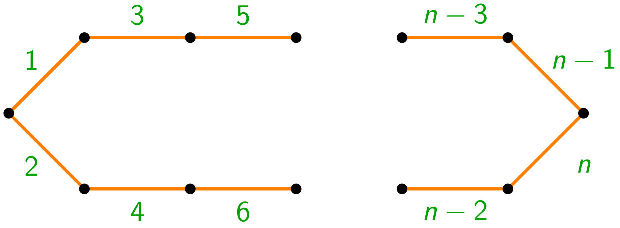

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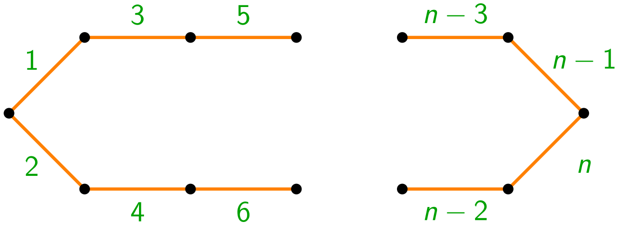

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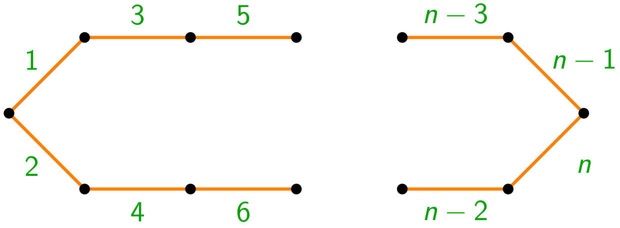

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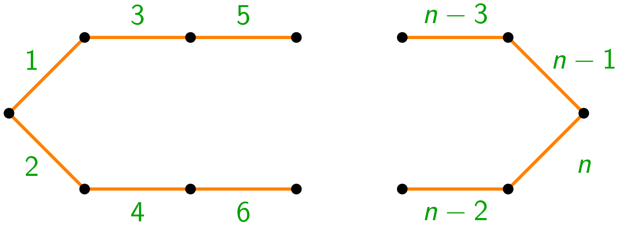

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Today, I'll prove the theorem for $k \geq 5$ odd.

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constructing an antimagic labeling: resolving all potential conflicts

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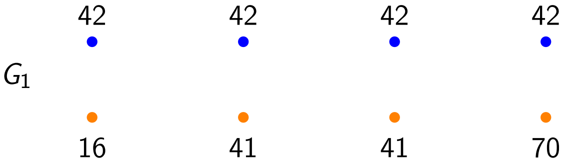

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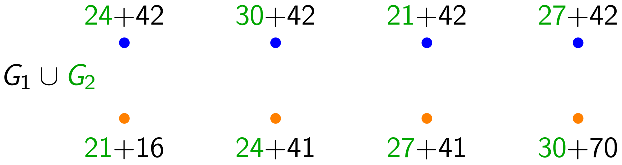

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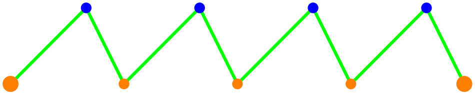

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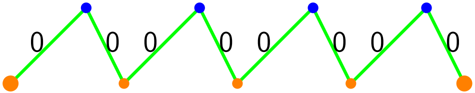

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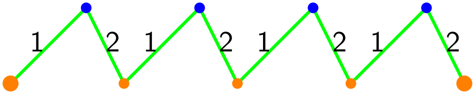

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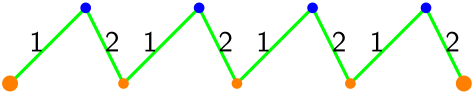

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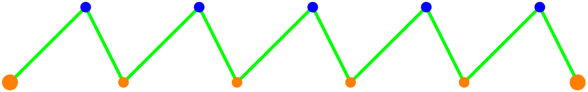

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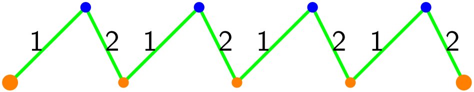

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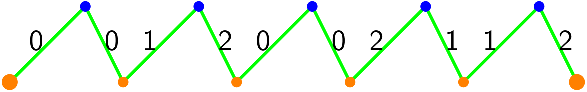

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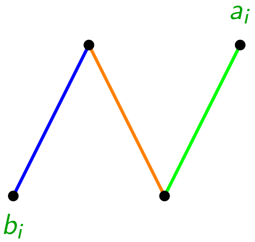

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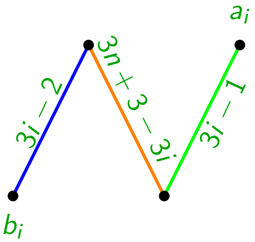

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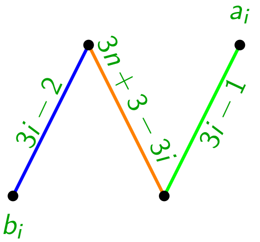

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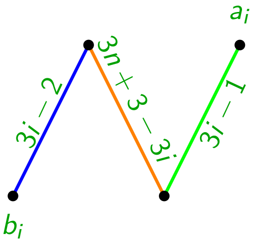

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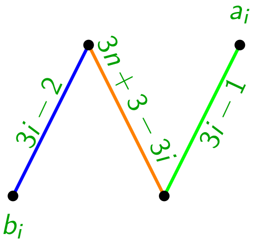

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Hence, every regular bipartite graph of odd degree ≥ 5 is antimagic.

Thank you!