The Hilton–Zhao Conjecture is True for Graphs with Maximum Degree 4

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Abstract

A simple graph G is *overfull* if $|E(G)| > \Delta \lfloor |V(G)|/2 \rfloor$. By the pigeonhole principle, every overfull graph G has $\chi'(G) > \Delta$. The *core* of a graph, denoted G_Δ, is the subgraph induced by its vertices of degree Δ. Vizing's Adjacency Lemma implies that if $\chi'(G) > \Delta$, then G_Δ contains cycles. Hilton and Zhao conjectured that if G_Δ has maximum degree 2 and $\Delta \geq 4$, then $\chi'(G) > \Delta$ precisely when G is overfull. We prove this conjecture for the case $\Delta = 4$.

1 Introduction and Proof Outline

A *proper edge-coloring* of a graph G assigns colors to its edges so that edges receive distinct colors whenever they share an endpoint. The *edge-chromatic number of* G, denoted $\chi'(G)$, is the smallest number of colors that allows a proper edge-coloring of G. Vizing showed that always $\chi'(G) \leq \Delta+1$, where Δ denotes the maximum degree of G. (In this paper, all graphs are *simple*, which means that every pair of vertices is joined by either 0 or 1 edges.) Since always $\chi'(G) \geq \Delta$, we call a graph *class 1* when $\chi'(G) = \Delta$ and call it *class 2* when $\chi'(G) = \Delta + 1$.

Erdős and Wilson [4] showed that almost every graph is class 1. In contrast, Holyer [8] showed that it is NP-hard to determine whether a graph is class 1 or class 2. As a result, most work in this area focuses on proving sufficient conditions for a graph to be either class 1 or class 2. A *k-vertex* is one of degree k, and a *k^--vertex* is one of degree at most k. A *k-neighbor* (and *k^--neighbor*) of a vertex v is defined analogously. A graph G is *overfull* if $|E(G)| > \left\lfloor \frac{|V(G)|}{2} \right\rfloor \Delta$. Every overfull graph is class 2, since it has more edges than can appear in Δ color classes. A graph G is *critical* if $\chi'(G) > \Delta$ and $\chi'(G-e) = \Delta$ for every edge $e \in E(G)$. It is easy to show that every class 2 graph G contains a critical subgraph H with the same maximum degree as G. Critical graphs are useful because they have more structure than general graphs. For example, Vizing proved the following.

Vizing's Adjacency Lemma (VAL). *Let G be critical. If vertices v and w are adjacent, then w has at least* $\max\{\Delta + 1 - d(v), 2\}$ *Δ-neighbors.*

The *core* of a graph G, denoted G_Δ, is the subgraph of G induced by Δ-vertices. VAL implies that if G is class 2, then G_Δ must contain cycles (this was also proved independently by Fournier [5]). So a natural question is which class 2 graphs have a core consisting of disjoint cycles. Hilton and Zhao [7] conjectured exactly when this can happen. Let P^* denote the Peterson graph with one vertex deleted.

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Hilton–Zhao Conjecture. *If G is a connected graph with $\Delta \geq 3$ and with core of maximum degree at most 2, then G is class 2 if and only if G is P^* or G is overfull.*

David and Gianfranco Cariolaro [1] proved this conjecture when $\Delta = 3$. Kral', Sereni, and Stiebitz [9, p. 57–63] gave an alternate proof. An easy counting argument shows that every graph satisfying the hypotheses of the conjecture has average degree at most $\Delta - 1 + \frac{\Delta-1}{2\Delta-3}$; when $\Delta = 3$, this is $\frac{8}{3}$. By Lemma 1 below, any counterexample to the conjecture must be critical. Thus, the case $\Delta = 3$ is also implied by our result [3] that every critical graph with $\Delta = 3$ (other than the Petersen graph with a vertex deleted) has average degree at least $\frac{46}{17} \approx 2.706$.

In this paper, we prove the conjecture when $\Delta = 4$. Let $\mathcal{G}_k$ denote the class of graphs with maximum degree k in which the core has maximum degree at most 2; that is, each k-vertex has at most two k-neighbors. Let $\mathcal{H}_k$ denote the class of graphs G such that (i) G has maximum degree k, (ii) G has minimum degree $k-1$, (iii) G_Δ is a disjoint union of cycles, and (iv) every vertex of G has a Δ-neighbor. Note that $\mathcal{H}_k \subseteq \mathcal{G}_k$. To prove our main result, we use a lemma of Hilton and Zhao [6], which follows from Vizing's Adjacency Lemma. To keep this paper self-contained, we include a proof.

Lemma 1. *If $G \in \mathcal{G}_k$ with $k \geq 3$ and $\chi'(G) > k$, then $G \in \mathcal{H}_k$ and G is critical.*

Proof. Let G satisfy the hypotheses and let H be a k-critical subgraph of G. Suppose H has a $(k-2)^-$-vertex v. By VAL, v has a k-neighbor w. Now w has at least $k+1-d(v) \geq k+1-(k-2) = 3$ neighbors of degree k, a contradiction, since $H \in \mathcal{G}_k$. Thus, H has no $(k-2)^-$-vertex.

Suppose that $V(H) \subsetneq V(G)$. Choose $v \in V(H)$ and $w \in V(G) \setminus V(H)$ such that $w \in N_G(v)$. If $d_G(v) \leq k-1$, then $d_H(v) \leq k-2$, a contradiction. So $d_G(v) = k$. Since H is critical, v has a k-neighbor w. But now w has at most two k-neighbors in G (since $G \in G_k$), one of which is v. So w has at most one k-neighbor in H, contradicting VAL. Thus, $V(H) = V(G)$. Finally, suppose there exists $e \in E(G) \setminus E(H)$. Now either H has a $(k-2)^-$-vertex or some k-vertex in H has at most one neighbor w in H with $d_H(w) = k$; both are contradictions. Thus, $E(G) = E(H)$. So G is critical. Now VAL implies that every vertex has at least two Δ-neighbors. Hence, $G \in \mathcal{H}_k$. □

Now we can prove our Main Theorem, subject to three reducibility lemmas, which we state and prove in the next section. In short, the lemmas say that a graph in $\mathcal{H}_4$ is class 1 whenever it contains at least one of the configurations in Figure 1 (not necessarily induced).

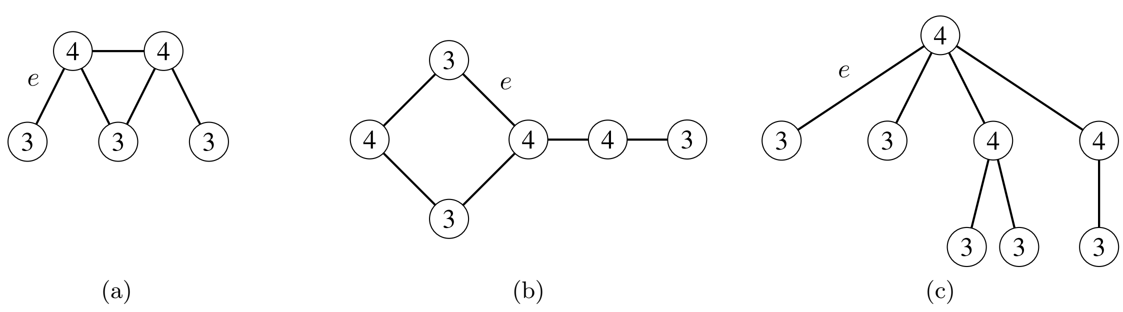


Figure 1: Each configuration cannot appear in a class 2 graph in $\mathcal{G}_4$. (The number at each vertex specifies its degree in G.)

Main Theorem. *A connected graph G with $\Delta = 4$ and with core of maximum degree at most 2 is class 2 if and only if G is $K_5 - e$. This implies the case $\Delta = 4$ of the Hilton–Zhao Conjecture.*

Proof. Let G be a graph with $\Delta = 4$ and with core of maximum degree at most 2. By Lemma 1, we assume $G \in \mathcal{H}_4$. Note that every 4-vertex in G has exactly two 3-neighbors and two 4-neighbors. Let v denote a 4-vertex and let $w_1, \ldots, w_4$ denote its neighbors, where $d(w_1) = d(w_2) = 3$ and $d(w_3) = d(w_4) = 4$. When vertices x and y are adjacent, we write $x \leftrightarrow y$. We assume that G contains no configuration in Figure 1 and show that G is $K_5 - e$.

First suppose that v has a 3-neighbor and a 4-neighbor that are adjacent. By symmetry, assume that $w_2 \leftrightarrow w_3$. Since Figure 1(a) is forbidden, we have $w_3 \leftrightarrow w_1$. Now consider w_4. If w_4 has a 3-neighbor distinct from w_1 and w_2, then we have a copy of Figure 1(b). Hence $w_4 \leftrightarrow w_1$ and $w_4 \leftrightarrow w_2$. If $w_3 \leftrightarrow w_4$, then G is $K_5 - e$. Suppose not, and let x be a 4-neighbor of w_4. Since G has no copy of Figure 1(b), x must be adjacent to w_1 and w_2. This is a contradiction, since w_1 and w_2 are 3-vertices, but now each has at least four neighbors. Hence, each of w_1 and w_2 is non-adjacent to each of w_3 and w_4.

Now consider the 3-neighbors of w_3 and w_4. If w_3 and w_4 have zero or one 3-neighbors in common, then we have a copy of Figure 1(c). Otherwise they have two 3-neighbors in common, so we have a copy of Figure 1(b). □

We first announced the Main Theorem in [2], and included the proof above. But we did not include proofs of the reducibility lemmas that we present in the next section.

2 Reducibility Lemmas

In this section we prove the reducibility of the three configurations in Figure 1. More precisely, suppose that $G \in G_4$ and G contains one of these configurations, H, as a subgraph, not necessarily induced (the number at each vertex of H denotes its degree in G). We show that $\chi'(G) = 4$. If not, then Lemma 1 implies that $G \in \mathcal{H}_4$ and G is critical. Thus, $\chi'(G - e) = 4$, where e is the edge denoted in the figure. For convenience, we write *coloring* to mean edge-coloring with colors 0, 1, 2, 3. Since $\chi'(G - e) = 4$, we begin with an arbitrary coloring φ of $G - e$. A priori, φ could restrict to many possible colorings of $H - e$. Starting from φ, we use repeated Kempe swaps (see below) to get a coloring of $G - e$ that restricts to one of a few colorings of $H - e$. We conclude by modifying the coloring of $H - e$ to transform the coloring of $G - e$ to a coloring of G. At each step, we call the current coloring φ. So, to change the color of some edge xy to i, we "let $\varphi(xy) = i$". In the figures that follow, we typically draw all edges incident to vertices of H. However, only the edges shown in Figure 1 are considered edges of H; the others are *pendant edges*.

An *(i, j)-chain at a vertex v* is the component containing v of the subgraph induced by edges colored i and j. If two vertices v and w are in the same (i, j)-chain, then v and w are *(i, j)-linked*; otherwise they are *(i, j)-unlinked*. Each (i, j)-chain P is a path or an even cycle. If P is a path that starts in $V(H)$, then P either ends in $V(H)$ or ends in $V(G) \setminus V(H)$. In the latter case, *P ends at ∞*. To *recolor* an (i, j)-chain P means to use color i on each edge colored j and vice versa (this is typically called a Kempe swap, but here we rarely use that term). Recoloring any chain in a coloring of $G - e$ yields another coloring of $G - e$. If each (i, j)-chain in $G - E(H)$ that starts in $V(H)$ ends at ∞, then we can recolor pendant edges independently, by recoloring the chain beginning with each pendant edge. If, instead, an (i, j)-chain beginning in $V(H)$ ends in $V(H)$, then its end edges (and endpoints) are paired, and recoloring one edge necessarily recolors the other. Choose $v, w \in V(H)$ that each begin an (i, j)-chain in $G - E(H)$; call the chains P_v and P_w. If P_v and P_w both end at ∞, the we can simulate that P_v ends at w, so $P_v = P_w$. To do so, whenever we recolor P_v we also recolor P_w. Thus, for any pair $(i,j) \subset \{0,1,2,3\}$, we can assume that at most one (i,j)-chain in $G - E(H)$ that starts in $V(H)$ ends at ∞.

During our process of modifying φ, we might want to recolor the (i,j)-chain P at v, but realize that this is no help if P ends at x. Similarly, we might also be happy to recolor the (i,j)-chain Q at w, but realize this also is no help if Q ends at x. Fortunately, we can make progress, since it is impossible for both P and Q to end at x. To get more control when recoloring, we frequently consider all (i,j)-chains in $G - E(H)$ that begin at vertices of $V(H)$. Now our analysis is similar, but more extensive. This approach is possible only when we know the color on every edge of $H - e$. We discuss this general technique further in [2].

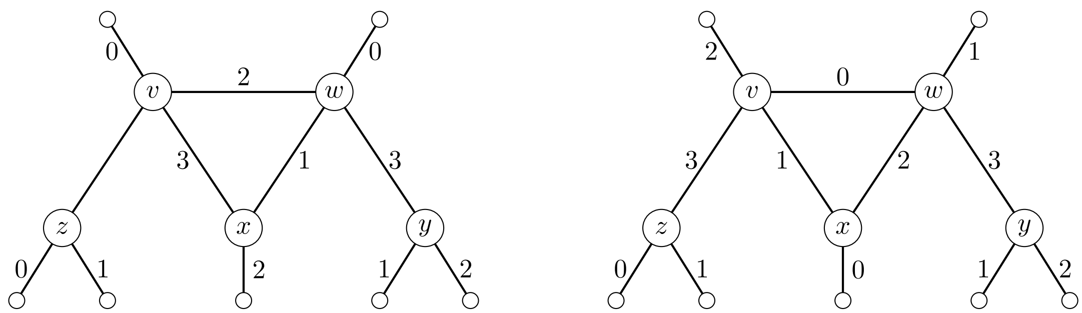


Figure 2: (a) A coloring of $G - vz$ in Case 1. (b) A coloring of G in Case 2.

Lemma 2. *Suppose that $\Delta = 4$ and G has the configuration in Figure 1(a) (reproduced in Figure 2, if we ignore the colors there). If $\chi'(G - vz) = 4$, then $\chi'(G) = 4$.*

Proof. We start with a coloring of $G - vz$ and assume that G has no coloring, which leads to a contradiction. We denote by v', w', and x' the sole unlabeled neighbors of v, w, and x, respectively. By symmetry, we assume that z sees 0 and 1, and v sees 0, 2, and 3. We repeatedly use that v and z must be (1,2)- and (1,3)-linked. We consider three cases: color 0 is used on vv', vx, or vw.

Case 1: 0 is used on vv'. Let φ be a coloring of $G - vz$, and suppose $\varphi(vv') = 0$. By symmetry, assume that $\varphi(vw) = 2$ and $\varphi(vx) = 3$, as in Figure 2(a). We show that WLOG all edges are colored as in Figure 2(a). Suppose $\varphi(wx) = 0$. Since v and z are (1,3)-linked, $\varphi(xx') = 1$. Now we (1,2)-swap at x, which makes v and z (1,3)-unlinked, a contradiction. So $\varphi(wx) = 1$. Since v and z are (1,2)-linked, $\varphi(xx') = 2$.

Suppose $\varphi(wy) = 0$, so $\varphi(ww') = 3$. Now y must see 1; otherwise a (0,1)-swap at y recolors wx with 0, and v and z become (1,3)-unlinked, a contradiction. So y misses 2 or 3. Now a (1,2)- or (1,3)-swap at y makes y miss 1, but nothing else has changed. So we are done.

So assume $\varphi(wy) = 3$ and $\varphi(ww') = 0$. Now y must see 1, since v and z are (1,3)-linked. If y misses 2, then a (1,2)-swap at y makes y miss 1, a contradiction. So y sees 2 and 1, and misses 0. Consider the (0,1)-chain P at y. P must contain either (a) $w'w, wx$ or (b) $v'v$; otherwise we (0,1)-swap at y and are done. If (a), then we (0,1)-swap at y and are done, since now v and z are (1,3)-unlinked. So assume (b). Now after a (0,1)-swap at y, let $\varphi(vx) = 0$ and $\varphi(vz) = 3$.

Case 2: 0 is used on vx. The following observation is useful. If no pendant edge uses 3 and edges vv', ww', xx' use distinct colors, then $\chi'(G) = 4$. By symmetry, assume that $\varphi(vv') = 2$, $\varphi(ww') = 1$, $\varphi(xx') = 0$, as in Figure 2(b). To extend the coloring to G, let $\varphi(vz) = 3$, $\varphi(wy) = 3$, $\varphi(vw) = 0$, $\varphi(wx) = 2$, $\varphi(vx) = 1$.

We now show that WLOG the edges are colored as in Figure 3(a). By symmetry, assume that $\varphi(vv') = 2$ and $\varphi(vw) = 3$. Note that $\varphi(wx) \in \{1,2\}$. We assume that $\varphi(wx) = 2$. Otherwise

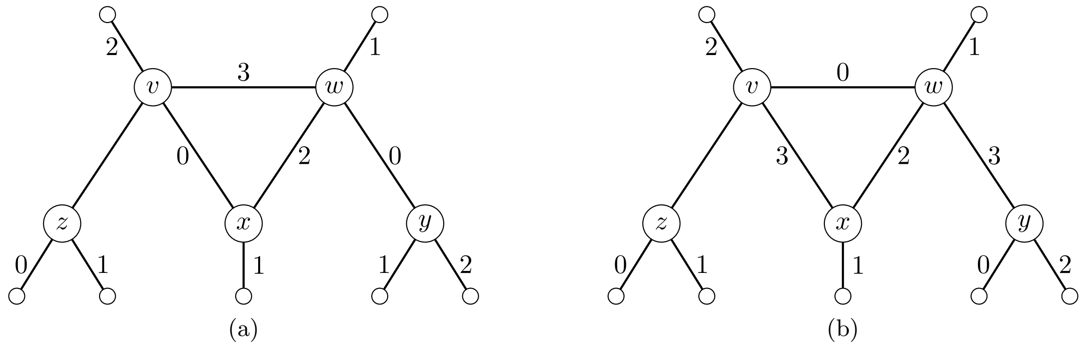


Figure 3: Two colorings of $G - vz$ in Case 2.

$\varphi(wx) = 1$ and $\varphi(xx') = 3$, so a $(1,2)$-swap at x gives $\varphi(wx) = 2$, as desired. Assume $\varphi(xx') = 3$; otherwise a $(1,3)$-swap at x yields this. Assume $\varphi(ww') = 0$ and $\varphi(wy) = 1$; otherwise a $(1,0)$-swap at w yields this. Since $\varphi(vw) = 3$ and $\varphi(wy) = 1$, vertex y must see 3. Also, y must see 2. If not, then we do a $(1,3)$-swap at x, followed by a $(1,2)$-swap at y, and the resulting $(1,3)$-chain at v ends at x. Now do a $(0,1)$-swap at y, followed by a $(1,3)$-swaps at x and y to ensure that x and y each see 1. Thus, all edges are colored as in Figure 3(a).

If a (0,1)-chain in $G - E(H)$ starts at either w or x and ends at ∞, then we are done by the observation at the start of Case 2. So we assume that the $(0,1)$-chain at y ends at ∞, and we recolor it. To maintain a coloring of $G - vz$, let $\varphi(vx) = \varphi(wy) = 3$ and $\varphi(vw) = 0$, as in Figure 3(b).

Consider the (1,2)-chains in $G - E(H)$ at v, w, x, y, z. Let P be the chain at w. If P ends at x, then we recolor it. To extend the coloring to G, let $\varphi(yw) = 1$, $\varphi(wx) = 3$, $\varphi(xv) = 1$, and $\varphi(vz) = 3$. If P instead ends at v, then we again recolor it; now the extension is the same as before, except that $\varphi(vx) = 2$. So we must consider three possibilities: the $(1,2)$-chain P at w ends at y, ends at z, or ends at ∞. In each case, we recolor P and show how to get a coloring of G.

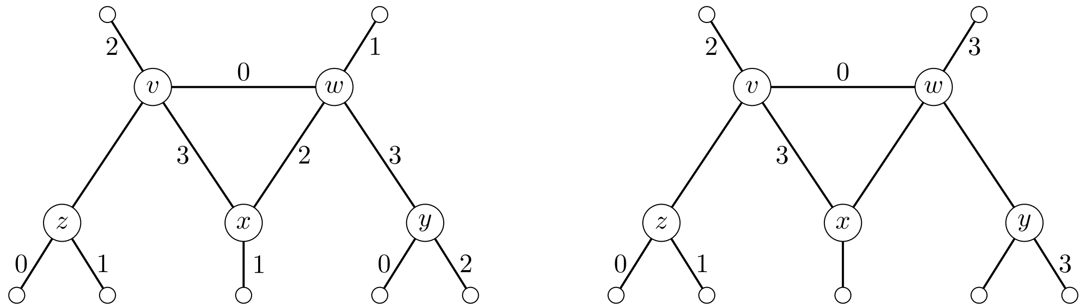
Suppose P ends at y. Recolor it, and let $\varphi(xw) = 0$ and $\varphi(wv) = 1$, to maintain a coloring of $G - vz$. Now consider the $(0,2)$-chain Q at w in $G - E(H)$. If Q ends at y or z, then we are done by the observation at the start of Case 2. So assume that Q ends at v. Thus, we assume the $(0,2)$-chain at y ends at z; now we recolor it and let $\varphi(vz) = 0$.

Suppose P ends at z. Recolor it, and again let $\varphi(xw) = 0$ and $\varphi(wv) = 1$. Now consider the $(0,1)$-chains in $G - E(H)$ that start at x, y, and z; one of them must end at ∞. If the chain at z ends at ∞, then recolor it and let $\varphi(vz) = 0$. If the chain at x ends at ∞, then recolor it, and let $\varphi(xw) = 1$, $\varphi(wv) = 0$, and $\varphi(vz) = 1$. Finally, if the chain at y ends at ∞, then recolor it, and let $\varphi(yw) = 0$, $\varphi(wx) = 3$, $\varphi(xv) = 0$, and $\varphi(vz) = 3$.

So assume P ends at ∞. As in the previous case, recolor P and let $\varphi(xw) = 0$ and $\varphi(wv) = 1$. Now consider the $(0,2)$-chains in $G - E(H)$ that start at v, w, and z; one such chain must end at ∞, so call it Q. If Q starts at v or w, then we recolor it and are done by the observation at the start of Case 2. Otherwise Q starts at z, so we recolor Q and let $\varphi(vz) = 0$. This completes Case 2.

Case 3: 0 is used on vw. By symmetry, assume that $\varphi(vv') = 2$ and $\varphi(vx) = 3$. Suppose that $\varphi(wy) = 3$, as in Figure 4(a). Now y must see 0, or else a (0,3)-swap at x reduces to Case 2. We also can assume that y sees 2 and misses 1, since v and z are (1,2)-linked. Now $\varphi(wx) \neq 1$, so $\varphi(wx) = 2$, $\varphi(ww') = 1$, $\varphi(xx') = 1$. But now we can (0,1)-swap at one of x and y without effecting vw. Afterwards, either x misses 1 or y misses 0; in both cases we are done.

So assume that $\varphi(ww') = 3$, as in Figure 4(b). Suppose that $\varphi(wx) = 2$, so $\varphi(wy) = \varphi(xx') = 1$. Now y must see 3 or else a (1,3)-swap at y takes us to the previous paragraph. Note that v and



(a) (b)

Figure 4: A coloring and a partial coloring of $G - vz$ in Case 3.

z must be (0,2)-linked, or else a (0,2)-swap at x reduces to Case 1. Thus, we can assume that y sees 2 and misses 0. However, now a (0,3)-swap at y makes y miss 3, a contradiction. So assume instead that $\varphi(wx) = 1$ and $\varphi(wy) = 2$. Also $\varphi(xx') = 2$, or else a (1,2)-swap at x makes v and z (1,3)-unlinked, a contradiction. Suppose y misses 0. Now y and z must be (0,2)-linked, or else a (0,2)-swap at y reduces to Case 2. But now a (0,2)-swap at x, followed by a (1,2)-swap at x makes v and z (1,3)-unlinked, a contradiction. Thus, y sees 0. Also, y sees exactly one of 1 and 3, and we can assume it is 3. But now a (1,2)-swap at y reduces to the case above, where $\varphi(wx) = 2$. This completes Case 3. □

Lemma 3. *Suppose that $\Delta = 4$ and G has the configuration in Figure 1(b) (reproduced in Figure 5, if we ignore the colors there). If $\chi'(G - ux) = 4$, then $\chi'(G) = 4$.*

Proof. We start with a coloring of $G - ux$ and assume that G has no coloring, which leads to a contradiction. We denote by u', w', and x' the sole unlabeled neighbors of u, w, and x, respectively. By symmetry, we assume that u sees 0 and 1, and x sees 0, 2, and 3. We repeatedly use that u and x must be (1,2)- and (1,3)-linked. We consider two cases: color 0 is used on uu' or on uv.

Case 1: 0 is used on uu'. We first show that WLOG the edges are colored as in Figure 5. By assumption $\varphi(uu') = 0$ and $\varphi(uv) = 1$. We show that w must miss 0. If $\varphi(vw) = 0$, then $\varphi(wx) = 2$ (by symmetry) and $\varphi(ww') = 1$, but after a (1,3)-swap at w, vertices u and x are (1,2)-unlinked. A similar argument works if $\varphi(wx) = 0$. If $\varphi(ww') = 0$, then $\varphi(wx) = 2$ (by symmetry), but now u and x are (1,2)-unlinked, a contradiction. So w misses 0, as claimed. So, by symmetry, we have $\varphi(vw) = 2$, $\varphi(wx) = 3$, $\varphi(ww') = 1$, as in Figure 5.

Now we show that $\varphi(xx') = 0$ and $\varphi(xy) = 2$. Suppose to the contrary that $\varphi(xx') = 2$ and $\varphi(xy) = 0$. If we can get $\varphi(yz) = 2$ and z missing 0, then a (0,2)-swap at z gives $\varphi(xx') = 0$ and $\varphi(xy) = 2$. Note that u and w must be (0,2)-linked; otherwise a (0,2)-swap at w gives $\varphi(vw) = 0$, a contradiction. Always u and x must be (1,2)- and (1,3)-linked. They must also be (0,3)-linked, since otherwise we get a coloring where w sees 0, a contradiction. Now we use a series of (0,2)-, (0,3)-, (1,2)-, and (1,3)-swaps at z to get $\varphi(yz) = 2$ and z missing 0. (If a (0,2)-swap ever recolors xx' and xy, then we accomplish our goal and are done, so we assume this never happens.) We write $(i;j)$ to denote that $\varphi(yz) = i$ and z misses j. Also $(i;j) \to (i';j')$ if one of the four swaps mentioned yields $(i';j')$ from $(i;j)$. We have $(3;0) \to (3;2) \to (3;1) \to (1;3) \to (1;0) \to (1;2) \to (2;1) \to (2;3) \to (2;0)$. So after a (2,0)-swap at z, we have $\varphi(xx') = 0$ and $\varphi(xy) = 2$, as desired.

Finally, we will show that WLOG $\varphi(yz) = 3$ and z misses 0. In the notation above, we want to reach the case (3;0). We can still use (0,3)-, (1,2)-, and (1,3)-swaps at z (but, in general, cannot use (0,2)-swaps). We have $(0;2) \to (0;1) \to (0;3) \to (3;0)$. We also have $(1;0) \to (1;3) \to (3;1) \to$

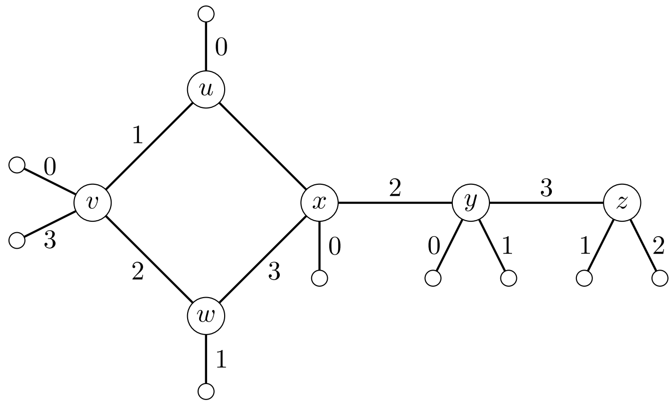


Figure 5: A coloring of $G - ux$ in Case 1 of the proof of Lemma 3.

(3; 2). Further, in (3;2), we can use a (0,2)-swap to reach (3;0). For, suppose it interchanges the colors 0 and 2 on xx' and xy. Now the (0,3)-chain at z ends at w. So a (0,3)-swap at z makes w see 0, a contradiction. Finally, consider (1;2). Now the (1,2)-chain at z ends at x. After a (1,2)-swap at z, we let $\varphi(ux) = 2$, to get a coloring of G. So, WLOG the edges are colored as in Figure 5.

Let H be the 6-edge-subgraph induced by $\{u, v, w, x, y, z\}$ of the configuration in Figure 5. Consider the (0,1)-chains in $G - E(H)$ at u, v, w, x, z. By parity, one chain must end at ∞. Recall that w and x must be (0,1)-linked in G, since w never sees 0. So if the (0,1)-chain at w or x ends at ∞ or z, then we reach a contradiction. If the (0,1)-chain at v ends at ∞ or z, then recolor it. Now let $\varphi(vw) = 0$, $\varphi(uv) = 2$, and $\varphi(ux) = 1$. So assume that the (0,1)-chain at z ends at ∞, and recolor it.

Consider the (1,2)-chains in $G - E(H)$ at w, y, and z. If the (1,2)-chain at w ends at ∞ or z, then recolor it, and let $\varphi(wv) = 1$, $\varphi(vu) = 2$, and $\varphi(ux) = 1$. So assume the (1,2)-chain at z ends at ∞, and recolor it. Finally, consider the (1,3)-chains in $G - E(H)$ that start at v, w, y, and z. If the chain at z ends at y, then recolor it, and let $\varphi(zy) = 2$, $\varphi(yx) = 1$, and $\varphi(xu) = 2$. If the chain at z ends at w, then recolor it and let $\varphi(zy) = 2$, $\varphi(yx) = 3$, $\varphi(xw) = 1$, and $\varphi(xu) = 2$. If the chain at y ends at w, then recolor it and let $\varphi(zy) = 2$, $\varphi(yx) = 1$, $\varphi(xw) = 2$, $\varphi(wv) = 1$, $\varphi(vu) = 2$, and $\varphi(ux) = 3$. This finishes Case 1.

Case 2: 0 is used on uv. We show that WLOG all edges are colored as in Figure 6. After that, the proof is easy. Suppose first that $\varphi(wx) = 0$. By possibly using a (1,2)- or (1,3)-swap at w, we assume that w misses 1. Now we let $\varphi(wx) = 1$, which reduces to Case 1. So we assume, by symmetry, that $\varphi(wx) = 2$. If w sees 0, then (possibly after a (1,3)-swap at w), vertex w misses 1, so u and x are (1,2)-unlinked, a contradiction. Thus, w misses 0. Suppose that $\varphi(vw) = 1$ and $\varphi(ww') = 3$. Now we uncolor wx and let $\varphi(ux) = 2$. This reduces to Case 1, with w in place of u (and 3 in place of 0). So we assume that $\varphi(vw) = 3$ and $\varphi(ww') = 1$, as in Figure 6. Assume that $\varphi(xx') = 3$ and $\varphi(xy) = 0$; otherwise, this follows from a (0,3)-swap at x.

As in Case 1, we write $(i; j)$ to denote that $\varphi(yz) = i$ and z misses j. Recall that (1,2)- and (1,3)-swaps at z do not change the colors on edges incident to u, v, w, and x. Neither do (0,2)-swaps, when $\varphi(yz) \neq 2$. (If w and u are (0,2)-unlinked, then we can get $\varphi(wx) = 0$, which reduces to Case 1, as in the previous paragraph.) In fact, we can also use (0,1)-swaps, as follows. Vertices u and x must be (0,1)-linked, or we reduce to Case 1. And w and x must be (0,1)-linked, or else we get w missing 1, which is a contradiction.

We write $(i; j) \to (i'; j')$ if, starting from $(i; j)$, we get $(i'; j')$ by using a (0,1)-, (0,2)-, (1,2)-, or (1,3)-swap at z. Our goal is to reach (2;0), as in Figure 6. When we do, the (0,2)-chain at z

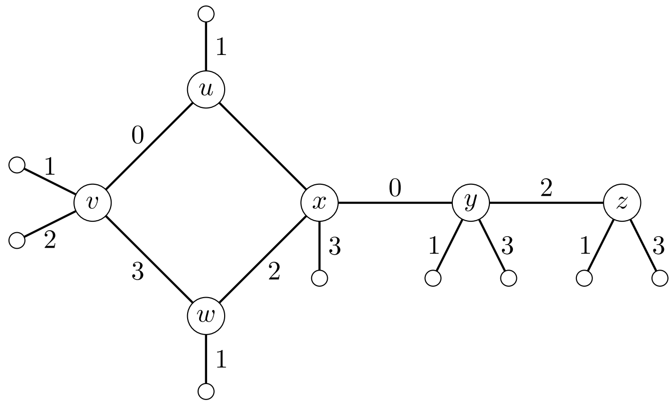


Figure 6: A coloring of $G - ux$ in Case 2 of the proof of Lemma 3.

ends at w. Now a (0,2)-swap at z gives $\varphi(wx) = 0$, which we handled in the first paragraph. Note that $(1;0) \to (1;2) \to (2;1) \to (2;0)$. Also, $(2;3) \to (2;1) \to (2;0)$. In any of these five cases, we are done. Note also that $(3;2) \to (3;0) \to (3;1) \to (1;3)$, so we can assume (1;3). Now we use (0,3)-swaps at x and z. So we have (1;0) and $\varphi(xx') = 0$ and $\varphi(xy) = 3$. We use a (0,2)-swap at z, followed by a (0,3)-swap at x. Now all edges are colored as in Figure 6, so we are done. $\square$

Lemma 4. *Suppose that $G \in \mathcal{H}_4$ and G has the configuration in Figure 1(c) (reproduced in Figure 7, if we ignore the colors there). If $\chi'(G - st) = 4$, then $\chi'(G) = 4$.*

Proof. We start with a coloring of $G - st$ and assume that G has no coloring, which leads to a contradiction. We denote by v' the unlabeled neighbor of v. By symmetry, we assume that t sees 0 and 1, and s sees 0, 2, and 3. We consider two cases: either 0 is used on su or it is not.

Case 1: 0 is used on su. By symmetry, assume that $\varphi(su) = 0$, $\varphi(sv) = 2$, $\varphi(sw) = 3$. Our plan is either to reach the coloring shown in Figure 7 or to reduce to Case 2: $\varphi(su) \neq 0$. Since s and t are (1,2)- and (1,3)-linked, we can use (1,2)- and (1,3)-swaps at u to get u missing 2 (without changing colors on edges incident to s and t).

We show that WLOG $\varphi(vx) = 0$. Suppose not; by symmetry between x and y, assume that $\varphi(vx) = 3$, $\varphi(vy) = 1$, and $\varphi(vv') = 0$. Now y sees 2, since s and t are (1,2)-linked. And u must be (0,2)-linked to t (possibly through w and z) or else we (0,2)-swap at u and reduce to Case 2. If y misses 0, then a (0,2)-swap at y makes y miss 2 (and thus s and t are (1,2)-unlinked). So assume y

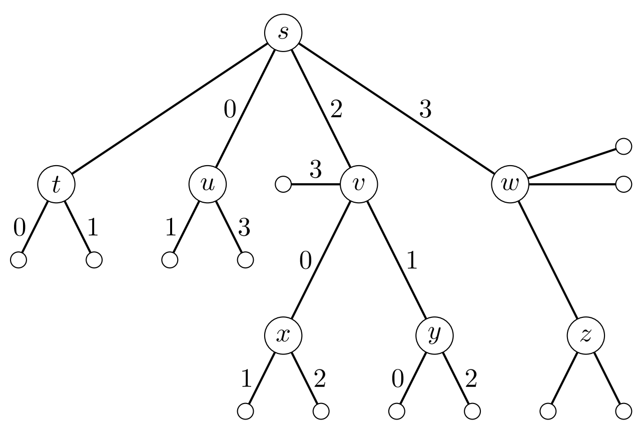


Figure 7: A partial coloring of $G - st$ in Case 1 of the proof of Lemma 4.

sees 0. After a (1,3)-swap at y, we have $\varphi(vy) = 3$ and $\varphi(vx) = 1$, with nothing else changed. So, by the argument above, x sees 0 and 2. But now the (1,3)-chain at x ends at y. So either s and t are currently (1,2)-unlinked, or else they become so after a (1,3)-swap at x. Thus, we conclude that $\varphi(vx) = 0$.

Now we show that WLOG $\varphi(vy) = 1$. Assume instead that $\varphi(vy) = 3$ and $\varphi(vv') = 1$. Now x sees 2, or else a (0,2)-swap at x reduces to Case 2. If necessary, use a (1,3)-swap at x to get x missing 3. Note that y sees 1, or else a (1,3)-swap at y gives $\varphi(vy) = 1$. If necessary, use a (0,2)-swap at y to get y missing 0. Now the (0,3)-chain at x ends at y. So either u and t are (0,2)-unlinked, or else they become so after a (0,3)-swap at x. In either case, we use a (0,2)-swap at u to reduce to Case 2. So we must have $\varphi(vx) = 0$, $\varphi(vy) = 1$, $\varphi(vv') = 3$.

Since s and t are (0,2)- and (1,2)-linked, both x and y see 2. Further, if y misses 0, then after a (0,2)-swap at y vertices s and t are (1,2)-unlinked. Thus, y sees 0 and 2, and misses 3. Suppose x misses 1. Now the (1,2)-chain P at x must end at u (possibly via w and z); otherwise we recolor P, which makes u and t (0,2)-unlinked, a contradiction. Now recolor P and let $\varphi(us) = 1$, $\varphi(sv) = 0$, $\varphi(vx) = 2$, $\varphi(st) = 2$. So instead x sees 1 and misses 3, as in Figure 7. Now recolor the (1,3)-chains at x and y (possibly the same chain). Again, the (1,2)-chain Q at x must end at u (possibly via w and z), since otherwise we recolor it and u and t are (0,2)-unlinked. Now Recolor Q; as before, let $\varphi(us) = 1$, $\varphi(sv) = 0$, $\varphi(vx) = 2$, $\varphi(st) = 2$. This completes Case 1.

Case 2: 0 is not used on *su*. By assumption s sees 0. We show that WLOG $\varphi(sw) = 0$. Suppose instead that $\varphi(sv) = 0$. Since $G \in \mathcal{H}_4$, vertex w has two 3-neighbors. If $u \in N(w)$ or $t \in N(w)$, then we have an instance of Figure 1(a), since $z \notin \{t, u\}$. So $\chi'(G) = 4$, by Lemma 2. Thus, we assume $t, u \notin N(w)$. Now we interchange the roles of v and w. (Vertices v and w could have a common 3-neighbor, but this is not a problem.) So $\varphi(sw) = 0$. By symmetry, assume that $\varphi(su) = 2$ and $\varphi(sv) = 3$. Since s and t are (1,2)-linked, u must see 1. If u misses 3, then after a (1,3)-swap at u, vertices s and t are (1,2)-unlinked. So u must miss 0. Thus, we have Figure 8(a), except for colors on edges incident to w and z.

We show that WLOG, we have either Figure 8(a) or else Figure 8(b) with y missing 3. Note that u and s must be (0,1)-linked, since otherwise we (0,1)-swap at u and finish as above. First, we get z missing 0. If z sees 0 and misses 1, then we (0,1)-swap at z. Otherwise, if z sees 0 it misses 2 or 3, so after a (1,2)- or (1,3)-swap, z misses 1. These swaps at z do not change the colors on edges

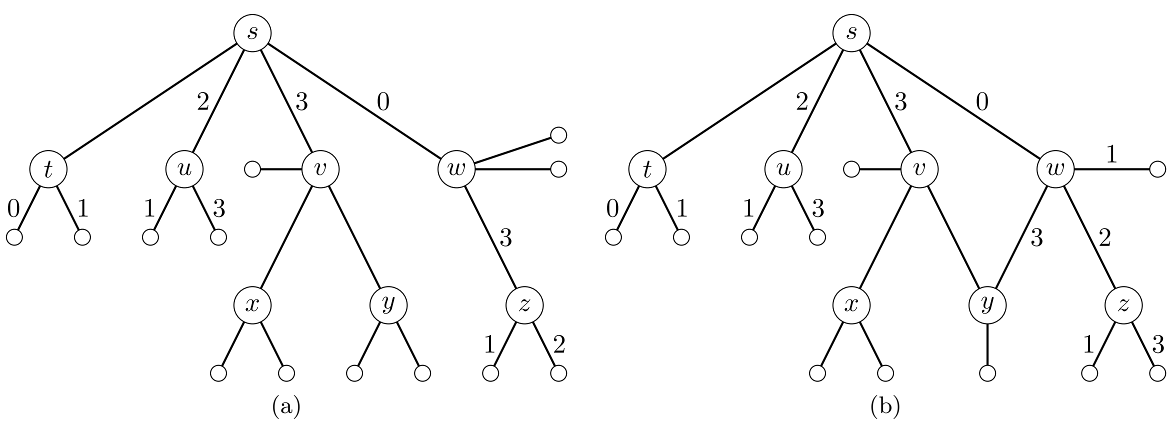


Figure 8: Two partial colorings of $G - st$ in Case 2 of Lemma 4. (a) A partial coloring of $G - st$. (b) A partial coloring of $G - st$, when G also has the edge wy.

incident to s, t, or u, since s and t are (1,2)- and (1,3)-linked and s and u are (0,1)-linked. Thus z misses 0. Now consider $\varphi(wz)$. If $\varphi(wz) = 1$, then the (0,1)-chain at z ends at s, so we recolor it and let $\varphi(su) = 0$ and $\varphi(st) = 2$. If $\varphi(wz) = 3$, then we are in Figure 8(a). So assume $\varphi(wz) = 2$.

Now consider the 3-neighbor $\widehat{z}$ of w, other than z. As in the first paragraph of Case 2, we know $\widehat{z} \notin \{t, u\}$. First suppose $\widehat{z} \notin \{x, y\}$. Now we essentially repeat the argument above, with $\widehat{z}$ in place of z. Suppose $\varphi(w\widehat{z}) = 3$. If $\widehat{z}$ misses 0, then we have Figure 8(a), with $\widehat{z}$ in place of z. If $\widehat{z}$ misses 1, then a (0,1)-swap at $\widehat{z}$ gives that $\widehat{z}$ misses 0, and we again reach Figure 8(a); this could make z miss 1, but that is irrelevant. If $\widehat{z}$ misses 2, then a (1,2)-swap at $\widehat{z}$ gives that $\widehat{z}$ misses 1. So instead assume $\varphi(w\widehat{z}) = 1$. If $\widehat{z}$ misses 0, then we use a (0,1)-swap at $\widehat{z}$, as above. If $\widehat{z}$ misses 2, then a (1,2)-swap at $\widehat{z}$ makes $\varphi(wz) = 1$ and z still misses 0, so we are done. So assume $\widehat{z}$ misses 3. Now a (1,3)-swap at $\widehat{z}$ reduces to the case above where $\varphi(w\widehat{z}) = 3$. This conclude the case where $\widehat{z} \notin \{x, y\}$. Now suppose $\widehat{z} \in \{x, y\}$; by symmetry, assume that $\widehat{z} = y$. This case is identical, except that we end in Figure 8(b) with y missing 0. Thus, WLOG we have either Figure 8(a) or Figure 8(b) with y missing 0. We first consider the latter case, since the argument is simpler.

Case 2a: we have Figure 8(b) with y missing 0. Clearly $\varphi(vx)$ is 2, 1, or 0. First suppose that $\varphi(vx) = 2$, which implies $\varphi(vy) = 1$. Now let $\varphi(vy) = 3$, $\varphi(yw) = 0$, $\varphi(ws) = 3$, $\varphi(sv) = 1$, $\varphi(su) = 0$, $\varphi(st) = 2$. So $\varphi(vx) \neq 2$. In what follows, we often use variations on this recoloring idea, typically letting $\varphi(vy) = \varphi(ws) = 3$ and $\varphi(wy) = 0$, and also recoloring some other edges.

Suppose instead that $\varphi(vx) = 1$, which implies $\varphi(vy) = 2$; recall that y misses 0. Now x must see 3, so x misses 2 or 0. If x misses 2, then let $\varphi(vx) = 2$, $\varphi(vy) = 3$, $\varphi(yw) = 0$, $\varphi(ws) = 3$, $\varphi(sv) = 1$, $\varphi(su) = 0$, and $\varphi(st) = 2$. So assume x sees 2 and misses 0. Consider the (0,2)-chains at t, v, and x, and let P be the chain that ends at ∞. If P starts at x, then recoloring P reduces to the previous case, where x misses 2. If P starts at v, then we use nearly the same coloring; the only difference is that we let $\varphi(vx) = 0$ (rather than $\varphi(vx) = 2$). So we assume that P starts at t, and recolor it.

Now consider the (0,1)-chains in $G - E(H)$ at t, u, v, w, y, and z. If the chain at t ends at u, v, y, or z, then we recolor it (and let $\varphi(vx) = 0$ if it ends at v) and let $\varphi(st) = 1$. So assume t and w are (0,1)-linked. Now consider the (0,1)-chain P at u. If P ends at z, then recolor it and let $\varphi(su) = 1$, $\varphi(sv) = 2$, $\varphi(vy) = 3$, $\varphi(yw) = 0$, $\varphi(ws) = 3$, $\varphi(st) = 0$. If P ends at v, then the only difference is that we also let $\varphi(vx) = 0$. If P ends at y, then nearly the same idea works. Now we recolor P, and let $\varphi(su) = 1$, $\varphi(sv) = 2$, $\varphi(vy) = 3$, $\varphi(yw) = 2$, $\varphi(wz) = 0$, $\varphi(ws) = 3$, $\varphi(st) = 0$. This finishes the case when $\varphi(vx) = 1$.

Finally, assume that $\varphi(vx) = 0$; again recall that y misses 0. If $\varphi(vy) = 1$, then $\varphi(vv') = 2$. In this case, let $\varphi(vy) = 3$, $\varphi(yw) = 0$, $\varphi(ws) = 3$, $\varphi(sv) = 1$, $\varphi(su) = 0$, and $\varphi(st) = 2$. So assume instead that $\varphi(vy) = 2$ and $\varphi(vv') = 1$. By using a (1,2)- and (1,3)-swap at x, we can assume that x misses 2. Consider the (0,1)-chains in $G - E(H)$ at u, v, w, x, y, z. Recall that u and w must be (0,1)-linked (possibly through v and x) or else we can recolor the (0,1)-chain at u and let $\varphi(us) = 1$ and $\varphi(st) = 2$. So clearly neither u nor w is (0,1)-linked to either y or z. Further, neither u nor w is (0,1)-linked to x, since we can recolor that (0,1)-chain, let $\varphi(vx) = 2$, $\varphi(vy) = 0$, and proceed as before. So u and w must be (0,1)-linked. Thus, v is (0,1)-linked to x, y, or z. Let P be the (0,1)-chain in $G - E(H)$ at v. If P ends at x or z, then recolor it and let $\varphi(vx) = 2$, $\varphi(vy) = 3$, $\varphi(yw) = 0$, $\varphi(ws) = 3$, $\varphi(sv) = 1$, $\varphi(su) = 0$, and $\varphi(st) = 2$. So assume instead that P ends at y. Now recolor P and let $\varphi(vx) = 2$, $\varphi(vy) = 3$, $\varphi(yw) = 2$, $\varphi(wz) = 0$, $\varphi(ws) = 3$, $\varphi(sv) = 1$, $\varphi(su) = 0$, and $\varphi(st) = 2$. This completes the Case 2a.

Case 2b: we have Figure 8(a). If a (1,3)-swap elsewhere ever recolors wz, then we can finish as in the second paragraph of Case 2, when $\varphi(wz) = 1$. So we assume this never happens. We show that WLOG all edges are colored as in Figure 9. After that, the proof is easy. First, we show that $\varphi(vx) = 1$. Suppose not. By symmetry between x and y, we assume that $\varphi(vx) = 2$, $\varphi(vy) = 0$,

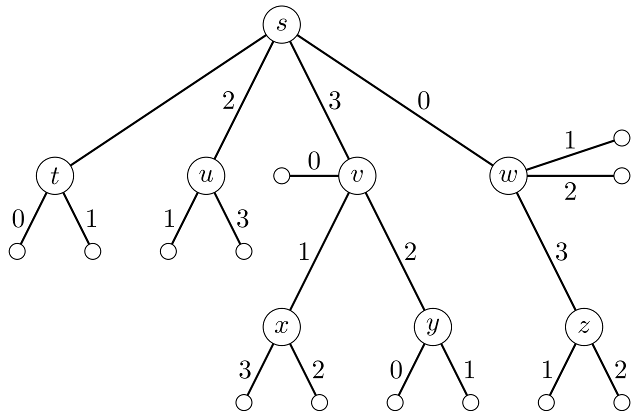


Figure 9: A coloring of $G - st$ in Case 2 of the proof of Lemma 4.

and $\varphi(vv') = 1$. If x misses 1, then after a (1,2)-swap at x, we have $\varphi(vx) = 1$, as desired. If x misses 3 and sees 1, then a (1,3)-swap at x gets x missing 1. So assume x misses 0. Note that u and t must be (0,3)-linked; otherwise we use a (0,3)-swap at u, followed by a (1,3)-swap at u, which results in s and t being (1,2)-unlinked, a contradiction. So y must see 3; otherwise a (0,3)-swap at x gives x missing 3 (since the (0,3)-chain at x does not interact with other edges shown colored 0 or 3). Now y also sees either 1 or 2. We assume y sees 2; otherwise, we use a (1,2)-swap at y (if this recolors vx, then we have $\varphi(vx) = 1$, as desired; so assume not). The (0,1)-chain at y must end at z; otherwise, we recolor it, and have $\varphi(vy) = 1$, as desired. So we can recolor the (0,1)-chain at x without effecting any other edges shown. Now after a (1,2)-swap at x, we have $\varphi(vx) = 1$, as desired. Since s and t are (1,3)-linked, x also sees 3.

Now we show that WLOG $\varphi(vy) = 2$ and $\varphi(vv') = 0$. Assume to the contrary that $\varphi(vy) = 0$ and $\varphi(vv') = 2$. If x misses 0 and y misses 3, then a (0,1)-swap at x makes s and t be (1,3)-unlinked, a contradiction. If x misses 0 and y misses 1, then a (1,3)-swap at y causes y to miss 3 (as in the previous sentence). So the four possibilities for the ordered pair of colors missed at x and y are (0,2), (2,1), (2,2), (2,3). We show that WLOG we are in the case (2,3). In the case (2,1), a (1,3)-swap at y yields (2,3), as desired. Suppose we are in the case (0,2), and use a (1,2)-swap at y. This must recolor the path through vx, since otherwise we are in the case (0,1), handled above. Now the (0,1)-chain at y must end at z, or we recolor it. So we can use a (0,1)-swap at x. Now (1,2)-swaps at x and y yield the case (2,2). So it suffices to reduce the case (2,2) to the case (2,3), and also handle the latter.

Suppose we are in the case (2,2), that is both x and y miss 2. Use (1,2)-swaps at x and y, followed by (1,3)-swaps at x and y. Now at x we use a (0,3)-swap, a (0,1)-swap, and a (1,2)-swap (the (0,1)-swap cannot recolor vy, since this makes s and t (1,3)-unlinked). This yields the case (2,3).

Finally, assume we are in the case (2,3). Consider the (0,1)-chains in $G - E(H)$ starting at u, w, x, y, z. Let P be the chain starting at x. If P does not end at w, then recolor P, and let $\varphi(vx) = 0$, $\varphi(vy) = 3$, $\varphi(vs) = 1$, and $\varphi(st) = 3$. So P must end at w. Let Q be the (0,1)-chain starting at u. If Q ends at z or ∞, then recolor Q and let $\varphi(us) = 1$ and $\varphi(st) = 2$. So Q must end at y. Now recolor P, and let $\varphi(yv) = 3$, $\varphi(vs) = 0$, $\varphi(sw) = 3$, $\varphi(wz) = 0$, $\varphi(us) = 1$, $\varphi(st) = 2$. Thus, we conclude that $\varphi(vy) = 2$, and so $\varphi(vv') = 0$.

Now we need only to show that the colors missing at x and y are as in Figure 9. If x misses 2 and y misses 3, then a (1,2)-swap at x makes s and t (1,3)-unlinked. If x misses 2 and y misses 1, then a (1,3)-swap at y takes us to the previous sentence. So suppose x misses 2 and y misses

0. Now a (1,2)-swap at x (and interchanging the roles of x and y) yields the case that x misses 0 and y misses 1. Thus, we assume that x misses 0. Suppose that y misses 0. Now x must be (0,3)-linked to v; otherwise a (0,3)-swap at x makes s and t be (1,3)-unlinked, a contradiction. Now we (0,3)-swap at y, after which y misses 3. This gives the colors in Figure 9. So assume x misses 0 and y misses 1. Now we (1,3)-swap at y, which again gives the colors in Figure 9.

Finally, we show that if the colors are as in Figure 9, then G has a coloring. Consider the (1,2)-chains in $G - E(H)$ that start at t, u, x, y. If the (1,2)-chain at y ends at x, then recolor it and let $\varphi(vx) = 2$ and $\varphi(vy) = 1$. Now s and t are (1,3)-unlinked, a contradiction. If the (1,2)-chain at y ends at u, then recolor it and let $\varphi(yv) = 3$, $\varphi(vs) = 2$, $\varphi(su) = 1$, $\varphi(st) = 3$. So assume the (1,2)-chain at y ends at t. Recolor it and let $\varphi(yv) = 3$, $\varphi(vs) = 2$, $\varphi(sw) = 3$, $\varphi(wz) = 0$, $\varphi(su) = 0$, $\varphi(st) = 1$. This completes Case 2b, and the proof. □

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