Acyclic edge-coloring of planar graphs: Δ colors suffice when Δ is large

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Abstract

An *acyclic edge-coloring* of a graph G is a proper edge-coloring of G such that the subgraph induced by any two color classes is acyclic. The *acyclic chromatic index*, $\chi'_a(G)$, is the smallest number of colors allowing an acyclic edge-coloring of G. Clearly $\chi'_a(G) \geq \Delta(G)$ for every graph G. Cohen, Havet, and Müller conjectured that there exists a constant M such that every planar graph with $\Delta(G) \geq M$ has $\chi'_a(G) = \Delta(G)$. We prove this conjecture.

1 Introduction

A *proper edge-coloring* of a graph G assigns colors to the edges of G such that two edges receive distinct colors whenever they have an endpoint in common. An *acyclic edge-coloring* is a proper edge-coloring such that the subgraph induced by any two color classes is acyclic (equivalently, the edges of each cycle receive at least three distinct colors). The *acyclic chromatic index*, $\chi'_a(G)$, is the smallest number of colors allowing an acyclic edge-coloring of G. In an edge-coloring φ, if a color α is used incident to a vertex v, then α is *seen by* v. For the maximum degree of G, we write $\Delta(G)$, and simply Δ when the context is clear. Note that $\chi'_a(G) \geq \Delta(G)$ for every graph G. When we write *graph*, we forbid loops and multiple edges. A *planar graph* is one that can be drawn in the plane with no edges crossing. A *plane graph* is a planar embedding of a planar graph. Cohen, Havet, and Müller [9, 4] conjectured that there exists a constant M such that every planar graph with $\Delta(G) \geq M$ has $\chi'_a(G) = \Delta(G)$. We prove this conjecture.

Main Theorem. *All planar graphs G satisfy $\chi'_a(G) \leq \max\{\Delta, 4.2 * 10^{14}\}$. Thus, $\chi'_a(G) = \Delta$ for all planar graphs G with $\Delta \geq 4.2 * 10^{14}$.*

We start by reviewing the history of acyclic coloring and acyclic edge-coloring. An *acyclic coloring* of a graph G is a proper vertex coloring of G such that the subgraph induced by any two color classes is acyclic. The fewest colors that allows an acyclic coloring of G is the *acyclic chromatic number*, $\chi_a(G)$. This concept was introduced in 1973 by Grünbaum [12], who conjectured that every planar graph G has $\chi_a(G) \leq 5$. This is best possible, as shown (for example) by the octahedron. After a flurry of activity, Grünbaum's conjecture was confirmed in 1979 by Borodin [6]. This result contrasts sharply with the behavior of $\chi_a(G)$ for a general graph G. Alon, McDiarmid,

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and Reed [1] found a constant C_1 such that for every Δ there exists a graph G with maximum degree Δ and $\chi_a(G) \geq C_1\Delta^{4/3}(\log \Delta)^{-1/3}$. This construction is nearly best possible, since they also found a constant C_2 such that $\chi_a(G) \leq C_2\Delta^{4/3}$ for every graph G with maximum degree Δ. The best known upper bound is $\chi_a(G) \leq 2.835\Delta^{4/3} + \Delta$, due to Sereni and Volec [14].

Now we turn to acyclic edge-coloring. In contrast to the results above, there does exist a constant C_3 such that $\chi'_a(G) \leq C_3\Delta$ for every graph G with maximum degree Δ. Using the Asymmetric Local Lemma, Alon, McDiarmid, and Reed [1] showed that we can take $C_3 = 64$. This constant has been improved repeatedly, and the current best bound is 3.74, due to Giotis et al. [11]. But this upper bound is still far from the conjectured actual value.

Conjecture 1. *Every graph G satisfies $\chi'_a(G) \leq \Delta + 2$.*

Conjecture 1 was posed by Fiamčík [10] in 1978, and again by Alon, Sudakov, and Zaks [2] in 2001. The value $\Delta+2$ is best possible, as shown (for example) by K_n when n is even. In an acyclic edge-coloring at most one color class can be a perfect matching; otherwise, two perfect matchings will induce some cycle, by the Pigeonhole principle. Now the lower bound $\Delta + 2$ follows from an easy counting argument.

For planar graphs, the best upper bounds are much closer to the conjectured value. Cohen, Havet, and Müller [9] proved $\chi'_a(G) \leq \Delta + 25$ whenever G is planar. The constant 25 has been frequently improved [3, 4, 13, 16, 15]. The current best bound is $\chi'_a(G) \leq \Delta + 6$, due to Wang and Zhang [15]. However, for planar graphs with Δ sufficiently large, Conjecture 1 can be strengthened further. This brings us to the previously mentioned conjecture of Cohen, Havet, and Müller [9].

Conjecture 2. *There exists a constant M such that if G is planar and $\Delta \geq M$, then $\chi'_a(G) = \Delta$.*

Our Main Theorem confirms Conjecture 2. For the proof we consider a hypothetical counterexample. Among all counterexamples we choose one with the fewest vertices, a *minimal counterexample*. In Section 2 we prove our Structural Lemma, which says that every 2-connected plane graph contains one of four configurations. In Section 3 we show that every minimal counterexample G must be 2-connected, and that G cannot contain any of these four configurations. This shows that no minimal counterexample exists, which finishes the proof of the Main Theorem.

2 The Structural Lemma

A vertex v is *big* if $d(v) \geq 8680$. For a graph G, a vertex v is *very big* if $d(v) \geq \Delta - 4(8680)$. A *$k$-vertex* (resp. *$k^+$-vertex* and *$k^-$-vertex*) is a vertex of degree k (resp. at least k and at most k). For a vertex v, a *k-neighbor* is an adjacent k-vertex; *k^+-neighbors* and *k^--neighbors* are defined analogously. Similarly, we define *k-faces*, *k^+-faces*, and *k^--faces*. For the length of a face f, we write $\ell(f)$.

A key structure in our proof, called a *bunch*, consists of two big vertices with many common 4^--neighbors that are embedded as successive neighbors (for both big vertices); see Figure 1 for an example. Let $x_0, \ldots, x_{t+1}$ denote successive neighbors of a big vertex v, that are also successive for a big vertex w. We require that $d(x_i) \leq 4$ for all $i \in [t]$, where $[t]$ denotes $\{1, \ldots, t\}$. Further, for each $i \in [t+1]$, we require that the 4-cycle vx_iwx_{i-1} is not separating; so, either the cycle bounds a 4-face, or it bounds the two 3-faces vx_ix_{i-1} and wx_ix_{i-1}. (Each 4-vertex in a bunch is incident to four 3-faces, each 3-vertex in a bunch is incident to a 4-face and two 3-faces, and each 2-vertex in a bunch is incident to two 4-faces.) For such a bunch, we call $x_1, \ldots, x_t$ its *bunch vertices*, and we

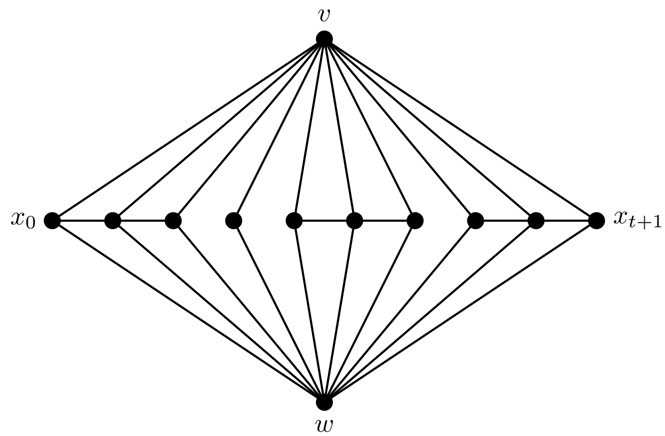

Figure 1: A bunch, with v and w as its parents.

call v and w the *parents* of the bunch. (When we refer to a bunch, we typically mean a maximal bunch.) For technical reasons, we exclude x_0 and x_{t+1} from the bunch. The *length* of the bunch is t. A *horizontal edge* is any edge $x_i x_{i+1}$, with $1 \le i \le t-1$. Each path vx_iw is a *thread*.

Borodin et al. [8] constructed graphs in which every 5^--vertex has at least two big neighbors. Begin with a truncated dodecahedron, and subdivide t times each edge that lies on two 10-faces. Now add a new vertex into every 4^+-face, making it adjacent to every vertex on the face boundary. The resulting plane triangulation has $\Delta = 5k + 10$, minimum degree 4, and every 5^--vertex has two Δ-neighbors. This final fact motivates our Structural Lemma, by showing that if we omit from it (RC3) and (RC4), then the resulting statement is false. (For illustrating that we cannot omit both (RC3) and (RC4), the above construction can be generalized. Rather than truncating a dodecahedron, we can start by truncating any 3-connected plane graph with all faces of length 5 or 6; the rest of the construction is the same.) Now we state and prove our Structural Lemma.

Structural Lemma. *Let G be a 2-connected plane graph. Let $k = \max\{\Delta, 5(8680)\}$. Now G contains one of the following four configurations:*

(RC1) a vertex v such that $\sum_{w \in N(v)} d(w) \le k$; or

(RC2) a big vertex v such that among those 5^--vertices with have v as their unique big neighbor we have either (i) at least $\max\{1, d(v) - \Delta + 8889\}$ 2-vertices or (ii) at least $\max\{1, d(v) - \Delta + 17655\}$ 3^--vertices or (iii) at least $\max\{1, d(v) - \Delta + 26401\}$ 4^--vertices or (iv) at least $\max\{1, d(v) - \Delta + 35137\}$ 5^--vertices; or

(RC3) a big vertex v such that $n_5 + 2n_6 \le 35$, where n_5 and n_6 denote the number of 5^--neighbors and 6^+-neighbors of v that are in no bunch with v as a parent; or

(RC4) a very big vertex v such that $n_5 + 2n_6 \le 141415$, where n_5 and n_6 denote the number of 5^--neighbors and 6^+-neighbors of v that are in no bunch with v as a parent.

Proof. We use discharging, assigning $d(v) - 6$ to each vertex v and $2\ell(f) - 6$ to each face f. By Euler's formula, the sum of these charges is -12. We assume that G contains none of the four configurations and redistribute charge so that each vertex and face ends with nonnegative charge, a contradiction. We use the following three discharging rules.

(R1) Let v be a 5^--vertex. If v has a single big neighbor w, then v takes $6 - d(v)$ from w. If v is in a bunch, then v takes 1 from each parent of the bunch. If v has exactly two big neighbors, and they are not its parents in a bunch, then v takes $\frac{1}{2}$ from each of these big neighbors.

(R2) Let v be a 5^--vertex with a big neighbor w, and let vw lie on a face f. If $\ell(f) = 4$, then v takes 1 from f. If $\ell(f) \geq 5$ and v has a second big neighbor along f, then v takes 2 from f. Otherwise, if $\ell(f) \geq 5$, then v takes 1 from f.

(R3) Every 5^--vertex on a 3-face with two big neighbors takes 2 from a central "bank"; each big vertex gives 12 to the bank.

If a vertex or face ends with nonnegative charge, then it ends *happy*. We show that each vertex and face (and the bank) ends happy. Let V_{big} denote the set of big vertices. The number of 5^--vertices that take 2 from the bank is at most $2|E(G[V_{big}])|$. Since $G[V_{big}]$ is planar, $|E(G[V_{big}])| < 3|V_{big}|$. So the bank ends happy, since it receives $12|V_{big}|$ and gives away less than this.

Consider a face f.

1. $\ell(f) \geq 6$. Rather than sending charge as in (R2), suppose that f sends 1 to each incident vertex, and then each big incident vertex sends 1 to its successor (in clockwise direction) around f. Now each 5^--vertex incident to f receives at least as much as in (R2), and f ends happy since $2\ell(f) - 6 - \ell(f) \geq 0$.

2. $\ell(f) = 5$. If f sends charge to at most two incident vertices, then f ends happy, since $2\ell(f) - 6 - 2(2) = 0$. So suppose f sends charge to at least three incident vertices. Now two of these receive only 1 from f. So f again ends happy, since $2\ell(f) - 6 - 2 - 2(1) = 0$.

3. $\ell(f) = 4$. Because f sends charge to at most two incident vertices, it ends happy, since $2(4) - 6 - 2(1) = 0$.

4. $\ell(f) = 3$. Now f ends happy, since it starts and ends with 0.

Now we consider vertices. A 5^--vertex with no big neighbor would satisfy (RC1), so each 5^--vertex has at least one big neighbor. Since G is 2-connected, it has minimum degree at least 2.

1. $d(v) = 2$. If v has only one big neighbor, w, then v receives 4 from w, so v finishes with $2 - 6 + 4 = 0$. So assume that v has two big neighbors, w_1 and w_2. Since G is 2-connected, the path w_1vw_2 lies on two (distinct) faces. If one of these is a 3-face, then v takes 2 from the bank, at least 1 from its other incident 4^+-face, and $\frac{1}{2}$ from each big neighbor; so v ends happy, since $2 - 6 + 2 + 1 + 2(\frac{1}{2}) = 0$. If one incident face is a 5^+-face, then v takes 2 from it, again ending happy. So assume that both incident faces are 4-faces. Now v is in a bunch with its two big neighbors, so takes 1 from each. Thus v ends with $2 - 6 + 2(1) + 2(1) = 0$.

2. $d(v) = 3$. If v has only one big neighbor, then it gives 3 to v, and v finishes with $3 - 6 + 3 = 0$. If v has three big neighbors, then for each incident face f, either v takes 1 from f or two from the bank, and v ends happy. So assume v has exactly two big neighbors, w_1 and w_2. If w_1vw_2 lies on a 3-face, then v takes 2 from the bank and $\frac{1}{2}$ from each w_i, ending happy, since $3 - 6 + 2 + 2(\frac{1}{2}) = 0$. So assume w_1vw_2 lies on a 4^+-face. If it lies on a 5^+-face, or if v lies on two 4^+-faces, then v receives at least 2 from its incident faces and $\frac{1}{2}$ from each w_i, again ending happy. So assume v lies on a 4-face with w_1 and w_2 and also on two 3-faces. Now v is in a bunch with w_1 and w_2, so takes 1 from each. Thus, v ends happy, since $3 - 6 + 1 + 2(1) = 0$.

3. $d(v) = 4$. If v has only one big neighbor, w, then v receives 2 from w, and v finishes with $4-6+2 = 0$. Suppose v has at least three big neighbors. So v has two big neighbors along at least two incident faces, f_1 and f_2. If either f_i is a 3-face, then v takes 2 from the bank and ends happy. Otherwise v takes at least 1 from each of f_1 and f_2, so ends happy. So assume that v has exactly two big neighbors, w_1 and w_2. Suppose that vw_1 and vw_2 are incident to the same face f. If f is a 3-face, then v takes 2 from the bank and ends happy. If f is a 4^+-face, then v takes at least 1 from f and at least $\frac{1}{2}$ from each big neighbor, ending happy since $4-6+1+2(\frac{1}{2}) = 0$. So assume that w_1 and w_2 do not appear consecutively among the neighbors of v. If v is incident to any 4^+-face f, then v takes at least 1 from f and $\frac{1}{2}$ from each of its big neighbors. Thus, we assume that v lies on four 3-faces. Now v is in a bunch with w_1 and w_2, so takes 1 from each, and ends happy.

4. $d(v) = 5$. If v has only one big neighbor, w, then v receives 1 from w, and ends happy. If v has exactly two big neighbors, w_1 and w_2, then v receives $\frac{1}{2}$ from each, ending with $5 - 6 + 2(\frac{1}{2}) = 0$. So assume that v has at least three big neighbors. By the Pigeonhole principle, v lies on at least one face, f, with two big neighbors. So v receives at least 1 from either f or from the bank. Thus, v finishes with at least $5 - 6 + 1 = 0$.

5. $d(v) \geq 6$, but v is not big. Now v ends happy, since $d(v) - 6 \geq 0$.

6. v is a big vertex but not a very big vertex. Suppose that v has a 5^--neighbor w such that v is the only big neighbor of w. Now $\sum_{x\in N(w)} d(x) \leq 4(8680) + d(v) \leq 4(8680) + (\Delta - 4(8680)) = \Delta \leq k$. Thus, w is an instance of (RC1), a contradiction. So v has no such 5^--neighbor. As a result, v sends at most 1 to each of its neighbors. Since G has no instance of (RC3), we have $n_5 + 2n_6 \geq 36$ (where n_5 and n_6 are defined as in (RC3)). Note that v sends at most $\frac{1}{2}$ to each vertex counted by n_5 and sends no charge to each vertex counted by n_6. Further, v sends at most 1 to each other neighbor. Also, v sends 12 to the bank. So v finishes with at least $d(v) - 6 - 12 - \frac{1}{2}n_5 - 1(d(v) - n_5 - n_6) = -18 + \frac{1}{2}(n_5 + 2n_6) \geq -18 + \frac{1}{2}(36) = 0$.

7. v is a very big vertex. Let $\mathcal{W}$ denote the set of 5^--vertices w for which v is the only big neighbor of w. Since G has no instance of (RC2), the numbers of 2-vertices, 3^--vertices, 4^--vertices, and 5^--vertices in $\mathcal{W}$ are (respectively) at most 8888, 17654, 26400, and 35136. So the total charge that v sends to these vertices is at most $8888+17654+26400+35136 = 88258$. Since G has no copy of (RC4), we have $n_5 + 2n_6 \geq 141416$. If w is counted by n_5 and is not in $\mathcal{W}$, then v sends w at most $\frac{1}{2}$. If w is counted by n_6, then v sends w nothing. So v ends happy, since $d(v)-6-12-88258-\frac{1}{2}(n_5-35136)-1(d(v)-n_5-n_6) = -18-88258+\frac{1}{2}n_5+\frac{1}{2}(35136)+n_6 = -70708 + \frac{1}{2}(n_5 + 2n_6) \geq -70708 + \frac{1}{2}(141416) = 0$. □

3 Reducibility

In this section we use the Structural Lemma to prove the Main Theorem (its second statement follows immediately from its first, so we prove the first). Throughout, we assume the Main Theorem is false and let G be a counterexample with the fewest vertices. Let $k = \max\{\Delta, 4.2 * 10^{14}\}$. We must show that $\chi'_a(G) \leq k$. In Lemma 1, we show that G is 2-connected, so we can apply the Structural Lemma to G. Thus, it suffices to show that G contains none of (RC1), (RC2), (RC3), and (RC4). Lemma 1 forbids (RC1), and Lemma 2 and Corollary 3 forbid (RC2). For (RC3) the argument is longer, so we pull out a key piece of it as Lemma 4, before finishing the proof in Lemma 5. Finally, we handle (RC4) in Lemma 6, using a proof similar to that of Lemma 5.

Lemma 1. *Let G be a minimal counterexample to the Main Theorem. Now G is 2-connected and has no instance of configuration (RC1). That is, every vertex v has $\sum_{w\in N(v)} d(w) > k$. In particular, every 5^--vertex has a big neighbor.*

Proof. Let G be a minimal counterexample. Note that G is connected, since otherwise one of its components is a smaller counterexample. Suppose G has a cut-vertex v, and let $G_1, G_2, \dots$ denote the components of $G - v$. For each i, let $H_i = G[V(G_i) \cup \{v\}]$, the subgraph formed from G_i by adding all edges between v and $V(G_i)$. By minimality, each G_i has a good coloring, say φ_i. By permuting colors, we can assume that the sets of colors seen by v in the distinct φ_i are disjoint. Now identifying the copies of v in each H_i gives a good coloring of G, a contradiction. Thus, G must be 2-connected.

Suppose that G has a vertex v such that $\sum_{w\in N(v)} d(w) \le k$. By minimality, $G - v$ has a good coloring φ. We greedily extend φ to each edge incident to v. We color these edges with distinct colors that do not already appear on some edge incident to a vertex w in $N(v)$. This is possible precisely because $\sum_{w\in N(v)} d(w) \le k$. Since each color seen by v is seen by only one neighbor of v, the resulting extension of φ is proper and has no 2-colored cycle containing v; thus, it is acyclic. This contradiction shows that $\sum_{w\in N(v)} d(w) > k$ for every vertex v. Finally, suppose some 5^--vertex v contradicts the final statement of the lemma. Now $\sum_{w\in N(v)} d(w) \le d(v)(8680) \le 5(8680) \le k$, a contradiction. Thus, the lemma is true. □

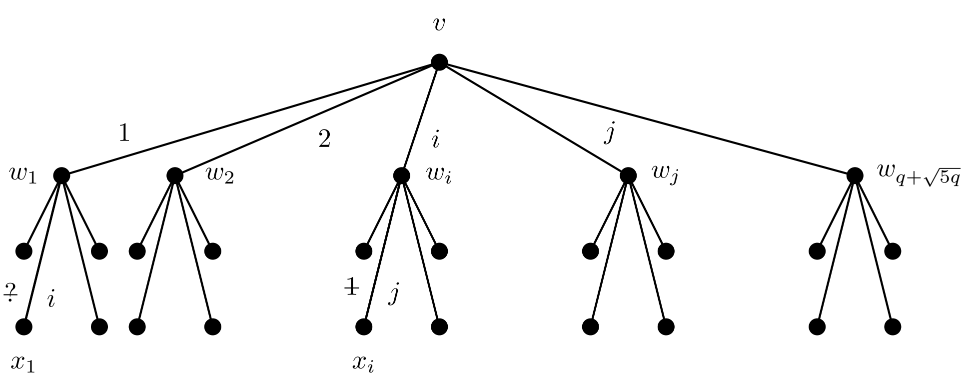

Figure 2: A big vertex v and its set $\mathcal{W}$ of 5^--neighbors with v as their unique big neighbor, as in Lemma 2.

Lemma 2. *Fix an integer q such that $q \ge 100$. Now G cannot have a vertex v such that $d(v) - \Delta + |\mathcal{W}| \ge q + \sqrt{5q}$, where $\mathcal{W}$ is the set of 5^--neighbors w of v such that $\sum_{x\in N(w)\setminus v} d(x) \le q$ and $\mathcal{W}$ is nonempty.*

Proof. Suppose the lemma is false, and that q, G, and v witness this. Let $\mathcal{W}$ be the set of these neighbors of v; see Figure 2 for an example. Pick an arbitrary $w_1 \in \mathcal{W}$ (here we use that $\mathcal{W}$ is non-empty). By minimality, $G - w_1$ has a good coloring, φ. We can greedily extend this coloring to $G - w_1 + vw_1$ (and we still call it φ). Let $w_1, w_2, \dots$ denote the vertices of $\mathcal{W}$. Let $\mathcal{S}$ be the set of colors either not used incident to v or else used on an edge from v to a neighbor in $\mathcal{W}$. For each neighbor w_i, by symmetry we assume that $\varphi(vw_i) = i$. For each w_i, let $\mathcal{C}_i$ be the set of colors used on edges incident to vertices in $N(w_i) \setminus v$. For each i, let $\mathcal{S}_i = \mathcal{S} \setminus \mathcal{C}_i$. Set $\mathcal{S}_i$ contains the colors that are potentially *safe* to use on an edge incident to w_i, as we explain below.

Let x_1 be an arbitrary neighbor in G of w_1, other than v. We now show how to extend the good coloring to w_1x_1. This will complete the proof, since the same argument can be repeated to extend the coloring to each other uncolored edge of G incident to w_1.

If we color w_1x_1 with any $i \in \mathcal{S}_1$, then any 2-colored cycle we create must use edges x_1w_1, w_1v, vw_i. Such a cycle is only possible if w_i sees color 1. So we assume w_i sees color 1, for every $i \in \mathcal{S}_1$ (otherwise we can extend the coloring to w_1x_1). Now for each $i \in \mathcal{S}_1$, define x_i such that $\varphi(w_ix_i) = 1$. (Note that $x_i = x_1$ for at most one value of x_i, so we can essentially ignore this case.)

Our goal is to find indices i and j such that $i \in \mathcal{S}_1$ and $j \in \mathcal{S}_i$ and w_j does not see color i. If we find such i and j, then we color w_1x_1 with i and recolor w_ix_i with j. This creates no 2-colored cycles, as we now show. Any 2-colored cycle using w_1x_1 must also use w_1v and vw_i (since $i \in \mathcal{S}_1$). Bu no such 2-colored cycle exists, since w_i no longer sees 1. Similarly, any 2-colored cycle using edge w_ix_i also uses w_iv and vw_j. Again, no such 2-colored cycle exists, since w_j does not see i.

Now we show that we can find such i and j. Suppose not. So for each $i \in \mathcal{S}_1$ and $j \in \mathcal{S}_i$ vertex w_j sees color i. Thus, among the at most $4|\mathcal{W}|$ edges incident to some w_i, but not to v, each color $i \in \mathcal{S}_1$ appears at least $|\mathcal{S}_i|$ times. Since $|\mathcal{S}_i| = |\mathcal{W}| - |\mathcal{C}_i| \geq |\mathcal{W}| - q$, we get $(|\mathcal{W}| - q)^2 \leq 4|\mathcal{W}|$. Solving this quadratic gives $|\mathcal{W}| \leq q + 2 + \sqrt{4q+4}$. But this quantity is less than $q + \sqrt{5q}$ when $q > 80$, a contradiction. □

Corollary 3. *Configuration (RC2) cannot appear in a minimal counterexample G. That is, G has no big vertex v such that among those 5^--vertices with v as their unique big neighbor we have either (i) at least* $\max\{1, d(v) - \Delta + 8889\}$ *2-vertices or (ii) at least* $\max\{1, d(v) - \Delta + 17655\}$ 3^-*-vertices or (iii) at least* $\max\{1, d(v) - \Delta + 26401\}$ 4^-*-vertices or (iv) at least* $\max\{1, d(v) - \Delta + 35137\}$ 5^-*-vertices.*

Proof. This is a direct application of the previous lemma. Each 5^--vertex w with v as its only big neighbor has $\sum_{x \in d(w) \setminus v} d(x) \leq (d(w) - 1)8680$. Thus, for 2-vertices, 3-vertices, 4-vertices, and 5-vertices, the sums are (respectively) at most 8680, 17360, 26040, and 34720. Now we are done, since $8680 + \sqrt{5(8680)} \leq 8889$; $17360 + \sqrt{5(17360)} \leq 17655$; $26040 + \sqrt{5(26040)} \leq 26401$; and $34720 + \sqrt{5(34720)} \leq 35137$. □

For a bunch B in a graph G, form G_B from G by deleting all horizontal edges of B (recall that this does not delete x_0x_1 and x_tx_{t+1}). Now B is *long* if, given any integer $k \geq 13$ and any acyclic k-edge-coloring of G_B, there exists an acyclic k-edge-coloring of G. Long bunches are crucial in our proofs that (RC3) and (RC4) cannot appear in a minimal counterexample to the Main Theorem.

Lemma 4. *In every planar graph, every bunch of length at least 11 is long.*

Proof. Consider a graph G with a bunch, B, of length at least 11. Fix an integer $k \geq 13$. By assumption, G_B has an acyclic k-edge-coloring; see Figure 3 for an example. Let v and w be the parents of the bunch and let $x_1, \ldots, x_t$ denote its vertices. We will reorder the threads of B so that (for each $i \in [t-1]$) no color appears incident to both x_i and x_{i+1}. (Technically, we reorder the pairs of colors on the edges vx_i and x_iw, while preserving, in each pair, which color is incident to v and which is incident to w; but this minor distinction will not trouble us.) We also require that the colors seen by x_2 not appear on edge x_0x_1 and, similarly, the colors seen by x_{t-1} not appear on

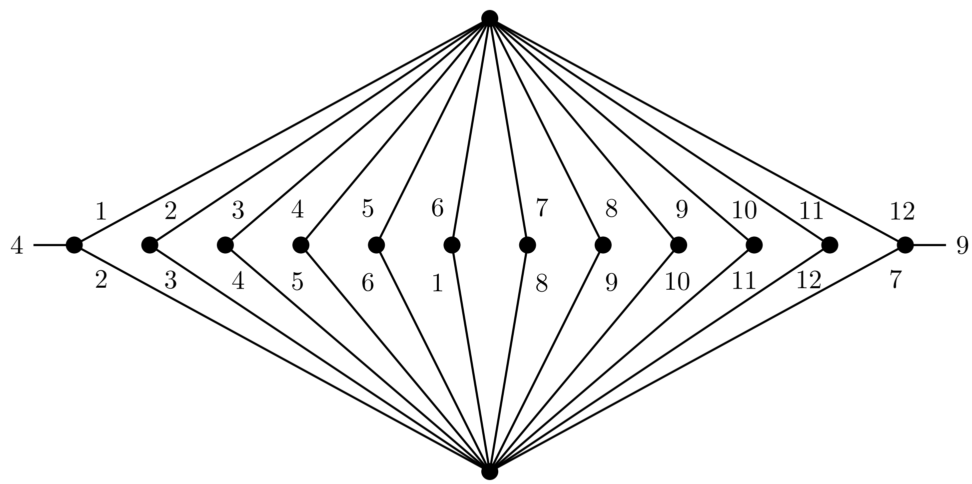

Figure 3: An acyclic edge-coloring of G_B, restricted to the edges incident to bunch vertices of B, where B is a bunch of length 12.

edge $x_t x_{t+1}$. If we can reorder the threads to achieve this property, then it is easy to extend the k-edge-coloring to G, as follows.

We greedily color the horizontal edges in any order, requiring that the color used on $x_i x_{i+1}$ not appear on any (colored) edge incident to x_{i-1}, x_i, x_{i+1}, or x_{i+2}. Each of these vertices has two incident edges on a thread, for a total of 8 edges. We must also avoid the colors on at most 4 horizontal edges. Thus, at most 12 colors are forbidden. Since $k \geq 13$, we greedily complete the coloring. Given an acyclic k-edge-coloring of G_B, suppose that we reorder the threads of G_B and greedily extend the coloring to the horizontal edges of B. Call the resulting k-edge-coloring φ. Clearly, φ is a proper edge-coloring. We must also show that it has no 2-colored cycles. Suppose, to the contrary, that φ has a 2-colored cycle, C. By the condition on our ordering of the threads of B, cycle C must use at least two successive horizontal edges of B. But now one of these horizontal edges $x_i x_{i+1}$ of C must share a color with an edge incident to x_{i-1} or x_{i+2}, a contradiction. Thus, φ is an acyclic k-edge-coloring of G, as desired. Hence, it suffices to show that we can reorder the threads of B so that no color appears incident to both x_i and x_{i+1}.

For each $i \in [t]$, we think of putting some thread vx_jw into position i (where also $j \in [t]$). We always put thread 1 into position 1 and thread t into position t. We will also initially put threads into the positions with i odd. Let $\mathcal{O}$ be the set of threads that we put in the odd positions (and thread t, whether or not t is odd); $\mathcal{O}$ is for odd. Note that $|\mathcal{O}| = \lceil (t+1)/2 \rceil$. Later, we put threads into the even positions. To do so, after putting threads into the odd positions, we build a bipartite graph, $H(B, \mathcal{O})$, where the vertices of one part are the even numbered positions (excluding t) and the vertices of the other part are those threads not yet placed. We add an edge between a thread vx_iw and a position j if no color used on the thread is also used on a thread already in position $j-1$ or $j+1$, or used on x_0x_1 when $j = 2$, or on $x_t x_{t+1}$ when $j = t-1$; see Figure 6 for an example. (The notation $H(B, \mathcal{O})$ is slightly misleading, since the edges of this graph depend not only on our choice of $\mathcal{O}$, but also on which threads we put where.) Thus, to place the remaining threads, it suffices to find a perfect matching in $H(B, \mathcal{O})$. When $t \geq 22$, we can put threads into the odd positions essentially arbitrarily, and we are guaranteed a perfect matching in $H(B, \mathcal{O})$ by a straightforward application of Hall's Theorem (this approach allows us to complete the proof, but requires that we replace $4.2*10^{14}$ with a larger constant). For smaller t, we use a similar approach, but need more detailed case analysis.

We build a *conflict graph*, B_{conf}, which has as its vertices the threads of B, that is, vx_iw, for all $i \in [t]$. Two vertices are adjacent in B_{conf} if their corresponding threads share a common color; see Figure 4 for an example. Note that B_{conf} is a disjoint union of paths and cycles, since every edge in a thread of B is incident to either v or w. We refer interchangeably to a thread and its corresponding vertex in B_{conf}. To form $\mathcal{O}$ we start with an empty set and repeatedly add vertices, subject to the following condition. Each component of B_{conf} with a vertex in $\mathcal{O}$ must have all of its vertices in $\mathcal{O}$, except for at most one component; if such a component exists, then its vertices that are in $\mathcal{O}$ must induce a path. Thus, at most two threads in $\mathcal{O}$ have neighbors in B_{conf} that are not in $\mathcal{O}$ (and if exactly two, then each has at most one such neighbor).

First suppose that threads 1 and t are in different components of B_{conf}. We begin by putting into $\mathcal{O}$ all threads in the smaller of these components, and then proceed to the other component, beginning with the thread in $\{1, t\}$. If threads 1 and t are in the same component of B_{conf}, then we start by putting into $\mathcal{O}$ all vertices on a shortest path in B_{conf} from 1 to t, and thereafter continue growing arbitrarily, such that when the set reaches size $\lceil (t+1)/2 \rceil$ it satisfies the desired property. The only exception is if the shortest path from 1 to t has more than $\lceil (t+1)/2 \rceil$ vertices. In this case the component of B_{conf} is a path; now we add a single edge in B_{conf} joining its endpoints, and proceed as above, which allows us to take a shorter path from 1 to t, including the edge we just added. Thus, we have constructed the desired $\mathcal{O}$.

Now we describe how to place the threads of $\mathcal{O}$ in the odd positions. (See Figure 5 for an example of these threads in position, and Figure 6 for the resulting graph $H(B, \mathcal{O})$.) Recall that at most two threads in $\mathcal{O}$ have neighbors in B_{conf} that are not in $\mathcal{O}$ (and if exactly two, then each has at most one such neighbor). Let r denote the size of each part in $H(B, \mathcal{O})$. Since $|\mathcal{O}| = \lceil (t+1)/2 \rceil$ and $t \geq 11$, we get that $r = \lfloor (t-1)/2 \rfloor \geq 5$.

Suppose that 1 and t are the two threads in $\mathcal{O}$ with neighbors in B_{conf} that are not in $\mathcal{O}$. We put threads 1 and t in their positions and we put the other threads of $\mathcal{O}$ in the odd positions arbitrarily, except that if t is even, then we pick a thread for position $t-1$ that does not conflict with thread t and does not conflict with the color on $x_t x_{t+1}$ (if it exists); this is easy, since $t \geq 11$. Now we must put the remaining threads into the even positions. At most three threads are forbidden from position 2, since at most one thread has a color used on thread 1 and at most two threads have colors used on $x_0 x_1$. Similarly, at most three threads are forbidden from position $t-1$. For all other positions, no threads are forbidden. Positions 2 and $t-1$ have degree at least $r-3 \geq 2$ in $H(B, \mathcal{O})$ and all other slots have degree r. Thus, by Hall's Theorem, $H(B, \mathcal{O})$ has a perfect matching. We now use similar arguments to handle the other possibilities for which vertices of $\mathcal{O}$ have neighbors

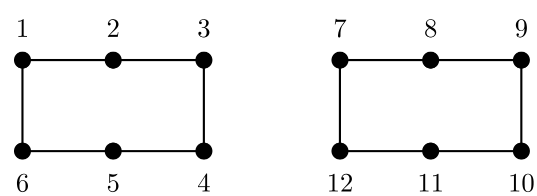

Figure 4: The conflict graph, B_{conf}. We label each vertex with the color the thread uses on the edge to its parent "above". Applying our algorithm to this instance of B_{conf} yields $\mathcal{O} = \{1, 7, 8, 9, 10, 11, 12\}$.

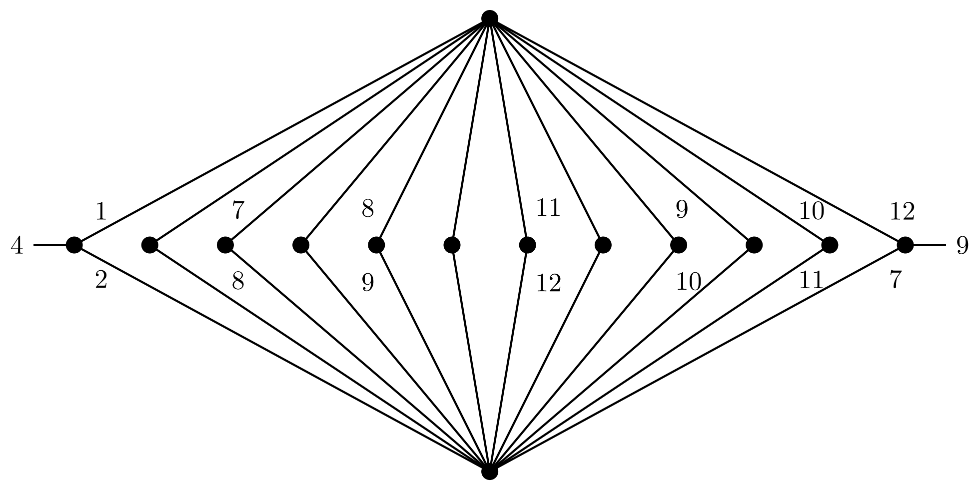

Figure 5: A partial acyclic edge-coloring of G_B, with the threads in $\mathcal{O}$ in position.

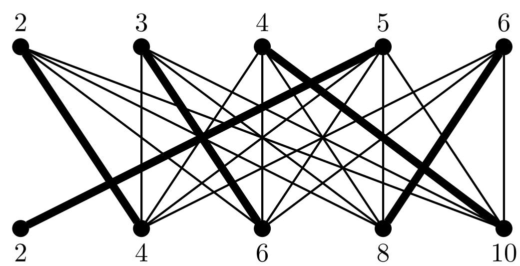

Figure 6: The auxiliary graph $H(B, \mathcal{O})$, with threads on top and positions on bottom, and a perfect matching shown in bold.

in B_{conf} that are not in $\mathcal{O}$.

Suppose that exactly one of threads 1 and t has a neighbor in B_{conf} that is not in $\mathcal{O}$. By symmetry, assume that it is 1. Further, assume that also $i \in \mathcal{O}$ and thread i has a neighbor in B_{conf} that is not in $\mathcal{O}$ (the case when no such i exists is easier). If t is odd, then we put thread i in position $t-2$, and fill the remaining odd positions arbitrarily from $\mathcal{O}$. If t is even, then we put thread i in position $t-2$, and fill odd positions 3 through $t-3$ arbitrarily from $\mathcal{O}$, except that we require that the thread in position $t-3$ not conflict with that in position $t-2$ (this is possible, since at most two threads in $\mathcal{O}$ conflict with thread i, and $|\mathcal{O}| \geq 7$). Again, we use Hall's Theorem to show that $H(B, \mathcal{O})$ has a perfect matching. Now positions 2 and $t-1$ each have degree at least $r-3 \geq 2$, and position $t-3$ has degree at least $r-1 \geq 4$. All other positions have degree r.

Suppose that one of threads 1 and t has two neighbors in B_{conf} that are not in $\mathcal{O}$, and the other has no such neighbors. (This will happen when B_{conf} consists of two cycles, each of length $t/2$.) By symmetry, assume that thread 1 has two neighbors in B_{conf} that are not in $\mathcal{O}$. We fill the odd positions arbitrarily with threads from $\mathcal{O}$ (here, and in the remaining cases, if t is even, then we also require that the thread in position $t-1$ not conflict thread t or with the color on $x_t x_{t+1}$). In $H(B, \mathcal{O})$, position 2 has degree at least $r-4 \geq 1$. Also, position $t-1$ has degree at least $r-2 \geq 3$. All other positions have degree r. So $H(B, \mathcal{O})$ has a perfect matching.

Now we can assume that neither of threads 1 and t has neighbors in B_{conf} that are not in $\mathcal{O}$.

Suppose that some thread, say i, in $\mathcal{O}$ has two neighbors in B_{conf} that are not in $\mathcal{O}$. We put thread i in position 3 and fill the remaining odd positions arbitrarily from $\mathcal{O}$. Position 2 has degree at least $r-4 \geq 1$, and position 4 has degree at least $r-2 \geq 3$. If t is odd, then position $t-1$ has degree at least $r-2 \geq 3$. All other positions have degree r. So $H(B, \mathcal{O})$ has a perfect matching.

Finally, suppose that two threads, i and j (neither of which is 1 or t), each have a neighbor in B_{conf} that is not in $\mathcal{O}$. Now we put thread i in position 3 and thread j in position 5, and fill the remaining odd positions from the rest of $\mathcal{O}$. Between them, threads i and j forbid at most two threads from position 4 and at most one thread each from positions 2 and 6. Thus, position 2 has degree at least $r-3 \geq 2$, position 4 has degree at least $r-2 \geq 3$, and position 6 has degree at least $r-1 \geq 4$. Once again $H(B, \mathcal{O})$ has a perfect matching. □

Lemma 5. *Configuration (RC3) cannot appear in a minimal counterexample G. That is, G cannot contain a big vertex v such that $n_5 + 2n_6 \leq 35$, where n_5 and n_6 denote the numbers of 5^--neighbors and 6^+-neighbors of v that are in no bunch with v as a parent.*

Proof. Suppose G is a minimal counterexample that contains such a vertex v. Form G' from G by deleting all horizontal edges of long bunches for which v is a parent. It suffices to find a good coloring of G' since, by definition, we can extend it to G. Let B be the longest bunch that has v as a parent, and let w be the other parent of this bunch. Let x be a bunch vertex in B. By minimality, we have a good coloring of $G' - x$; we can greedily extend this to $G' - x + wx$, and we call this coloring φ. We construct a set of colors $\mathcal{C}_{good}(v)$ as follows. Remove from $[k]$ every color that is used on an edge incident to v leading to a vertex that is not a bunch vertex of some bunch with v as a parent. Further, for each such color α used on an edge vp we do the following. Remove either (i) all other colors used incident to p or (ii) every colors used on an edge vu, whenever u is a 2-vertex incident to an edge colored with α; for each color α, we pick either (i) or (ii), giving preference to the option that removes fewer colors. Finally, we remove all colors used on edges incident to v that are in short bunches. This completes the construction of $\mathcal{C}_{good}(v)$. Starting from φ, we remove all

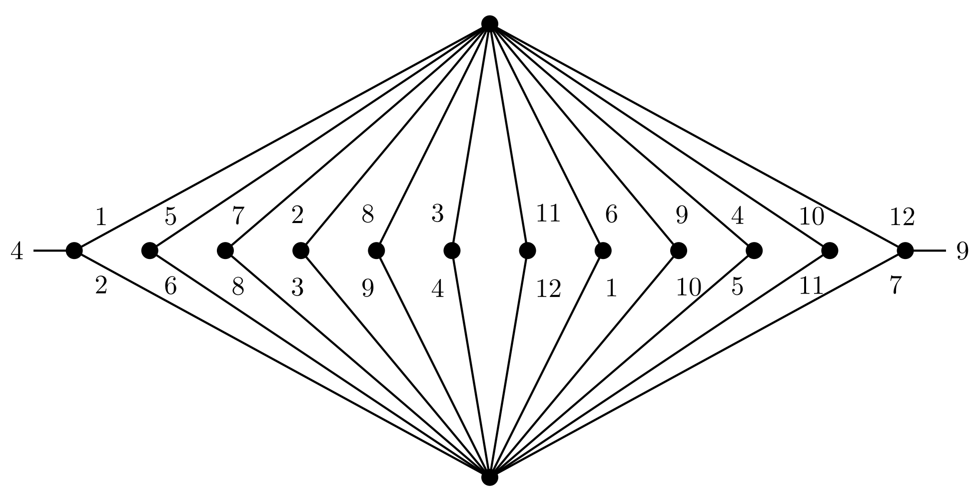

Figure 7: The desired acyclic edge-coloring of G_B, ready to be extended greedily to the horizontal edges of B.

colors in $\mathcal{C}_{good}(v)$ that are used on edges incident to v. We will gradually recolor all of these edges, as well as vx (first with a proper coloring, and eventually with an acyclic coloring).

Suppose that $\varphi(wx)$ is already used on some edge vy in bunch B. To avoid creating any 2-colored cycles through x, it suffices to color vx with any color in $\mathcal{C}_{good}(v) \setminus \{\varphi(wx), \varphi(wy)\}$, which is easy. So assume $\varphi(wx)$ is not used on any edge vy in B. (The hardest case is when $\varphi(wx)$ is used on some edge incident to v leading to a non-bunch vertex. This case motivates most of our effort, so the reader will do well to keep it in mind.) Our goal is to find some color, say α, other than $\varphi(wx)$, such that $\alpha \in \mathcal{C}_{good}(v)$ and α is already used on an edge wy of B. Given such an α, we use it to color vx, and color vy with some color in $\mathcal{C}_{good}(v) \setminus \{\varphi(wx), \alpha\}$. This ensures that each of vx and wx will never appear in a 2-colored cycle, no matter how we further extend the coloring. Such an α exists by the Pigeonhole principle, because $\text{length}(B) + |\mathcal{C}_{good}(v)| \geq k+2$. We defer the computation proving this to the end of the proof. Now we greedily extend our coloring to a proper (not necessarily acyclic) k-edge-coloring of G', essentially assigning the colors of $\mathcal{C}_{good}(v)$ to the uncolored edges arbitrarily. We only require that the final two edges we color are in some bunch with at least two uncolored edges incident to v. (For each edge the only forbidden color is the one already used incident to the endpoint of degree 2.) If we get stuck on the final edge, then we can backtrack slightly and complete the coloring.

Now we modify this proper edge-coloring to make it acyclic. It is important to note that any 2-colored cycle must pass through v. Further, it must use some edges e_1, e_2, e_3, e_4, where v is the common endpoint of e_2 and e_3 and the common endpoints of edges e_1 and e_2 and of edges e_3 and e_4 are both 2-vertices (this follows from our construction of $\mathcal{C}_{good}$). Suppose that such a 2-colored cycle exits, say with colors β_1, β_2. One of these colors must be in $\mathcal{C}_{good}(v)$, since the 2-colored cycle did not exist before assigning these colors; say it is β_1. Suppose that a second such 2-colored cycle exists, with colors γ_1, γ_2; by symmetry, assume that $\gamma_1 \in \mathcal{C}_{good}$. To fix both cycles, we swap colors β_1 and γ_1 on the edges incident to v where they are used. We repeat this process until we have only at most one 2-colored cycle through v. Suppose we have one, with edges colored β_1, β_2 (and $\beta_1 \in \mathcal{C}_{good}$); when we state the colors on edges of a thread, we always start with the edge incident to v. Now we look for some other thread with edges colored γ_1, γ_2 (and $\gamma_1 \in \mathcal{C}_{good}(v)$) such that no thread incident to v has edges colored γ_2, β_1. If we find such a thread, then we swap colors β_1 and γ_1 on the edges incident to v where they appear, and this fixes the 2-colored cycle. Since v is a parent in at most 35 bunches, at most 35 incident threads have edges colored γ_2, β_1, for some choice of γ_2. Further, for each choice of γ_2, v has at most 35 incident threads colored γ_1, γ_2, for some choice of γ_1. Thus, at most $35^2 = 1225$ of these threads are forbidden. However, it is straightforward to check that $k - |\mathcal{C}_{good}(v)| \leq 2000$ (in fact the difference is much smaller, but this is unimportant). Now we have the desired thread incident to v since $d(v) - 1225 - 2000 > 0$. Thus, we can recolor the edge colored β_1 to get an acyclic edge-coloring of G', as desired.

Now we prove that $\text{length}(B) + |\mathcal{C}_{good}(v)| \geq k+2$. Note that $k - |C_{good}(v)| + 2 \leq 5n_5 + n_6(n_5 + n_6 + 1 - s) + 10s + 2$, where s is the number of short bunches with v as a parent. This is because each short bunch causes us to remove at most 10 colors, each vertex counted by n_5 causes us to remove at most 5 colors, and each counted by n_6 causes us to remove at most $n_5 + n_6 + 1 - s$ colors. We must show that the right side of the latter inequality is at most $\text{length}(B)$. In fact, we will show that it is no more than the average length of the long bunches (rounded up). Since the number of bunches is at most $n_5 + n_6$, we want

$$\frac{d(v) - (5n_5 + n_6(n_5 + n_6 + 1 - s) + 10s)}{n_5 + n_6 - s} > 5n_5 + n_6(n_5 + n_6 + 1 - s) + 10s) + 1,$$

which is implied by

$$d(v) \geq (5n_5 + n_6(n_5 + n_6 + 1 - s) + 10s + 1)(n_5 + n_6 - s + 1).$$

Since $n_5 + 2n_6 \leq 35$, it suffices to have

$$d(v) \geq (5(35 - 2n_6) + n_6((35 - 2n_6) + n_6 + 1 - s) + 10s + 1)(35 - n_6 - s + 1).$$

If we maximize the right side over all integers n_6 and s such that $0 \leq n_6 \leq 17$ and $0 \leq s \leq 35 - n_6$ (using nested For loops, for example), then we get 8680. $\square$

Lemma 6. *Configuration (RC4) cannot appear in a minimal counterexample G. That is, G cannot contain a very big vertex v such that $n_5 + 2n_6 \leq 141415$, where n_5 and n_6 denote the numbers of 5^--neighbors and 6^+-neighbors of v that are in no bunch with v as a parent.*

Proof. Most of the proof is identical to that of Lemma 4, that (RC3) cannot appear in a minimal counterexample. The only difference is our argument showing that $\text{length}(B) + |\mathcal{C}_{good}(v)| \geq k + 2$, which we give now. As in the previous lemma, it suffices to have

$$d(v) \geq (5n_5 + n_6(n_5 + n_6 + 1 - s) + 10s + 1)(n_5 + n_6 - s + 1).$$

By hypothesis, we have $n_5 \leq 141415 - 2n_6$. Now substituting for n_5, we get that it suffices to have

$$\begin{aligned} d(v) &\geq (5(141415 - 2n_6) + n_6((141415 - 2n_6) + n_6 + 1 - s) + 10s + 1)((141415 - 2n_6) + n_6 - s + 1) \\ &= n_6^3 + 2n_6^2 s - 282822n_6^2 + n_6 s^2 - 282832n_6 s \\ &\quad + 19996363820n_6 - 10s^2 + 707084s + 99991859616. \end{aligned} \tag{1}$$

We must upper bound the value of (1) over the region where $0 \leq n_6 \leq 70707$ and $0 \leq s \leq n_5 + n_6 - 1 \leq 141415 - n_6$. Since this domain is much larger than in the previous lemma, we relax the integrality constraints and solve a multivariable calculus problem. The only critical point for this function is outside the domain, so it suffices to find the maximum along the boundary. This occurs when $s = 0$ and $n_6 \approx 47134$; the value is approximately $4.19 * 10^{14}$. Recall that v is very big, so we have $d(v) \geq \Delta - 4(8680)$. Since we need $d(v) \geq 4.19 * 10^{14}$, it suffices to require that $\Delta \geq 4.2 * 10^{14}$. This completes the proof. $\square$

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