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PLANAR GRAPHS ARE 9/2-COLORABLE AND HAVE INDEPENDENCE RATIO AT LEAST 3/13

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Abstract. We show that every planar graph G has a 2-fold 9-coloring. In particular, this implies that G has fractional chromatic number at most $\frac{9}{2}$. This is the first proof (independent of the 4 Color Theorem) that there exists a constant $k < 5$ such that every planar G has fractional chromatic number at most k. We also show that every n-vertex planar graph has an independent set of size at least $\frac{3n}{13}$. This improves on Albertson's bound of $\frac{2n}{9}$.

1. Introduction

To fractionally color a graph G, we assign to each independent set in G a nonnegative weight, such that for each vertex v the sum of the weights on the independent sets containing v is 1. A graph G is *fractionally k-colorable* if G has such an assignment of weights where the sum of the weights is at most k. The minimum k such that G is fractionally k-colorable is its *fractional chromatic number*, denoted $\chi_f(G)$. (If we restrict the weight on each independent set to be either 0 or 1, we return to the standard definition of chromatic number.) In 1997, Scheinerman and Ullman [12, p. 75] succintly described the state of the art for fractionally coloring planar graphs. Not much has changed since then.

> The fractional analogue of the four-color theorem is the assertion that the maximum value of $\chi_f(G)$ over all planar graphs G is 4. That this maximum is no more than 4 follows from the four-color theorem itself, while the example of K_4 shows that it is no less than 4. Given that the proof of the four-color theorem is so difficult, one might ask whether it is possible to prove an interesting upper bound for this maximum without appeal to the four-color theorem. Certainly $\chi_f(G) \leq 5$ for any planar G, because $\chi(G) \leq 5$, a result whose proof is elementary. But what about a simple proof of, say, $\chi_f(G) \leq \frac{9}{2}$ for all planar G? The only result in this direction is in a 1973 paper of Hilton, Rado, and Scott [6] that predates the proof of the four-color theorem; they prove $\chi_f(G) < 5$ for any planar graph G, although they are not able to find any constant $c < 5$ with $\chi_f(G) < c$ for all planar graphs G. This may be the first appearance in print of the invariant χ_f.

In Section 2, we give exactly what Scheinerman and Ullman asked for—a simple proof that $\chi_f(G) \leq \frac{9}{2}$ for every planar graph G. In fact, this result is an immediate corollary of a stronger statement in our first theorem. Before we can express it precisely, we need another

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definition. A *k-fold ℓ-coloring* of a graph G assigns to each vertex a set of k colors, such that adjacent vertices receive disjoint sets, and the union of all sets has size at most ℓ. If G has a k-fold ℓ-coloring, then $\chi_f(G) \leq \frac{\ell}{k}$. To see this, consider the ℓ independent sets induced by the color classes; assign to each of these sets the weight $\frac{1}{k}$. Now we can state the theorem.

Theorem 1. *Every planar graph G has a 2-fold 9-coloring. In particular, $\chi_f(G) \leq \frac{9}{2}$.*

In an intuitive sense, Theorem 1 sits somewhere between the 4 Color Theorem and the 5 Color Theorem. It is certainly implied by the former, but it does not immediately imply the latter. The *Kneser graph* $K_{n:k}$ has as its vertices the k-element subsets of $\{1, \dots, n\}$ and two vertices are adjacent if their corresponding sets are disjoint. Saying that a graph G has a 2-fold 9-coloring is equivalent to saying that it has a homomorphism to the Kneser graph $K_{9:2}$. To claim that a coloring result for planar graphs is between the 4 and 5 Color Theorems, we would like to show that every planar graph G has a homomorphism to a graph H, such that H has clique number 4 and chromatic number 5. Unfortunately, $K_{9:2}$ is not such a graph. It is easy to see that $\omega(K_{n:k}) = \lfloor n/k \rfloor$; so $\omega(K_{9:2}) = 4$, as desired. However, Lovász [8] showed that $\chi(K_{n:k}) = n - 2k + 2$; thus $\chi(K_{9:2}) = 9 - 2(2) + 2 = 7$. Fortunately, we can easily overcome this problem.

The *categorical product* (or *universal product*) of graphs G_1 and G_2, denoted $G_1 \times G_2$ is defined as follows. Let $V(G_1 \times G_2) = \{(u, v) | u \in V(G_1) \text{ and } v \in V(G_2)\}$; now (u_1, v_1) is adjacent to (u_2, v_2) if $u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)$. Let $H = K_5 \times K_{9:2}$. It is well-known [5] that if a graph G has a homomorphism to each of graphs G_1 and G_2, then G also has a homomorphism to $G_1 \times G_2$ (the image of each vertex in the product is just the products of its images in G_1 and G_2). The 5 Color Theorem says that every planar graph has a homomorphism to K_5; so if we prove that every planar graph G has a homomorphism to $K_{9:2}$, then we also get that G has a homomorphism to $K_5 \times K_{9:2}$.

It is easy to check that for any G_1 and G_2, we have $\omega(G_1 \times G_2) = \min(\omega(G_1), \omega(G_2))$ and $\chi(G_1 \times G_2) \leq \min(\chi(G_1), \chi(G_2))$. To prove this inequality, we simply color each vertex (u, v) of the product with the color of u in an optimal coloring of G_1, or the color of v in an optimal coloring of G_2. (It is an open problem whether this inequality always holds with equality [11].) When $H = K_5 \times K_{9:2}$ we get $\omega(H) = 4$ and $\chi(H) = 5$. Earlier work of Naserasr [9] and Nešetřil and Ossona de Mendez [10] also constructed graphs H, with $\omega(H) = 4$ and $\chi(H) = 5$, such that every planar graph G has a homomorphism to H; however, their examples had more vertices than ours. Naserasr gave a graph with size $63\binom{62}{4} = 35,144,235$ and the construction in [10] was still larger. In contrast, $|K_5 \times K_{9:2}| = 5\binom{9}{2} = 180$.

Wagner [13] characterized K_5-minor-free graphs. The Wagner graph is formed from an 8-cycle by adding an edge joining each pair of vertices that are distance 4 along the cycle. Wagner showed that every maximal K_5-minor-free graph can be formed recursively from planar graphs and copies of the Wagner graph by pasting along copies of K_2 and K_3 (see also [4, p. 175]). Since the Wagner graph is 3-colorable, it clearly has a 2-fold 9-coloring. To show that every K_5-minor-free graph is 2-fold 9-colorable, we color each smaller planar graph and copy of the Wagner graph, then permute colors so that the colorings agree on the vertices that are pasted together.

Hajós conjectured that every graph is $(k-1)$-colorable unless it contains a subdivision of K_k. This is known to be true for $k \leq 4$ and false for $k \geq 7$. The cases $k = 5$ and

$k = 6$ remain unresolved. Since this problem seems difficult, we offer the following weaker conjecture: Every graph with no K_5-subdivision is 2-fold 9-colorable.

In Section 3 we study the independence number of planar graphs. An immediate consequence of the 4 Color Theorem is that every n-vertex planar graph has an independent set of size at least $\frac{n}{4}$ (and this is best possible, as shown by the disjoint union of many copies of K_4). In 1968, Erdös [2] suggested that perhaps this corollary could be proved more easily than the full 4 Color Theorem. And in 1976, Albertson [1] showed (independently of the 4 Color Theorem) that every n-vertex planar graph has an independent set of size at least $\frac{2n}{9}$. Our second theorem improves this bound to $\frac{3n}{13}$.

Theorem 2. *Every n-vertex planar graph has an independent set of size at least $\frac{3n}{13}$.*

The proof of Theorem 2 is heavily influenced by Albertson's proof. One apparent difference is that our proof uses the discharging method, while his does not. However, this distinction is largely cosmetic. To demonstrate this point, we include an appendix with a short discharging version of the final step in Albertson's proof, which he verified using edge-counting (the reader unfamiliar with discharging arguments may prefer to start with this appendix). Although the arguments are essentially equivalent, the discharging method is somewhat more flexible. In part it was this added flexibility that allowed us to push his ideas further.

The proofs of our two main results both follow the same general outline. The bulk of the work in each proof consists in showing that certain configurations are *reducible*, i.e., they cannot appear in a minimal counterexample to the theorem. In each case, the proof concludes via a discharging argument, where we show that every planar graph contains one of the forbidden configurations; hence, it is not a minimal counterexample. In the proof of the fractional coloring bound, the details are much simpler, so we present that result first.

Before the proofs, we need a few definitions. A *k-vertex* is a vertex of degree k; similarly, a k^--vertex (resp. k^+-vertex) has degree at most (resp. at least) k. A k-neighbor of a vertex v is a k-vertex that is a neighbor of v; and k^--neighbors and k^+-neigbors are defined analogously. A k-cycle is a cycle of length k. A vertex set V_1 in a connected graph G is *separating* if $G \setminus V_1$ has at least two components. A cycle C is separating if $V(C)$ is separating. Finally, an *independent k-set* is an independent set (or stable set) of size k.

2. Fractional Coloring of Planar Graphs

The goal of this section is to prove Theorem 1, that every planar graph has a 2-fold 9-coloring. Our proof will use the methods of reducibility and discharging. Throughout most of this section, we will prove that certain properties must hold for every minimal counterexample to the theorem (by "minimal" we mean having the fewest vertices and, subject to that, the fewest non-triangular faces). To conclude the section, we give a counting argument, via the discharging method, showing that every planar graph violates one of these properties. Thus, no minimal counterexample exists, so the theorem is true.

Throughout the rest of the section we will let G denote a minimal counterexample to the theorem. To remind the reader of this assumption, we will often refer to a *minimal* G. Throughout when we say "a coloring", we mean a 2-fold 9-coloring. Note that G is a plane triangulation (if not, then adding an edge contradicts our choice of G as having the fewest non-triangular faces).

Lemma 3. *A minimal G has no separating clique. Specifically, G has no separating 3-cycle.*

Proof. Suppose G has a separating clique X and let $C_1, \ldots, C_k$ be the components of $G \setminus X$. By minimality of $|G|$, we have colorings of $G[V(C_i) \cup X]$ for each $i \in \{1, \ldots, k\}$. Permute the colors on each subgraph $G[V(C_i) \cup X]$ so the colorings agree on X. Now identifying the copies of X in each $G[V(C_i) \cup X]$ gives a coloring of G, a contradiction. □

Although it was easy to prove, Lemma 3 will play a crucial role in our proof. We will often want to identify two neighbors u_1 and u_2 of a vertex v and color the smaller graph by minimality. To do so, we must ensure that u_1 and u_2 are indeed non-adjacent; these arguments typically use the fact that if u_1 and u_2 were adjacent, then u_1u_2v would be a separating 3-cycle.

Lemma 4. *G has minimum degree 5.*

Proof. Since G is a plane triangulation, it has minimum degree at least 3 and at most 5. If G contains a 3-vertex, then its neighbors induce a separating 3-cycle, contradicting Lemma 3. If G contains a 4-vertex v, then some pair of its neighbors are non-adjacent, since K_5 is non-planar. Form G' from G by deleting v and contracting a non-adjacent pair of its neighbors. Color G' by minimality, then lift the coloring back to G; only v is uncolored. Since two of v's neighbors have the same colors, we can extend the coloring to G. □

The following fact will often allow us to extend a 2-fold 9-coloring to the uncolored vertices of an induced $K_{1,3}$. It will be useful in verifying that numerous configurations are forbidden from a minimal G. We will also often apply it when the uncolored subgraph is simply P_3.

Fact 1. *Let $H = K_{1,3}$. If each leaf has a list of size 3 and the center vertex has a list of size 5, then we can choose 2 colors for each vertex from its lists such that adjacent vertices get disjoint sets of colors.*

Proof. Let v denote the center vertex and u_1, u_2, u_3 the leaves. Since $2|L(v)| > |L(u_1)| + |L(u_2)| + |L(u_3)|$, some color $c \in L(v)$ appears in $L(u_i)$ for at most one u_i. If such a u_i exists, then by symmetry, say it is u_1; now color v with c and some color not in $L(u_1)$. Otherwise color v with c and an arbitrary color. Now color each u_i arbitrarily from its at least 2 available colors. □

We use the same approach to prove each of Lemmas 5, 6, and 7. Our idea is to contract some edges of G to get a smaller planar graph G', which we color by minimality. In particular, in G' we identify some pairs of non-adjacent vertices of G that each have a common neighbor. When we lift the coloring of G' to G this means that some of the uncolored vertices will have neighbors with both colors the same, reducing the number of colors used on the neighborhood of each such uncolored vertex.

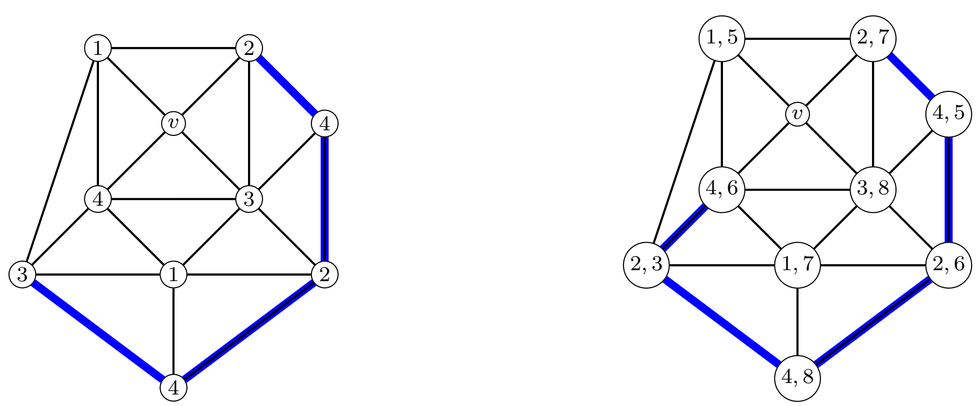
One early example of this technique is Kainen's proof [7] of the 5 Color Theorem. If G is a planar graph, then by Euler's Theorem, G has a 5^--vertex v. If $d(v) \leq 4$, then we 5-color $G - v$ by minimality; now, since $d(v) \leq 4$, we can extend the 5-coloring to v. Suppose instead that $d(v) = 5$. Since K_6 is non-planar, v has two neighbors u_1 and u_2 that are non-adjacent;

form G' by contracting the edges vu_1 and vu_2, and again 5-color G' by minimality. To extend the 5-coloring to v, we note that even though $d(v) = 5$, at most four colors appear on the neighbors of v (since u_1 and u_2 have the same color). This completes the proof.

Because a minimal G has no separating 3-cycles, if vertices u_1 and u_2 have a common neighbor v and do not appear sequentially on the cycle induced by the neighborhood of v, then u_1 and u_2 are non-adjacent. The numeric labels in the figures denote pairs (or more) of vertices that are identified in G' when we delete any vertices labeled v, u_1, u_2 or u_3; vertices with the same numeric label get identified.

Typically, it suffices to verify that the vertices receiving a common numeric label are pairwise nonadjacent. One potential complication is if two vertices that are drawn as distinct are in fact the same vertex. This usually cannot happen if the vertices have a common neighbor v, since then the degree of v would be too small. Similarly, it cannot happen if they are joined by a path of length three, since then we would get a separating 3-cycle.

For 4-coloring, Birkhoff [3] showed how to exclude separating 4-cycles and 5-cycles. Excluding separating 4-cycles would simplify our arguments below since we would not need to worry about vertices at distance at most four being the same. The proof excluding 4-cycles for 4-coloring is quite easy, but it does not work in our context because standard Kempe chain arguments break down for 2-fold coloring. The problem is illustrated in Figure 1. Figure 1(A) shows the situation for 1-fold coloring; here the 13-path blocks the 24-path. Figure 1(B) shows the situation for 2-fold coloring, here the 24-path can get through because on the 13-path, a vertex has color 2 as well as color 1.

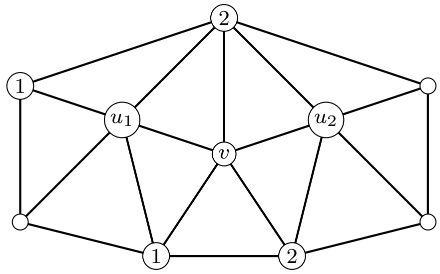


(A) The 2, 4-path is blocked by the 1, 3-path. (B) The 2, 4-path gets through.

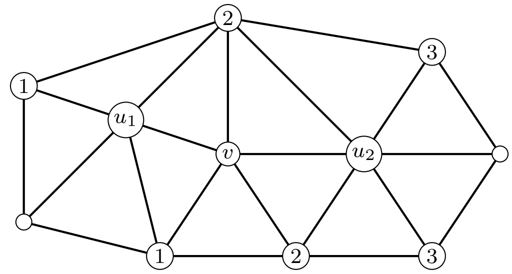
FIGURE 1. The problem with Kempe chains for 2-fold coloring.

Lemma 5. *G has no 5-vertex with a 5-neighbor and a non-adjacent 6^--neighbor.*

Proof. We first consider the case where a 5-vertex v has non-adjacent 5-neighbors u_1 and u_2, as shown in Figure 2(A). We color G' by minimality, then lift the coloring to G. (Recall that to form G', we delete v and all u_i and for each pair (or more) of vertices with the same label, we identify them.) Now in G each u_i has a list of at least 3 colors and v has a list of at least 5 colors; so, by Fact 1, we can extend the coloring to G.



(A) A 5-vertex, v, with non-adjacent 5-neighbors, u_1 and u_2.

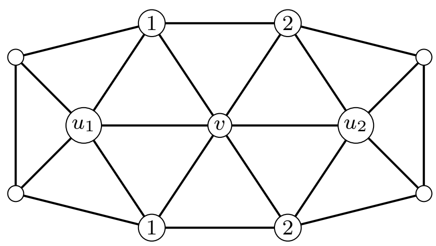


(B) A 5-vertex, v, with a non-adjacent 5-neighbor, u_1, and 6-neighbor, u_2.

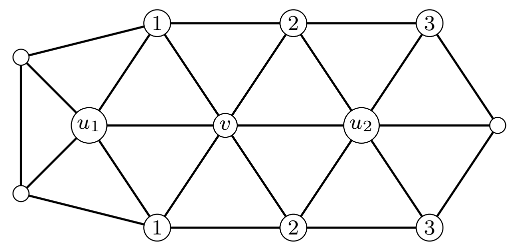
Figure 2. The cases of Lemma 5.

Now we consider the case where a 5-vertex v has a 5-neighbor and a 6-neighbor that are non-adjacent, as shown in Figure 2(B). Again, when we lift the coloring of G' to G, v has a list of size 5 and each of its uncolored neighbors has a list of size 3. Hence, by Fact 1, we can extend the coloring of G' to G. Here no pair of labeled vertices can be identified, since each such pair is drawn at distance three or less (and G has no separating 3-cycle). □

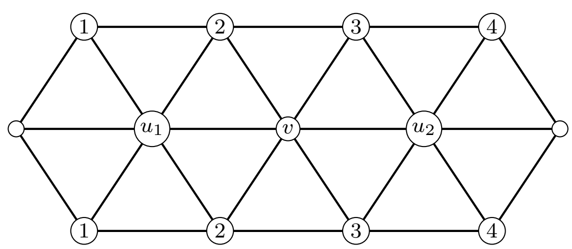
Lemma 6. *G has no 6-vertex with non-adjacent 6^--neighbors.*



(A) A 6-vertex, v, with non-adjacent 5-neighbors, u_1 and u_2, that are across from each other.



(B) A 6-vertex, v, with a non-adjacent 5-neighbor, u_1, and 6-neighbor, u_2, that are across from each other.

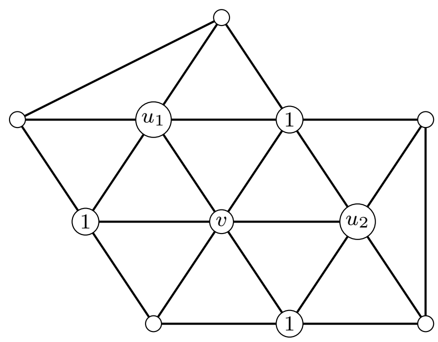


(C) A 6-vertex, v, with non-adjacent 6-neighbors, u_1 and u_2, that are across from each other.

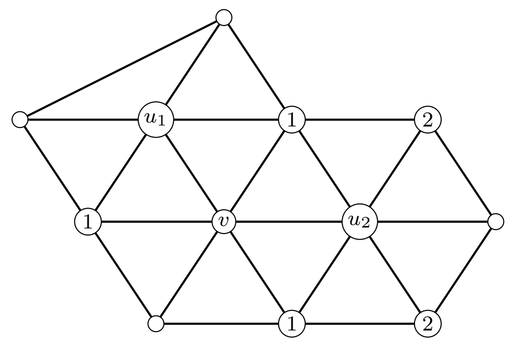
Figure 3. The "across" cases of Lemma 6.

Proof. Let v be a 6-vertex with two non-adjacent 6^--neighbors, u_1 and u_2. We have three possibilities for the degrees of these 6^--neighbors: two 5-vertices, a 5-vertex and a 6-vertex, and two 6-vertices. For each choice of degrees for the u_is, we have two possibilities for their relative location; they could be "across" from each other (at distance three along the cycle induced by the neighbors of v) or "offset" from each other (at distance two along the same cycle). This yields a total of six possibilities; the three across possibilities are shown in Figure 3 and the three offset possibilities are shown in Figure 4.

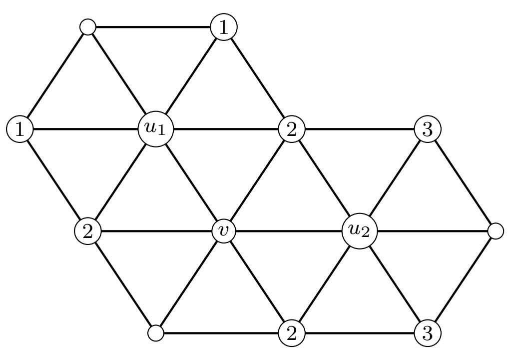
In Figures 3(A,B), all of the vertices with numeric labels (those that will be identified in G') must be distinct, since they are drawn within distance three of each other. The only complication is in Figure 3(C): a vertex labeled 1 might be the same as a vertex labeled 4 that is drawn at distance four; call this vertex x. By symmetry, assume that x is formed by identifying the vertex in the top left labeled 1 and the vertex in the bottom right labeled 4. This is only a problem if also a vertex labeled 1 is adjacent to one labeled 4; so suppose this happens. Note that the vertex in the top right labeled 4 cannot be adjacent to the vertex in the bottom left labeled 1; they are on opposite sides of the cycle xu_1vu_2. So, again by symmetry, we assume that x is adjacent to the vertex in the bottom left labeled 1. However, now we have a separating 3-cycle (consisting of x, its neighbor labeled 1, and their common neighbor u_1); this contradicts Lemma 3. This contradiction finishes the across cases.



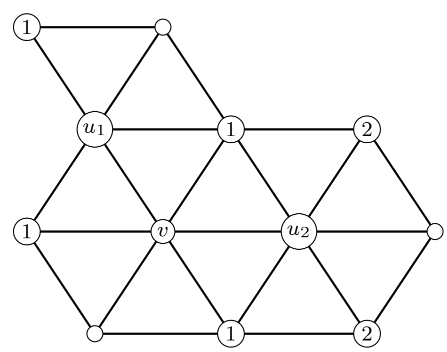
(A) A 6-vertex, v, with non-adjacent 5-neighbors, u_1 and u_2, that are offset from each other.



(B) A 6-vertex, v, with a non-adjacent 5-neighbor, u_1, and 6-neighbor, u_2, that are offset from each other.



(C) A 6-vertex, v, with non-adjacent 6-neighbors, u_1 and u_2, that are offset from each other (case i).



(D) A 6-vertex, v, with non-adjacent 6-neighbors, u_1 and u_2, that are offset from each other (case ii).

FIGURE 4. The "offset" cases of Lemma 6.

Now we consider the three offset cases, which are shown in Figure 4. As with the across cases, in Figures 4(A,B), all of the vertices with numeric labels must be distinct, since they are drawn within distance 3 of each other.

The only complication in is the third case, shown in Figures 4(C,D): the vertices labeled 1 and 3 that are drawn at distance four in Figure 4(C) might be the same; if so, then call this vertex x. In this case we switch to the identifications shown in Figure 4(D); we omit from Figure 4(D) a few edges incident to x, to keep the picture pretty. Now all vertices with numeric labels are at distance at most three, due to the extra edges incident to x. Also, the two vertices labeled 1 that are drawn at distance three are non-adjacent, since they are separated by cycle u_1vu_2x. This finishes the offset cases. □

Lemma 7. *G has no 7-vertex with a 5-neighbor and two other 6^--neighbors such that all three are pairwise non-adjacent.*

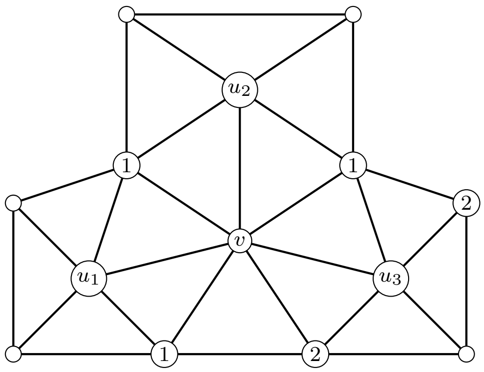
Proof. Figure 5(A) shows a 7-vertex with three pairwise non-adjacent 5-neighbors. Here, all pairs of vertices with numeric labels are at distance at most three, so they must be distinct.

In Figure 5(B), all pairs of vertices with numeric labels are again at distance at most three, except for one vertex labeled 1 which is drawn at distance four from each vertex labeled 3. The only possible problem is if one pair of vertices labeled 1 and 3 are actually the same vertex, while another pair labeled 1 and 3 are adjacent; these pairs must be disjoint, since otherwise we have a separating 3-cycle. The pair that are adjacent must be drawn at distance at least three, to avoid a separating 3-cycle. Hence, we need only consider the case where the vertices labeled 1 and 3 drawn at distance three are adjacent, and the other pair labeled 1 and 3 are the same vertex x. However, this is impossible, since then the adjacent pair are on opposite sides of the cycle u_1vu_3x.

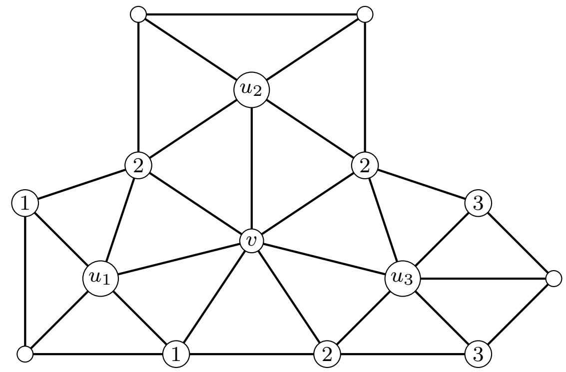
In Figure 5(C), all pairs of vertices with numeric labels are at distance at most three, so they must be distinct.

Consider Figure 5(D). None of the vertices labeled 3 can be the same as any other numerically labeled vertices since they are all distance at most 3 apart. Similarly, none of the vertices labeled 1 and 2 can be the same. So we need only consider the case that vertices labeled 4 are the same as those labeled 1 or 2. If a pair of vertices labeled 2 and 4 are the same, then it must be the pair that are drawn at distance 4; call this vertex x. In this case, we unlabel the vertices labeled 4 and label w_3 with 3. Now, thanks to x, all labeled vertices are distance at most three apart; hence, they must be distinct. So we may assume that vertices labeled 4 are not the same as those labeled 2.

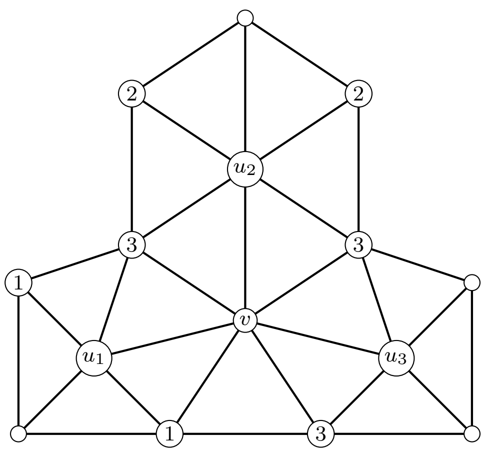
Suppose instead that a vertex labeled 4 is the same as one labeled 1; call this vertex y. This is only a problem if also some pair of vertices labeled 1 and 4 are adjacent. But this is impossible as follows. Since the pair of vertices labeled 1 have a common neighbor, they cannot be adjacent; similarly for the pair labeled 4. So the pairs that are identified and adjacent must be disjoint. Further, the identified pair must contain the rightmost vertex labeled 1. If it is identified with the bottom vertex labeled 4, then the remaining vertices cannot be adjacent, since they are on opposite sides of the 4-cycle u_1vu_3y. If it is identified with the top vertex labeled 4, then the remaining pair cannot be adjacent, since they have a common neighbor.



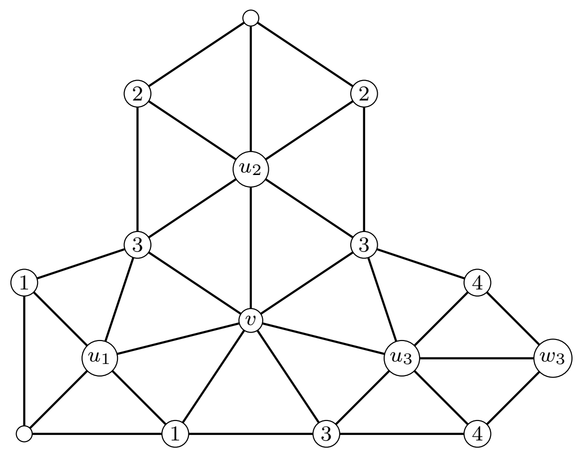
(A) A 7-vertex, v, with non-adjacent 5-neighbors, u_1, u_2, and u_3.



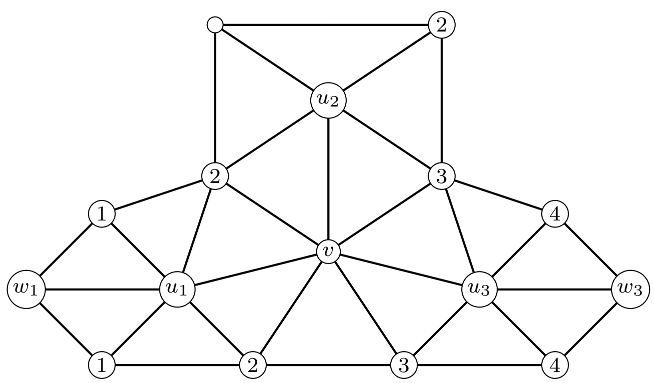
(B) A 7-vertex, v, with a 6-neighbor, u_3, and two 5-neighbors, u_1 and u_2, with all pairs of u_is non-adjacent.



(C) A 7-vertex, v, with a 6-neighbor, u_2, and two 5-neighbors, u_1 and u_3, with all pairs of u_is non-adjacent.



(D) A 7-vertex, v, with a 5-neighbor, u_1, and two 6-neighbors, u_2 and u_3, with all pairs of u_is non-adjacent.



(E) A 7-vertex, v, with a 5-neighbor, u_2, and two 6-neighbors, u_1 and u_3, with all pairs of u_is non-adjacent.

Figure 5. The five cases of Lemma 7.

Finally, consider Figure 5(E). By horizontal symmetry (and planarity), we assume that the vertices labeled 2 that are drawn at distance three are indeed non-adjacent; furthermore, we can assume that the vertices labeled 1 and 2 that are drawn at distance four are distinct. If not, then we reflect across the edge u_2v. Hence, in forming G' we can contract the vertices labeled 2 to a single vertex (we can also contract the vertices labeled 3 to a single vertex). So we only need to consider the vertices labeled 1 and 4. The only possible problem is if some pair of vertices labeled 1 and 4 that are drawn at distance four are actually the same vertex x. Suppose this is the case. If w_1 and w_3 are distinct, then we neglect the vertices labeled 1 and 4 altogether; instead we label w_1 as 2 and w_3 as 3. Due to x, all pairs of vertices with numeric labels are now distance at most three. (Also, we can assume that w_1 is not adjacent to the vertex labeled 2 that is drawn at distance 4; if not, then we again reflect across edge u_1v.) So assume that w_1 and w_3 are identified. Now we switch the vertex identifications we use to form G'. Contract the two vertices labeled 4 onto u_3; also contract onto u_1 its two neighbors labeled 2, the topmost vertex labeled 3, and $w_{1/3}$. As usual, we color this smaller graph by minimality; when we lift this coloring to G, vertex v and each vertex u_i has enough available colors that we can extend the coloring by Fact 1. This finishes Figure 5(E) and completes the proof of the lemma. □

Now we use discharging to prove that every planar graph has a 2-fold 9-coloring.

Theorem 1. *Every planar graph G has a 2-fold 9-coloring. In particular, $\chi_f(G) \leq \frac{9}{2}$.*

Proof. The second statement follows from the first, which we prove now. Let G be a minimal counterexample to the theorem. We will use the discharging method with initial charge $d(v) - 6$ for each vertex v. We write $\text{ch}(v)$ to denote the initial charge and $\text{ch}^*(v)$ to denote the charge after redistributing. By Euler's Formula, $\sum_{v \in V(G)} \text{ch}(v) = -12$. By assuming that G satisfies the conditions stipulated in Lemmas 5–7, we redistribute the charge (without changing its sum) so that every vertex finishes with nonnegative charge. This yields the obvious contradiction $-12 = \sum_{v \in V(G)} \text{ch}(v) = \sum_{v \in V(G)} \text{ch}^*(v) \geq 0$.

We need a few definitions. For a vertex v, let H_v denote the subgraph induced by the 5-neighbors and 6-neighbors of v. If some $w \in V(H_v)$ has $d_{H_v}(w) = 0$, then w is an *isolated* neighbor of v; otherwise w is a *non-isolated* neighbor. A non-isolated 5-neighbor of a vertex v is *crowded* (with respect to v) if it has two 6-neighbors in H_v. We use crowded 5-neighbors in the discharging proof to help ensure that 7-vertices finish with sufficient charge, specifically to handle the configuration in Figure 6. We redistribute charge via the following four rules; they are applied simultaneously, wherever applicable.

- (R1) Each 8^+-vertex gives charge $\frac{1}{2}$ to each isolated 5-neighbor and charge $\frac{1}{4}$ to each non-isolated 5-neighbor.
- (R2) Each 7-vertex gives charge $\frac{1}{2}$ to each isolated 5-neighbor, charge 0 to each crowded 5-neighbor and charge $\frac{1}{4}$ to each remaining 5-neighbor.
- (R3) Each 7^+-vertex gives charge $\frac{1}{4}$ to each 6-neighbor.
- (R4) Each 6-vertex gives charge $\frac{1}{2}$ to each 5-neighbor.

To show that every vertex v finishes with nonnegative charge, we consider $d(v)$.

$\mathbf{d(v) \geq 8}$: We will show that v gives away charge at most $\frac{d(v)}{4}$. Since $d(v) \geq 8$, we have $\text{ch}(v) = d(v) - 6 \geq \frac{d(v)}{4}$, so this will imply $\text{ch}^*(v) \geq 0$. Rather than giving away charge by rules (R1) and (R3), instead let v give charge $\frac{1}{4}$ to each neighbor. Now let each isolated 5-neighbor w take also the charge $\frac{1}{4}$ that v gave to the neighbor that clockwise around v succeeds w. Now each neighbor of v has received at least as much charge as by rules (R1) and (R3) and v has given away charge $\frac{d(v)}{4}$. Thus, when v gives away charge according to rules (R1) and (R3), this charge is at most $\frac{d(v)}{4}$, so $\text{ch}^*(v) \geq 0$.

$\mathbf{d(v) = 7}$: First, suppose that v has an isolated 5-neighbor w. Let $x, y \in N(v)$ be the two 7^+-vertices that are common neighbors of v and w. We will show that the total charge that v gives to $N(v) \setminus \{x, y\}$ is at most $\frac{1}{2}$. By Lemma 7, these four remaining vertices include at most two 6^--vertices. So, if v gives them a total of more than $\frac{1}{2}$, then one of them must be another isolated 5-neighbor. But now the final 6^--vertex must be at distance 2 from each of the previous 5-neighbors, violating Lemma 7.

So instead assume that v has no isolated 5-neighbors. Thus, if v loses total charge more than 1, then it must have at least five 6^--neighbors that receive charge from it (since they each take charge $\frac{1}{4}$). So assume that $|H_v| \geq 5$. This implies that H_v consists of either (i) a 7-cycle or (ii) a single path or (iii) two paths. Recall from Lemma 6, that no 6-vertex has non-adjacent 6^--neighbors. This means that every vertex of degree 2 in H_v is a 5-vertex; in other words, every vertex on a cycle or in the interior of a path in H_v is a 5-vertex.

Now in each of cases (i)–(iii), H_v has an independent 3-set containing at least one 5-vertex; the only exception is if H_v consists of a path on two vertices and a path on three vertices, and the only 5-vertex is the internal vertex on the longer path. However, in this case the 5-vertex is a crowded neighbor of v, as in Figure 6, so it receives no charge from v. Thus, $\text{ch}^*(v) \geq 0$.

$\mathbf{d(v) = 6}$: By Lemma 6, we know that v has at most two 6^--neighbors (and if exactly two, then they are adjacent). Now (R3) implies that $\text{ch}^*(v) \geq 0 + 4(\frac{1}{4}) - 2(\frac{1}{2}) = 0$.

$\mathbf{d(v) = 5}$: If v has at least two 6-neighbors, then $\text{ch}^*(v) \geq -1 + 2(\frac{1}{2}) = 0$; so assume that v has at most one 6-neighbor. Now if v has at least four 6^+-neighbors, then $\text{ch}^*(v) \geq -1 + 4(\frac{1}{4}) = 0$ (since v has at most one 6-neighbor, v is not a crowded neighbor for any of its 7-neighbors); so v must have at least two 5-neighbors. By Lemma 5, these 5-neighbors must be adjacent and v has no 6-neighbors. But now one of v's three 7^+-neighbors sees v as an isolated 5-neighbor, so sends v charge $\frac{1}{2}$. Thus, $\text{ch}^*(v) \geq -1 + \frac{1}{2} + 2(\frac{1}{4}) = 0$. This completes the proof. □

3. Independence Ratio of Planar Graphs

We write $\alpha(G)$ to denote the maximum size of an independent set in a graph G. The *independence ratio* of a graph G is the quantity $\frac{\alpha(G)}{|V(G)|}$. Albertson [1] proved that every planar graph has independence ratio at least $\frac{2}{9}$. In this section, we strengthen his result by showing that every planar graph has independence ratio at least $\frac{3}{13}$. The discharging rules we use here are similar to (but more complicated than) those in the previous section. The reducible configurations are also often different. Even for those configurations which are common to both sections, the proofs of reducibility differ.

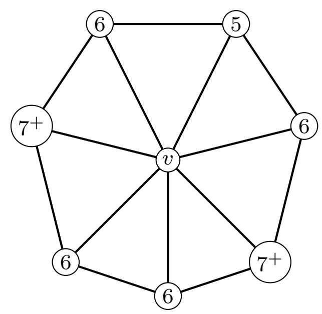


FIGURE 6. A 7-vertex v gives no charge to any crowded 5-neighbor.

As we mentioned in the introduction, our proof of the main result in this section follows the same general outline as the proof in the previous section. Specifically, we will use the discharging method, where each vertex v has initial charge $d(v) - 6$. We will redistribute charge so that every vertex finishes with nonnegative charge. To this end, we want to show that a minimal counterexample to the theorem cannot contain too many 6^--vertices near each other, since 5-vertices will need to receive charge and 6-vertices will have no spare charge to give away. Most of the work in this section goes into proving various formalizations of this intuition. We call this portion of the proof *reducibility*.

3.1. **Reducibility.** As in the previous proof, it is useful to know that a minimal counterexample has no separating 3-cycle; we prove this in Lemma 8. When proving coloring results, such a lemma is nearly trivial. However, for independece results, it requires much more work. Albertson proved an analogous lemma when showing that planar graphs have independence ratio at least $\frac{2}{9}$. Our proof generalizes his to the broader context of showing that all minor-closed graphs have independence ratio at least c for some rational c. We will apply this lemma to planar graphs and will let $c = \frac{3}{13}$.

Lemma 8. *Let $c > 0$ be rational. Let $\mathcal{G}$ be a minor-closed family of graphs. If G is a minimal counterexample to the statement that every n-vertex graph in $\mathcal{G}$ has an independent set of size at least cn, then G has no separating 3-cycle.*

Proof. Suppose to the contrary that G has a separating 3-cycle X. Let A_1 and A_2 be induced subgraphs of G with $V(A_1) \cap V(A_2) = X$ and $A_1 \cup A_2 = G$.

Our plan is to find big independent sets in two smaller graphs in $\mathcal{G}$ (by minimality) and piece those independent sets together to get an independent set in G of size at least $c|G|$. More precisely, we consider independent sets in each A_i, either with X deleted, or with some pair of vertices in X contracted. In Claims 1–3, we prove lower bounds on $\alpha(G)$ in terms of $|A_1|$ and $|A_2|$. In Claim 4, we examine $|A_1|$ and $|A_2|$ modulo b, where $c = \frac{a}{b}$ in lowest terms. In each case, we show that one of the independent sets constructed in Claims 1–3 has size at least $c|G|$. Our proof relies heavily on the fact that $\alpha(H)$ is an integer (for every graph H), which often allows us to gain slightly over $c|H|$.

Claim 1. $\alpha(G) \geq \lceil c(|A_1| - 3) \rceil + \lceil c(|A_2| - 3) \rceil$.

The union of the independent sets obtained by applying minimality of G to $A_1 \setminus X$ and $A_2 \setminus X$ is independent in G.

Claim 2. *$\alpha(G) \geq \lceil c(|A_i|-2)\rceil + \lceil c|A_j|\rceil - 1$ whenever $\{i,j\} = \{1,2\}$.*

For concreteness, let $i = 1$ and $j = 2$; the other case is analogous. Apply minimality to A_2 to get an independent set I_2 in A_2 with $|I_2| \geq \lceil c|A_2|\rceil$. Form A_1' from A_1 by contracting X to a single vertex u. Apply minimality to A_1' to get an independent set I_1 in A_1' with $|I_1| \geq \lceil c(|A_1|-2)\rceil$. If $u \in I_1$, then $I_1 \cup I_2 \setminus \{u\}$ is independent in G and has the desired size. Otherwise, $I_1 \cup I_2 \setminus X$ is an independent set of the desired size in G.

Claim 3. *$\alpha(G) \geq \lceil c(|A_1|-1)\rceil + \lceil c(|A_2|-1)\rceil - 1$.*

Let $X = \{x_1, x_2, x_3\}$. For each $k \in \{1,2\}$ and $t \in \{2,3\}$, form $A_{k,t}$ from A_k by contracting x_1x_t to a vertex $x_{k,t}$. Applying minimality to $A_{k,t}$ gives an independent set $I_{k,t}$ in $A_{k,t}$ with $|I_{k,t}| \geq \lceil c(|A_k|-1)\rceil$.

If at most one of $I_{1,t}$ and $I_{2,t}$ contains a vertex of X (or a contraction of two vertices in X), then to get a big independent set, we take their union, discarding this at most one vertex. Formally, if $\{x_{k,t}, x_{5-t}\} \cap I_{k,t} = \emptyset$, then $I_{k,t} \cup I_{3-k,t} \setminus X$ is an independent set in G of the desired size. So assume that each of $I_{1,t}$ and $I_{2,t}$ contains a vertex (or a contraction of an edge) of X.

Now we look for a vertex x_ℓ of X such that each of $I_{1,t}$ and $I_{2,t}$ contains x_ℓ or a contraction of x_ℓ. Formally, if $x_{5-t} \in I_{k,t} \cap I_{3-k,t}$, then $I_{k,t} \cup I_{3-k,t} \setminus X$ is an independent set in G of the desired size. Similarly, if $x_{k,t} \in I_{k,t}$ and $x_{3-k,t} \in I_{3-k,t}$, then $I_{k,t} \cup I_{3-k,t} \cup \{x_1\} \setminus \{x_{k,t}, x_{3-k,t}\}$ is an independent set in G of the desired size.

So, by symmetry, we may assume that $x_{1,2} \in I_{1,2}$ and $x_3 \in I_{2,2}$. Also, either $x_{1,3} \in I_{1,3}$ or $x_{2,3} \in I_{2,3}$. If $x_{1,3} \in I_{1,3}$, then $I_{2,2} \cup I_{1,3} \setminus \{x_{1,3}\}$ is an independent set in G of the desired size. Otherwise, $x_{2,3} \in I_{2,3}$ and $I_{1,2} \cup I_{2,3} \cup \{x_1\} \setminus \{x_{1,2}, x_{2,3}\}$ is an independent set in G of the desired size.

Claim 4. *The lemma holds.*

Let a and b be positive integers such that $c = \frac{a}{b}$ and $\gcd(a,b) = 1$. For each $i \in \{1,2\}$, let $N_i = |A_i| - 3$ and for each $j \in \{0,1,2,3\}$, choose k_i^j such that $1 \leq k_i^j \leq b$ and $k_i^j \equiv a(N_i + j) \pmod b$. In other words, $\lceil c(N_i + j)\rceil = \frac{a}{b}(N_i + j) + \frac{b - k_i^j}{b}$. Intuitively, if there exist i and j such that k_i^j is small compared to b, then we improve our lower bound on the independence number (in some smaller graph) by the fact that the independence number is always an integer. In the present claim, we show that if some k_i^j is small, then G has an independent set of the desired size. In contrast, if all k_i^j are big, then we get a contradiction.

By symmetry, we may assume that $k_1^0 \leq k_2^0$.

Subclaim 4a. *$k_1^0 + k_2^0 \geq 2b + 1 - 3a$ and $k_1^1 + k_2^3 \geq b + a + 1$ and $k_1^3 + k_2^1 \geq b + a + 1$ and $k_1^2 + k_2^2 \geq b + a + 1$.*

If any independent set constructed in Claims 1–3 has size at least $c|G|$, then we are done. So we assume not; more precisely, we assume that each of these independent sets has size at most $\frac{a|G|-1}{b}$. Each of the four desired bounds follow from simplifying the inequalities in Claims 1–3. Note that $|G| = N_1 + N_2 + 3$.

By Claim 1, we have $\alpha(G) \geq \lceil c(|A_1|-3)\rceil + \lceil c(|A_2|-3)\rceil = \frac{a}{b}(N_1 + N_2) + \frac{b-k_1^0}{b} + \frac{b-k_2^0}{b} = \frac{a}{b}|G| + \frac{2b-3a-k_1^0-k_2^0}{b}$. Hence $k_1^0 + k_2^0 \geq 2b + 1 - 3a$.

By Claim 2, we have $\alpha(G) \geq \lceil c(|A_1|-2)\rceil + \lceil c|A_2|\rceil - 1 = \frac{a}{b}(N_1 + 1 + N_2 + 3) + \frac{b-k_1^1}{b} + \frac{b-k_2^3}{b} - 1 = \frac{a}{b}|G| + \frac{2b+a-k_1^1-k_2^3}{b} - 1$. Hence $k_1^1 + k_2^3 \geq b + a + 1$. Similarly, $k_1^3 + k_2^1 \geq b + a + 1$.

By Claim 3, we have $\alpha(G) \ge \lceil c(|A_1|-1)\rceil + \lceil c(|A_2|-1)\rceil - 1 \ge \frac{a}{b}(N_1+2+N_2+2) + \frac{b-k_1^2}{b} + \frac{b-k_2^2}{b} - 1 = \frac{a}{b}|G| + \frac{2b+a-k_1^2-k_2^2}{b} - 1$. Hence $k_1^2 + k_2^2 \ge b+a+1$.

Now to get a contradiction, it suffices to show that $k_i^j \le a$ for some $i \in \{1,2\}$ and some $j \in \{1,2,3\}$; since $k_i^j \le b$ for all i and j, this will contradict one of the equalities above.

Subclaim 4b. *Either $k_2^1 \le a$ or $k_2^2 \le a$. In each case we get a contradiction, so the claim is true, and the lemma holds.*

By Subclaim 4a, we have $k_1^0 + k_2^0 \ge 2b+1-3a$. By symmetry, we assumed $k_2^0 \ge k_1^0$, so we have $k_2^0 \ge \frac{2b+1-3a}{2}$. Since, $k_2^2 \equiv k_2^0 + 2a \pmod b$ and $\frac{2b+1-3a}{2} + 2a > b$, we have $k_2^2 \le k_2^0 + 2a - b$. Now we consider two cases, depending on whether $k_2^0 \le b-a$ or $k_2^0 \ge b-a+1$. If $k_2^0 \le b-a$, then $k_2^2 \le k_2^0 + 2a - b \le (b-a) + 2a - b = a$, a contradiction. Suppose instead that $k_2^0 \ge b-a+1$. Now $k_2^1 \equiv k_2^0 + a \pmod b$. Since $k_2^0 \ge b-a+1$, we see that $k_2^0 + a \ge b+1$, so $k_2^1 \le k_2^0 + a - b \le a$, a contradiction. □

Now we turn to proving a series of lemmas showing that G can't have too many 6^--vertices near each other. Many of these lemmas will rely on applications of the following result, which we think may be of independent interest. The idea for the proof is to find big independent sets for two smaller graphs, and piece them together to get a big independent set in G.

For $S \subseteq V(G)$, let the *interior* of S be $\mathcal{I}(S) = \{x \in S | N(x) \subseteq S\}$. For vertex sets $V_1, V_2 \subset V(G)$ we write $V_1 \leftrightarrow V_2$ if there exists an edge $v_1v_2 \in E(G)$ with $v_1 \in V_1$ and $v_2 \in V_2$; otherwise, we write $V_i \not\leftrightarrow V_j$.

Lemma 9. *Let $\mathcal{G}$ be a minor-closed family of graphs. Let G be a minimal counterexample to the statement that every n-vertex graph in $\mathcal{G}$ has an independent set of size at least cn (for some fixed $c > 0$). Let $S_1, \ldots, S_t$ be pairwise disjoint subsets of a nonempty set $S \subseteq V(G)$ such that $t < |S|$ and $G[S_i]$ is connected for all $i \in \{1, \ldots, t\}$. Now there exists $X \subseteq \{1, ..., t\}$ such that $S_i \not\leftrightarrow S_j$ for all distinct $i, j \in X$ and $\alpha\left(G\left[\mathcal{I}(S) \cup \bigcup_{i\in X} S_i\right]\right) < |X| + \lceil c(|S|-t)\rceil$.*

Proof. Suppose to the contrary that $\alpha\left(G\left[\mathcal{I}(S) \cup \bigcup_{i\in X} S_i\right]\right) \ge |X| + \lceil c(|S|-t)\rceil$ for all $X \subseteq \{1, ..., t\}$ such that $S_i \not\leftrightarrow S_j$ for all distinct $i, j \in X$. Create G' from G by contracting S_i to a single vertex w_i for each $i \in \{1, \ldots, t\}$ and removing the rest of S. (Note that we allow $t = 0$.) Since $t < |S|$, we have $|G'| < |G|$ and hence minimality of G gives an independent set I in G' with $|I| \ge c|G'| = c(|G| - |S| + t)$. Let $W = I \cap \{w_1, \ldots, w_t\}$. By assumption, we have $\alpha\left(G\left[\mathcal{I}(S) \cup \bigcup_{w_i\in W} S_i\right]\right) \ge |W| + \lceil c(|S|-t)\rceil$. If T is a maximum independent set in $G\left[\mathcal{I}(S) \cup \bigcup_{w_i\in W} S_i\right]$, then $(I \setminus W) \cup T$ is an independent set in G of size at least $|I| - |W| + |T| \ge c(|G| - |S| + t) - |W| + (|W| + \lceil c(|S|-t)\rceil) \ge c|G|$, a contradiction. □

We will often apply Lemma 9 with $S = J \cup N(J)$ for an independent set J. In this case, we always have $J \subseteq \mathcal{I}(S)$. We state this case explicitly in Lemma 10

Lemma 10. *Let $\mathcal{G}$ be a minor-closed family of graphs. Let G be a minimal counterexample to the statement that every n-vertex graph in $\mathcal{G}$ has an independent set of size at least cn (for some fixed $c > 0$). No independent set J of G and nonnegative integer k satisfy the following conditions.*

(1) $|J| \ge c(|N(J)| + k)$.

(2) *For at least* $|J|-k$ *vertices* $x \in J$*, there is an independent set* $\{u_x, v_x\}$ *of size 2 in* $N(x) \setminus \bigcup_{y \in J\setminus\{x\}} N(y)$.

Proof. Suppose the lemma is false. Let $S = J \cup N(J)$ and $t = |J| - k$. Pick $x_1, \ldots, x_t \in J$ satisfying condition (2). For $i \in \{1, \ldots, t\}$, let $S_i = \{x_i, u_{x_i}, v_{x_i}\}$. Applying Lemma 9, we get $X \subseteq \{1, \ldots, t\}$ such that $S_i \not\leftrightarrow S_j$ for all distinct $i, j \in X$ and $\alpha\left(G\left[J \cup \bigcup_{i\in X} S_i\right]\right) < |X| + \lceil c(|S| - t)\rceil$. By (2), we have $\alpha\left(G\left[J \cup \bigcup_{i\in X} S_i\right]\right) \geq |(J \setminus X) \cup \bigcup_{x\in X}\{u_x, v_x\}| \geq (|J| - |X|) + 2|X| = |X| + |J|$. Hence $|X| + \lceil c(|S| - t)\rceil > |X| + |J|$, giving $\lceil c(|S| - t)\rceil > |J| \geq \lceil c(|N(J)| + k)\rceil$ by (1). But $|S| - t = (|J| + |N(J)|) - (|J| - k) = |N(J)| + k$; so $\lceil c(|S| - t)\rceil = \lceil c(|N(J)| + k)\rceil$, contradicting the previous inequality. This contradiction finishes the proof. □

As a simple example of how to apply Lemma 10, we note that it immediately implies that every planar graph G has independence ratio at least $\frac{1}{5}$. By Euler's theorem, G has a 5^--vertex v. If $d(v) \leq 4$, then let $G' = G \setminus (v \cup N(v))$. Let I' be an independent set in G' of size at least $(n-5)/5$, and let $I = I' \cup \{v\}$. If instead $d(v) = 5$, then apply Lemma 10, with $c = \frac{1}{5}$, $J = \{v\}$, and $k = 0$; since K_6 is non-planar, v has some pair of non-adjacent neighbors. This completes the proof.

Lemma 11. *Let* $\mathcal{G}$ *be a minor-closed family of graphs. Let* G *be a minimal counterexample to the statement that every* n*-vertex graph in* $\mathcal{G}$ *has an independent set of size at least* cn *(for some fixed* $c > 0$*). For any non-dominating independent set* J *in* G*, we have*

$$|N(J)| \geq \left\lfloor \frac{1-c}{c}|J| \right\rfloor + 2.$$

Proof. Assume the lemma is false and choose a counterexample J minimizing $|J|$. Suppose $G[J \cup N(J)]$ is not connected. Now we choose a partition $\{J_1, \ldots, J_k\}$ of J, minimizing k, such that $k \geq 2$ and $G[J_i \cup N(J_i)]$ is connected for each $i \in \{1, \ldots, k\}$. Applying the minimality of $|J|$ to each J_i we conclude that $|N(J_i)| \geq \left\lfloor \frac{1-c}{c}|J_i| \right\rfloor + 2$ for each $i \in \{1, \ldots, k\}$. The minimality of k gives $|N(J)| = \left|\bigcup_{i=1}^k N(J_i)\right| = \sum_{i=1}^k |N(J_i)|$, so $|N(J)| \geq 2k + \sum_{i=1}^k \left\lfloor \frac{1-c}{c}|J_i| \right\rfloor \geq k + \sum_{i=1}^k \frac{1-c}{c}|J_i| \geq 2 + \frac{1-c}{c}|J|$, a contradiction. Hence, $G[J \cup N(J)]$ is connected.

Let $S = J \cup N(J)$. Apply Lemma 9 with $t = 1$ and $S_1 = S$. This shows that either $|J| \leq \alpha(G[\mathcal{I}(S)]) < \lceil c(|S| - 1)\rceil$ or $\alpha(G[S]) < 1 + \lceil c(|S| - 1)\rceil$, since the only possibilities are $X = \emptyset$ and $X = \{1\}$. By assumption J is a counterexample, so $|N(J)| \leq \left\lfloor \frac{1-c}{c}|J| \right\rfloor + 1$, which implies that $|S| = |J| + |N(J)| \leq |J| + \left\lfloor \frac{1-c}{c}|J| \right\rfloor + 1 = \left\lfloor \frac{|J|}{c} \right\rfloor + 1$. Now $\lceil c(|S| - 1)\rceil \leq \left\lceil c\left(\left(\left\lfloor \frac{|J|}{c} \right\rfloor + 1\right) - 1\right) \right\rceil = \left\lceil c \left\lfloor \frac{|J|}{c} \right\rfloor \right\rceil \leq \lceil |J| \rceil = |J|$. Hence, we cannot have $X = \emptyset$ in Lemma 9.

Instead, we must have $X = \{1\}$, which implies that $\alpha(G[S]) < 1 + \lceil c(|S| - 1)\rceil$. Since J is non-dominating, we have $S \neq V(G)$, so we may apply minimality of G to $G[S]$ to conclude that $\alpha(G[S]) \geq \lceil c|S|\rceil$. Combining this inequality with the previous one, we have $\lceil c|S|\rceil = \lceil c(|S| - 1)\rceil$. Now the bound on $\lceil c(|S| - 1)\rceil$ gives $\lceil c|S|\rceil = \lceil c(|S| - 1)\rceil \leq \left\lceil c \left\lfloor \frac{|J|}{c} \right\rfloor \right\rceil \leq |J|$. Finally, applying Lemma 9 with $t = 0$ (simply deleting $J \cup N(J)$) shows that $|J| < \lceil c(|S|)\rceil$. These two final inequalities contradict each other, which finishes the proof. □

Lemmas 8–11 hold in a more general setting than just $c = \frac{3}{13}$, as we showed. In the rest of this section, we consider only a planar graph G that is minimal among those with independence ratio less than $\frac{3}{13}$. To remind the reader of this, we often call it a *minimal* G. Applying Lemma 11 with $c = \frac{3}{13}$ gives the following corollary.

Lemma 12. *For any non-dominating independent set J in a minimal G, we have*

$$|N(J)| \geq \left\lfloor \frac{10}{3}|J| \right\rfloor + 2.$$

In particular, if $|J| = 1$, then $|N(J)| \geq 5$; if $|J| = 2$, then $|N(J)| \geq 8$; and if $|J| = 3$, then $|N(J)| \geq 12$.

The case $|J| = 1$ shows that G has minimum degree 5, and this is the best we can hope for when $|J| = 1$. Since G is a planar triangulation, we can improve the bound when $|J| = 2$ to $|N(J)| \geq 9$. Similarly, in many cases we can improve the bound when $|J| = 3$ to $|N(J)| \geq 13$. These improvements are the focus of the next ten lemmas. In many instances, the proofs are easy applications of Lemma 9. First, we need a few basic facts about planar graphs.

Lemma 13. *If G is a plane triangulation with no separating 3-cycle and $\delta(G) = 5$, then*

(a) *If $v \in V(G)$, then $G[N(v)]$ is a cycle; and*
(b) *G is 4-connected with $|G| \geq 12$; and*
(c) *If $v, w \in V(G)$ are distinct, then $G[N(v) \cap N(w)]$ is the disjoint union of copies of K_1 and K_2.*

Proof. Plane triangulations are well-known to be 3-connected. Property (a) follows by noting that $G \setminus \{v\}$ is 2-connected and hence each face boundary is a cycle; so $G[N(v)]$ has a hamiltonian cycle. This cycle must be induced since G has no separating 3-cycle.

For (b), suppose that G has a separating set $\{x, y, z\}$. Since G has no separating 3-cycle, we assume that $xy \notin E(G)$. Since G is 3-connected, and $G \setminus \{x, y, z\}$ is disconnected, the vertices of $N(x)$ must be disconnected in $G \setminus \{x, y, z\}$. Since $xy \notin E(G)$, we get that $G[N(x)]$ has a separating set contained in $\{z\}$, contradicting (1). Since G is a plane triangulation and $\delta(G) = 5$, we have $5|G| \leq 2|E(G)| = 6|G| - 12$, so $|G| \geq 12$.

By (a) and $\delta(G) = 5$, it follows that no neighborhood contains K_3 or C_4. If $G[N(v) \cap N(w)]$ had an induced P_3, then the neighborhood of the center of this P_3 would contain K_3 or C_4. This proves (c). □

Lemma 14. *Every independent set J in a minimal G with $|J| = 2$, satisfies $|N(J)| \geq 9$.*

Proof. By Lemma 13(b), $|G| \geq 12$; so J cannot be dominating when $|N(J)| \leq 7$. Hence, by Lemma 12, we may assume $|N(J)| = 8$. Let $J = \{x, y\}$. If we can apply Lemma 10 with $k = 0$, then we are done. If we cannot, then by symmetry we may assume that there is no independent 2-set in $N(x) \setminus N(y)$. So $N(x) \setminus N(y)$ is a clique. Since $d(x) \geq 5$ and $N(x)$ induces a cycle, $|N(x) \setminus N(y)| \leq 2$. Now, since x is a 5^+-vertex, $G[N(x) \cap N(y)]$ induces P_3; this contradicts Lemma 13(c). □

A direct consequence of Lemma 14 is the following useful fact.

Lemma 15. *A minimal G has no non-adjacent 5-vertices u and w with at least two common neighbors. In particular, each vertex v in G has $\frac{d(v)}{2}$ or more 6^+-neighbors.*

Proof. The first statement follows immediately from Lemma 14. Now we consider the second. Let v be a vertex with $d(v) = k$ and neighbors $u_1, \ldots, u_k$ in clockwise order. If more than $k/2$ neighbors of v are 5-vertices, then (by Pigeonhole) there exists an integer i such that u_i and u_{i+2} are 5-vertices (subscripts are modulo k). Now we apply Lemma 14 to u_i and u_{i+2}. □

Now we consider the case when $|J| = 3$. Lemma 12 gives $|N(J)| \geq 12$. Our next few lemmas show certain conditions under which we can conclude that $|N(J)| \geq 13$.

Lemma 16. *Let J be an independent set in a minimal G with $|J| = 3$ and $|N(J)| \geq 12$. Choose $S_1, S_2 \subseteq J \cup N(J)$ such that $S_1 \cap S_2 = \emptyset$ and both $G[S_1]$ and $G[S_2]$ are connected. If $\alpha(G[S_i \cup J]) \geq 4$ for each $i \in \{1, 2\}$, then $|N(J)| \geq 13$.*

Proof. Suppose not and choose a counterexample minimizing $|J \cup N(J)| - |S_1 \cup S_2|$. Clearly $|N(J)| = 12$. First we show that $S_1 \cup S_2 = J \cup N(J)$. It suffices to show that $G[J \cup N(J)]$ is connected, since then we can add to either S_1 or S_2 any vertex in $N(S_1 \cup S_2) \setminus (S_1 \cup S_2)$. In particular, we show that every $x \in J$ satisfies $(x \cup N(x)) \cap (\cup_{y \in (J \setminus \{x\})}(y \cup N(y))) \neq \emptyset$. If not, then $| \cup_{y \in J \setminus \{x\}} (y \cup N(y))| \leq |J \cup N(J)| - (d(x) + 1) \leq 15 - 6 = 9$. However, now $J \setminus \{x\}$ violates Lemma 14. So, we must have $S_1 \cup S_2 = J \cup N(J)$. Similarly, $S_1 \leftrightarrow S_2$.

Now we apply Lemma 9 with $S = J \cup N(J)$, $t = 2$, and S_1 and S_2 as above. Since $S_1 \leftrightarrow S_2$, we have $|X| \leq 1$. By hypothesis, $\alpha(G[S_i \cup J]) \geq 4$ for each $i \in \{1, 2\}$, so suppose that $X = \emptyset$. Now we have $\alpha(G[J]) \geq |J| = 3 = \left\lceil \frac{3}{13}(|J \cup N(J)| - 2) \right\rceil = \left\lceil \frac{3}{13}(3 + 12 - 2) \right\rceil$. This contradiction completes the proof. □

Lemma 17. *Let $J = \{u_1, u_2, u_3\}$. If J is an independent set in a minimal G where*

(1) $N(u_1) \setminus (N(u_2) \cup N(u_3))$ *contains an independent 2-set; and*
(2) $\alpha(G[J \cup N(u_2) \cup N(u_3)]) \geq 4$,

then $|N(J)| \geq 13$.

Proof. Since G is a planar triangulation with minimum degree 5 and at least three 6^+-vertices by Lemma 15, we have $5|G| + 3 \leq 2|E(G)| = 6|G| - 12$ and hence $|G| \geq 15$. Thus J cannot be dominating when $|N(J)| \leq 11$. So, by Lemma 12, we know that $|N(J)| \geq 12$. Let I be an independent set of size 2 in $N(u_1) \setminus (N(u_2) \cup N(u_3))$.

First, suppose $N(u_2) \cap N(u_3) \neq \emptyset$. We apply Lemma 16 with $S_1 = \{u_1\} \cup I$ and $S_2 = \{u_2, u_3\} \cup N(u_2) \cup N(u_3)$. Clearly, $G[S_1]$ is connected. Also, $G[S_2]$ is connected since $N(u_2) \cap N(u_3) \neq \emptyset$, by assumption. The set $I \cup \{u_2, u_3\}$ shows that $\alpha(G[S_1 \cup J]) \geq 4$ and hypothesis (2) shows that $\alpha(G[S_2 \cup J]) \geq 4$. So the hypotheses of Lemma 16 are satisfied, giving $|N(J)| \geq 13$.

Instead, suppose $N(u_2) \cap N(u_3) = \emptyset$. This implies $N(u_2) \setminus (N(u_1) \cup N(u_3)) = N(u_2) \setminus N(u_1)$. If $N(u_2) \setminus N(u_1)$ contains an independent 2-set as well, then applying Lemma 10 with $k = 1$ gives $|N(J)| \geq 13$, as desired. Otherwise, $|N(u_2) \setminus N(u_1)| \leq 2$, so $G[N(u_2) \cap N(u_1)]$ contains P_3, contradicting Lemma 13(c). □

One particular case of Lemma 17 is very easy to verify in our applications, so we state it separately, as Lemma 19. First, we need the following lemma.

Lemma 18. *Let v be a 7^+-vertex in G. If $S \subseteq V(G)$ with $\{v\} \cup N(v) \subseteq S$ and $|S| \geq 10$, then $\alpha(G[S]) \geq 4$.*

Proof. If $d(v) \geq 8$, then we are done, since the neighbors of v induce an 8^+-cycle, which has independence number at least 4. So suppose $d(v) = 7$. Let $u_1, \ldots, u_7$ denote the neighbors of v in clockwise order; recall that $G[N(v)]$ is a 7-cycle. Pick $w_1, w_2 \in S \setminus (\{v\} \cup N(v))$. Let $H_i = G[N(v) \setminus N(w_i)]$ for each $i \in \{1, 2\}$. If H_i contains an independent 3-set for some $i \in \{1, 2\}$, then $J \cup \{w_i\}$ is the desired independent 4-set, so we are done. Therefore, we must have $|H_i| \leq 4$ for each $i \in \{1, 2\}$. So, $|N(v) \cap N(w_i)| \geq 3$ and hence Lemma 13(c) shows that $N(v) \cap N(w_i)$ has at least two components; therefore, so does H_i. It must have exactly two components or we get an independent 3-set in H_i. Similarly, if $|H_i| = 4$, then H_i has no isolated vertex. So, either H_i is $2K_2$ or $|H_i| \leq 3$. Now in each case we get a subdivision of $K_{3,3}$; the branch vertices of one part are v, w_1, w_2 and the branch vertices of the other are three of the u_i. This contradiction finishes the proof. □

Lemma 19. *Let $J = \{u_1, u_2, u_3\}$. If J is an independent set in a minimal G where*

(1) $N(u_1) \setminus (N(u_2) \cup N(u_3))$ *contains an independent 2-set; and*
(2) $G[J \cup N(u_2) \cup N(u_3)]$ *contains a 7^+-vertex and its neighborhood,*

then $|N(J)| \geq 13$.

Proof. We apply Lemma 17 using Lemma 18 to verify hypothesis (2). To do so, we need that $|J \cup N(u_2) \cup N(u_3)| \geq 10$; this is immediate from Lemma 14. □

Lemma 20. *Let J be an independent 3-set in G. Choose $S_1, S_2, S_3 \subseteq J \cup N(J)$ such that $G[S_i]$ is connected and $S_i \cap S_j = \emptyset$ for all distinct $i, j \in \{1, 2, 3\}$. If $|N(J)| \leq 13$, then either*

(1) $S_i \not\leftrightarrow S_j$ *for some* $\{i, j\} \subseteq \{1, 2, 3\}$; *or*
(2) $\alpha(G[S_i \cup J]) \leq 3$ *for some* $i \in \{1, 2, 3\}$.

Proof. This is an immediate corollary of Lemma 9 with $S = J \cup N(J)$ and $t = 0$. If $S_i \leftrightarrow S_j$ for all $\{i, j\} \in \{1, 2, 3\}$, then in Lemma 9 either $|X| = 1$ or $|X| = 0$. We cannot have $|X| = 0$, since $\alpha(G[\mathcal{I}(S)]) \geq \alpha(G[J]) \geq |J| = 3 = \left\lceil \frac{3}{13}(13 + 3 - 3) \right\rceil$. Hence $|X| = 1$, which implies (2). □

The next lemma can be viewed as a variant on the result we get by applying Lemma 10 with $|J| = 3$ and $k = 0$ (and $c = \frac{3}{13}$). As in that case, we require that each of $N(u_1) \setminus (N(u_2) \cup N(u_3))$ and $N(u_2) \setminus (N(u_1) \cup N(u_3))$ contains an independent 2-set. However, here we do not require that $N(u_3) \setminus (N(u_1) \cup N(u_2))$ contains an independent 2-set. Instead, we have hypothesis (2) below. Not surprisingly, the proof is similar to that of Lemma 10.

Lemma 21. *Let $J = \{u_1, u_2, u_3\}$. If J is an independent set in a minimal G such that*

(1) $N(u_i) \setminus (N(u_j) \cup N(u_3))$ *contains an independent 2-set M_i for all* $\{i, j\} = \{1, 2\}$; *and*
(2) $\alpha(G[J \cup V(H)]) \geq 4$, *where H is u_3's component in* $G[\{u_3\} \cup N(J)] \setminus (M_1 \cup M_2)$,

then $|N(J)| \geq 14$.

Proof. First, we show that u_3 is distance two from each of u_1 and u_2. Suppose not; by symmetry, assume that u_3 is distance at least three from u_1. Now $N(u_3) \setminus (N(u_1) \cup N(u_2)) = N(u_3) \setminus N(u_2)$. By Lemma 13, $N(u_3) \cap N(u_2)$ consists of disjoint copies of K_1 and K_2. Thus, since $d(u_3) \geq 5$, we see that $N(u_3) \setminus (N(u_1) \cup N(u_2))$ contains an independent 2-set. Now, if $|N(J)| \leq 13$, then applying Lemma 10 with $k = 0$ gives a contradiction. Hence, u_3 is distance two from each of u_1 and u_2.

Choose disjoint subsets $S_1, S_2, S_3 \subset J \cup N(J)$ where $G[S_i]$ is connected for all $i \in \{1,2,3\}$ and $\{u_i\} \cup M_i \subseteq S_i$ for each $i \in \{1,2\}$ and $u_3 \in S_3$, first maximizing $|S_3|$ and subject to that maximizing $|S_1| + |S_2| + |S_3|$. Since $J \subseteq S_1 \cup S_2 \cup S_3$, maximality of $|S_1| + |S_2| + |S_3|$ gives $S_1 \cup S_2 \cup S_3 = J \cup N(J)$.

Now we apply Lemma 9, with $S = S_1 \cup S_2 \cup S_3$. To get a contradiction, we need only verify, for each possible X, that $\alpha(G[\mathcal{I}(S) \cup \bigcup_{i \in X} S_i]) \geq |X| + \left\lceil \frac{3}{13}(|S| - |J|) \right\rceil = |X| + 3$. Since $S_3 \leftrightarrow S_1$ and $S_3 \leftrightarrow S_2$, either $|X| \leq 1$ or else $X = \{1,2\}$. In the latter case, $M_1 \cup M_2 \cup \{u_3\}$ is the desired independent 5-set. If instead $X = \emptyset$, then J is the desired independent 3-set.

So we must have $X = \{i\}$ for some $i \in \{1,2,3\}$. If $i \in \{1,2\}$, then $M_i \cup \{u_3, u_{3-i}\})$ is the desired independent set. So instead assume that $X = \{3\}$. But, by the maximality of $|S_3|$, $G[J \cup S_3]$ contains u_3's component in $G[\{u_3\} \cup N(J)] \setminus M_1 \setminus M_2$. So by (2), $G[J \cup S_3]$ has an independent 4-set, as desired. □

Again, one particular case of Lemma 21 is easy to verify, so we state it separately.

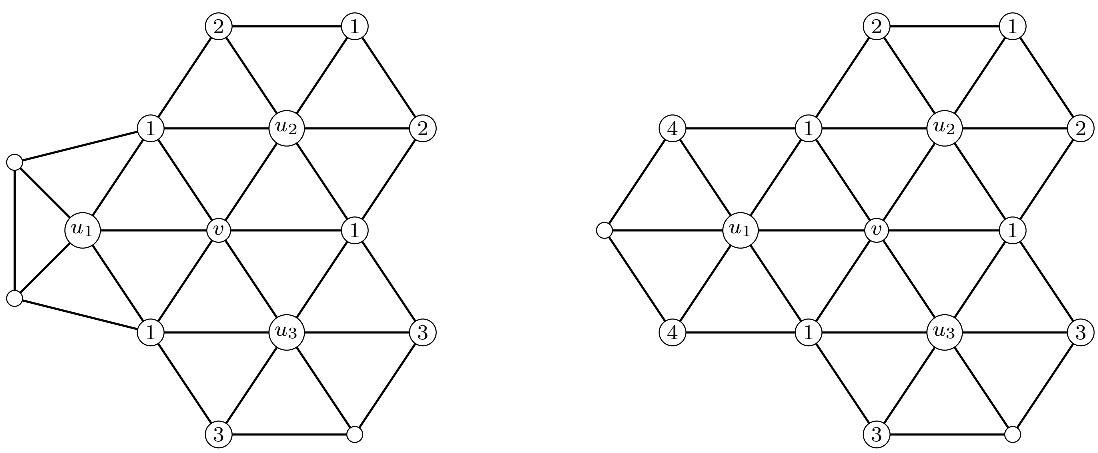
Lemma 22. *Let $J = \{u_1, u_2, u_3\}$. If J is an independent set in a minimal G such that*

(1) *$N(u_i) \setminus (N(u_j) \cup N(u_3))$ contains an independent 2-set M_i for all $\{i,j\} = \{1,2\}$; and*
(2) *u_3's component H in $G[\{u_3\} \cup N(J)] \setminus (M_1 \cup M_2)$ satisfies $|J \cup V(H)| \geq 10$ and $G[J \cup V(H)]$ contains a 7^+ vertex and its neighborhood,*

then $|N(J)| \geq 14$.

Proof. We apply Lemma 21, using Lemma 18 to verify hypothesis (2). □

Thus far, we our lemmas have not focused much on the actual planar embedding of G. At this point we transition and start analyzing the embedding, as well.



(A) A 6-vertex, v, with non-adjacent neighbors u_1, u_2, and u_3 such that $d(u_1) = 5$ and $d(u_2) = d(u_3) = 6$.

(B) A 6-vertex, v, with non-adjacent 6-neighbors u_1, u_2, and u_3.

Figure 7. The two cases of Lemma 23.

Lemma 23. *Every minimal G has no 6-vertex v with 6^--neighbors u_1, u_2 and u_3 that are pairwise non-adjacent.*

Proof. Lemma 12 yields $12 \leq |N(\{u_1, u_2, u_3\})| \leq d(u_1) + d(u_2) + d(u_3) - 5$. Hence, by symmetry, assume that the vertices are arranged as in Figure 7(A) with all vertices distinct as drawn or as in Figure 7(B) with at most one pair of vertices identified.

The first case is impossible by Lemma 10 with $k = 1$, using the vertices labeled 2 for u_2 and those labeled 3 for u_3. When the vertices in Figure 7(B) are distinct as drawn, we apply Lemma 10 with $k = 0$, using the vertices labeled 2 for u_2, the vertices labeled 3 for u_3, and those labeled 4 for u_1. Otherwise, by symmetry and the fact that G contains no separating 3-cycle, assume that the vertices labeled 2 and 3 that are drawn at distance four are identified. Now the pairs of vertices labeled 1 each have a common neighbor, so the vertices labeled 1 must be an independent set, to avoid a separating 3-cycle. Now, to get a contradiction, apply Lemma 17, using the vertices labeled 4 for the independent 2-set. □

Lemma 24. *Every minimal G has no 6-vertex v with pairwise non-adjacent neighbors u_1, u_2, and u_3, where $d(u_1) = 5$, $d(u_2) \leq 6$, and $d(u_3) = 7$.*

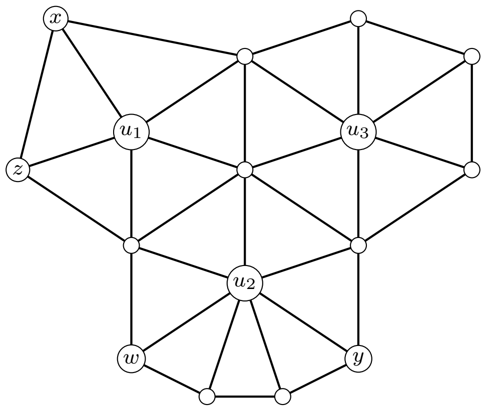
Proof. Let $J = \{u_1, u_2, u_3\}$. By Lemma 12, $12 \leq |N(J)| \leq 5 + 6 + 7 - 5 = 13$, so at most one pair of vertices in Figure 8(A) are identified.

First, suppose the vertices in the figure are distinct as drawn. Suppose $x \not\leftrightarrow y$, as in Figure 8(B). For each $i \in \{1, 2, 3\}$, let S_i consist of the vertices labeled i. Now for each $i \in \{1, 2, 3\}$, $G[S_i]$ is connected. Clearly, for each $i \in \{1, 2\}$ the vertices labeled I_i form an independent 4-set. Since $x \not\leftrightarrow y$, the vertices labeled I_3 also form an independent 4-set. Note that $S_1 \leftrightarrow S_3$ and $S_2 \leftrightarrow S_3$; however, possibly $S_1 \not\leftrightarrow S_2$. If $S_1 \leftrightarrow S_2$, then we can apply Lemma 20 to get a contradiction. So, we assume that $S_1 \not\leftrightarrow S_2$. But now we have an independent 5-set consisting of u_1, the two vertices labeled $\{1, I_1\}$ and the two vertices labeled $\{2, I_2\}$; hence $\alpha(G[S_1 \cup S_2 \cup J]) \geq 5$. So, we can apply Lemma 9 to get a contradiction. So, instead we assume $x \leftrightarrow y$.

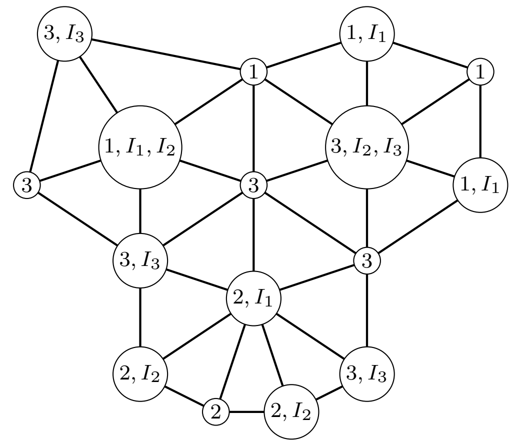
Suppose $w \not\leftrightarrow z$, as in Figure 8(C). For each $i \in \{1, 2, 3\}$, let S_i consist of the vertices labeled i . Clearly $G[S_i]$ is connected for each $i \in \{2, 3\}$. Also, $G[S_1]$ is connected because $x \leftrightarrow y$. Note that for each $i \in \{1, 3\}$, the vertices labeled I_i form an independent 4-set. Since $x \leftrightarrow y$ and $w \not\leftrightarrow z$, the vertices labeled I_2 also form an independent 4-set. Note that $S_1 \leftrightarrow S_2$ and $S_2 \leftrightarrow S_3$; however, possibly $S_1 \not\leftrightarrow S_3$. If $S_1 \leftrightarrow S_3$, then we apply Lemma 20 to get a contradiction. So instead we assume that $S_1 \not\leftrightarrow S_3$. But now we again have an independent 5-set, consisting of u_1, the two vertices labeled $\{1, I_1\}$, and the two vertices labeled $\{3, I_3\}$; hence $\alpha(G[S_1 \cup S_3 \cup J]) \geq 5$. So, again we apply Lemma 9 to get a contradiction. Thus, we instead assume $w \leftrightarrow z$.

Now consider Figure 8(D). For each $i \in \{1, 2, 3\}$, let S_i consist of the vertices labeled i. Note that $G[S_i]$ is connected for each $i \in \{2, 3\}$. Also, $G[S_1]$ is connected because $x \leftrightarrow y$ and $w \leftrightarrow z$. Clearly, the vertices labeled I_i form an independent 4-set for each $i \in \{1, 3\}$. Since $x \leftrightarrow y$, the vertices labeled I_2 also form an independent 4-set. Note that $S_1 \leftrightarrow S_2$ and $S_2 \leftrightarrow S_3$; however, possibly $S_1 \not\leftrightarrow S_3$. If $S_1 \leftrightarrow S_3$, then we apply Lemma 20 to get a contradiction. So, instead we assume that $S_1 \not\leftrightarrow S_3$. But now we have an independent 5-set, consisting of u_1, the two vertices labeled $\{1, I_1\}$, and the two vertices labeled $\{3, I_3\}$; hence $\alpha(G[S_1 \cup S_3 \cup J]) \geq 5$. So, we apply Lemma 9 to get a contradiction.

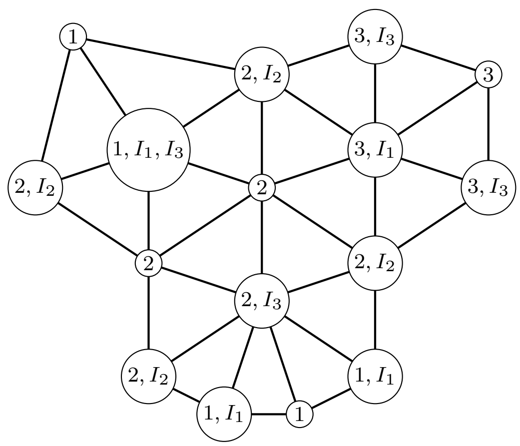
Hence, we may assume that exactly one pair of vertices in Figure 8(A) is identified. No neighbor of u_1 can be identified with a neighbor of u_3, since then u_1 and u_3 would have three



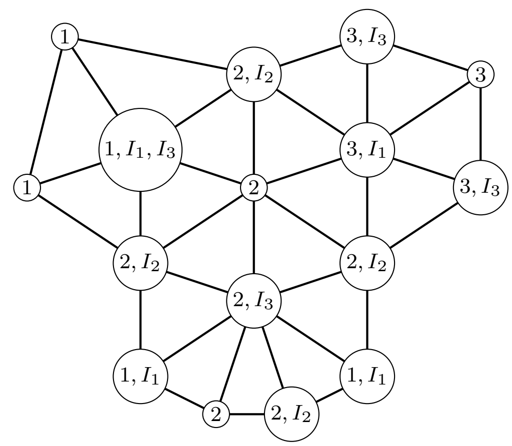
(A) The forbidden configuration.



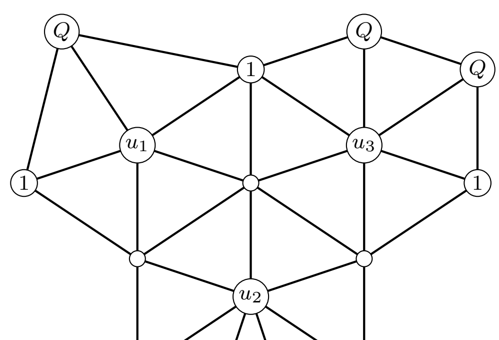
(B) The $x \not\leftrightarrow y$ case.



(C) The $x \leftrightarrow y$ and $w \not\leftrightarrow z$ case.



(D) The $x \leftrightarrow y$ and $w \leftrightarrow z$ case.



(E) The case of one identified pair.

Figure 8. The case of Lemma 24.

common neighbors, violating Lemma 14. Hence, to avoid separating 3-cycles, we assume that a vertex labeled $2a$ is identified with a vertex labeled Q (the case where a vertex labeled $2b$ is identified with a vertex lableled Q is nearly identical, so we omit the details). But now the rightmost vertex labeled 1 and the leftmost vertex labeled 1 are on opposite sides of a separating cycle and hence non-adjacent. Therefore, u_2 together with the vertices labeled 1 is an independent 4-set. So, now we apply Lemma 17 to get a contradiction, using the vertices labeled $2b$ for the independent 2-set. □

Lemma 25. *Let u_1 be a 6-vertex with non-adjacent vertices u_2 and u_3 each at distance two from u_1, where u_2 is a 5-vertex and u_3 is a 6^--vertex. A minimal G cannot have u_1 and u_2 with two common neighbors, and also u_1 and u_3 with two common neighbors.*

Proof. Figure 9 shows the possible arrangements when u_3 is a 6-vertex. The case when u_3 is a 5-vertex is similar, but easier. In particular, when u_3 is a 5-vertex, we already know $|N(\{u_1, u_2, u_3\})| \leq 12$, so all vertices in the corresponding figures must be distinct as drawn. Furthermore, it now suffices to apply Lemma 10 with $k = 1$. We omit further details. So suppose instead that $d(u_3) = 6$.

First, suppose all vertices in the figures are distinct as drawn. Now Figures 9(A,C) are impossible by Lemma 10 with $k = 0$; for each $i \in \{1, 2, 3\}$, we use the vertices labeled i as the independent 2-set for u_i. For Figure 9(B), let I_1 be the vertices labeled u_2 or $1a$ and let I_2 be the vertices labeled u_2 or $1b$. To avoid a separating 3-cycle, at least one of I_1 or I_2 is independent. Hence Figure 9(B) is impossible by Lemma 21; for the independent 4-set, use I_1 or I_2 and for each $i \in \{2, 3\}$, use the vertices labeled i as the independent 2-set for u_i.

By Lemma 12, $|N(J)| \geq 12$, so exactly one pair of vertices is identified in one of Figures 9(A,B,C). First, consider Figures 9(A,C) simultaneously. Since G has no separating 3-cycle, the identified pair must contain a vertex labeled 3. Now we apply Lemma 10 with $k = 1$, using the vertices labeled $3b$ in place of those labeled 3.

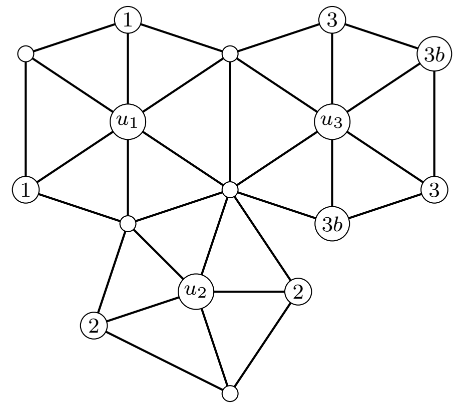
Finally, for Figure 9(B), we apply Lemma 17. For the independent 2-set we use either the vertices labeled 3 or the vertices labeled 4; at least one of these pairs contains no identified vertex. For the independent 4-set, we use either u_3 and the vertices labeled $5a$ or else u_3 and the vertices labeled $5b$. Since G has no separating 3-cycle, at least one of these 4-sets will be independent. □

Lemma 26. *Every minimal G has no 7-vertex v with a 5-neighbor and two other 6^--neighbors, u_1, u_2, and u_3, that are pairwise non-adjacent. In other words, Figures 5(A–E) are forbidden.*

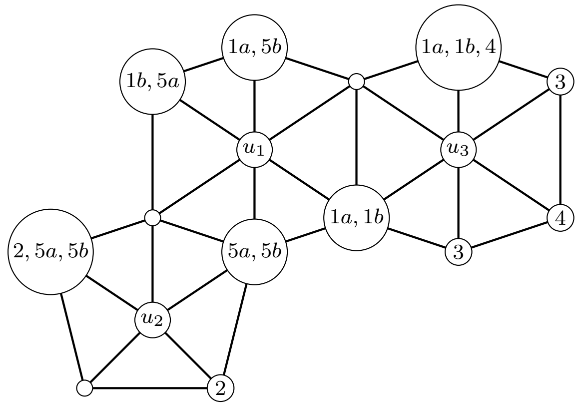
Proof. Lemma 12 yields $12 \leq |N(\{u_1, u_2, u_3\})| \leq d(u_1)+d(u_2)+d(u_3)-4 \leq 5+6+6-4 \leq 13$. So, by symmetry, we assume that the vertices are arranged as in Figures 5(B,C) with all vertices distinct as drawn or as in Figures 5(D,E) with at most one pair of vertices identified.

First suppose the vertices are disinct as drawn. For Figures 5(B,C,D), we apply Lemma 10; for (B) and (C) we use $k = 1$, and for (D) we use $k = 0$. For Figure 5(E), we apply Lemma 22, using the vertices labeled 1 for M_1 and the those labeled 4 for M_2.

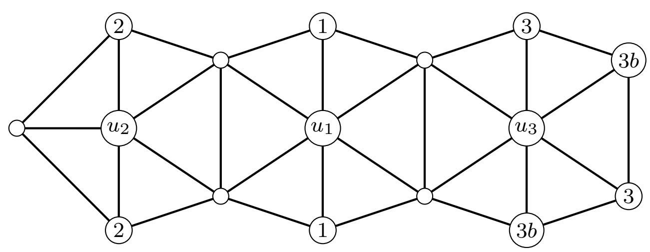
So, instead suppose that a single pair of vertices is identified in one of Figures 5(D,E). First consider (D). If a vertex labeled 1 is identified with another vertex, then we apply Lemma 19 using the vertices labeled 2 for the independent 2-set (vertices labeled 1 and 2



(A) Here u_2 and u_3 have a common neighbor in $N(u_1)$.



(B) Here u_2 and u_3 have adjacent neighbors in $N(u_1)$.



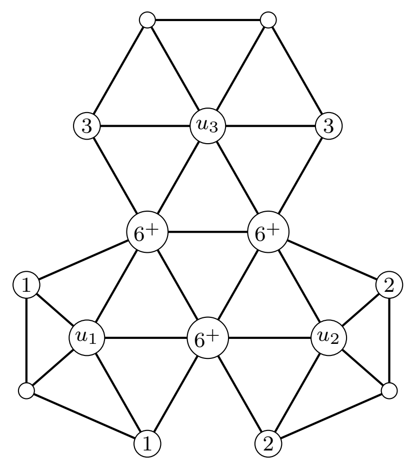
(C) Here u_2 and u_3 have neighbors at distance 2 in $N(u_1)$.

FIGURE 9. The cases of Lemma 25. The three possibilities for an independent 3-set $\{u_1, u_2, u_3\}$ where $d(u_1) = 6$, $d(u_2) \leq 6$, $d(u_3) = 5$, and each of u_2 and u_3 has two neighbors in common with u_1.

cannot be identified, since they are drawn at distance at most 3). Otherwise, the identified vertices must be those labeled 2 and 4 that are drawn at distance four. Now the vertices labeled w_3 or 3 are pairwise at distance two, so must be an independent 4-set. Now we get a contradiction, by applying Lemma 17 using the vertices labeled 1 for the independent 2-set.

Finally, consider (E). Again we apply Lemma 10, with $k = 1$. Since u_1 has three possibilities for its pair of non-adjacent neighbors, and no neighbor of u_1 appears in all three of these pairs, u_1 satisfies condition (2). Similarly, u_3 also satisfies condition (2). □

Lemma 27. *Let v_1, v_2, v_3 be the corners of a 3-face, each a 6^+-vertex. Let u_1, u_2, u_3 be the other pairwise common neighbors of v_1, v_2, v_3, i.e., u_1 is adjacent to v_1 and v_2, u_2 is adjacent to v_2 and v_3, and u_3 is adjacent to v_3 and v_1. We cannot have $|N(\{u_1, u_2, u_3\})| \leq 13$. In particular, we cannot have $d(u_1) = d(u_2) = 5$ and $d(u_3) \leq 6$.*

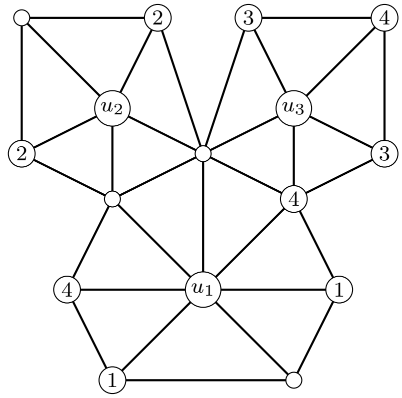


(A) A 3-face $v_1v_2v_3$, such that the pairwise common neighbors of v_1, v_2, v_3 have degrees 5, 5, and at most 6.

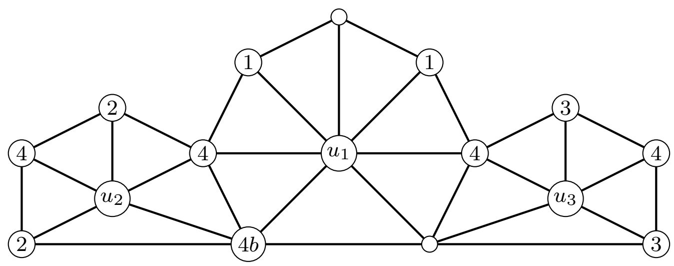
FIGURE 10. The key case of Lemma 27.

Proof. If the only pairwise common neighbors of the u_i are the v_i, then two u_i are 5-vertices and the third is a 6^--vertex. The case where the u_i have more pairwise common neighbors is nearly identical, and we remark on it briefly at the end of the proof. So suppose that $d(u_1) = d(u_2) = 5$ and $d(u_3) = 6$, as shown in Figure 10; the case where $d(u_3) = 5$ is nearly identical. We will apply Lemma 10 with $J = \{u_1, u_2, u_3\}$ and $k = 0$. Clearly, J is an independent set. Now we verify that each vertex of J satisfies condition (2). Since G has no separating 3-cycle, the two vertices in each pair with a common label (among $\{1, 2, 3\}$) are distinct and non-adjacent. Similarly, the vertices with labels in $\{1, 2, 3\}$ are distinct, since they are drawn at pairwise distance at most three, and G has no separating 3-cycle. Thus, we can apply Lemma 10, as desired.

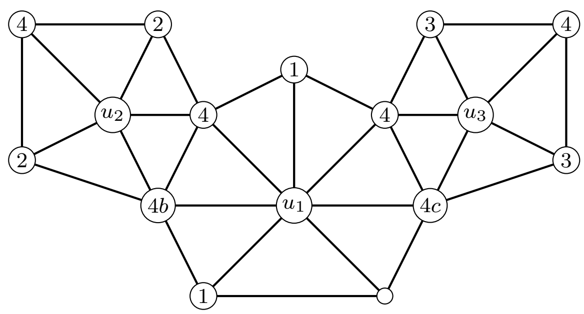
In the more general case where the u_i have pairwise common neighbors in addition to the v_i, the argument above still shows that the vertices with labels in $\{1, 2, 3\}$ are distinct. So again, we can apply Lemma 10 with $k = 0$. □



(A) Here u_2 and u_3 have a common neighbor in $N(u_1)$.



(B) Here u_2 and u_3 have adjacent neighbors in $N(u_1)$.



(C) Here u_2 and u_3 have neighbors at distance 2 in $N(u_1)$.

FIGURE 11. The cases of Lemma 28. The three possibilities for an independent 3-set $\{u_1, u_2, u_3\}$ where $d(u_1) = 7$, $d(u_2) = d(u_3) = 5$, and each of u_2 and u_3 has two neighbors in common with u_1.

Lemma 28. *Let u_1 be a 7-vertex with non-adjacent 5-vertices u_2 and u_3 each at distance two from u_1. A minimal G cannot have u_1 and u_2 with two common neighbors and also u_1 and u_3 with two common neighbors.*

Proof. This situation is shown in Figures 11(A,B,C), where at most one pair of vertices drawn as distinct are identified. If all vertices labeled 2 or 3 are distinct as drawn, then we apply

Lemma 22 and get a contradiction. By Lemma 12, the only other possibility is that exactly one pair of vertices is identified. Such a pair must consist of vertices labeled 2 and 3 that are drawn at distance four (otherwise we apply Lemma 10, with $k = 1$). In Figure 11(A), this is impossible, since the two 5-vertices u_2 and u_3 would have two neighbors in common, violating Lemma 15.

Now we consider the cases shown in Figures 11(B,C) simultaneously. We apply Lemma 17 using the vertices labeled 1 for the independent 2-set. Let I_1 be the set of vertices labeled 4. If I_1 is independent, then we are done; so assume not. Recall that a vertex labeled 2 is identified with a vertex labeled 3.

Suppose the vertices labeled 4 in $N(u_2) \setminus N(u_1)$ and $N(u_3) \setminus N(u_1)$ are not adjacent. Now by symmetry, we may assume that the vertex labeled 4 in $N(u_1) \cap N(u_2)$ is adjacent to the vertex labeled 4 in $N(u_3) \setminus N(u_1)$. Let I_2 be the set made from I_1 by replacing the vertex labeled 4 in $N(u_1) \cap N(u_2)$ with the vertex labeled $4b$. If I_2 is independent, then we are done; so assume not. Now the vertex labeled $4b$ must be adjacent to the vertex labeled 4 in $N(u_3) \setminus N(u_1)$, but this makes a separating 3-cycle (consisting of two vertices labeled 4 and one labeled $4b$), a contradiction.

So, we may assume that the vertices labeled 4 in $N(u_2) \setminus N(u_1)$ and $N(u_3) \setminus N(u_1)$ are adjacent. Suppose the topmost vertex labeled 2 is identified with the topmost vertex labeled 3. Now again we are done; our independent 4-set consists of the two neighbors of u_1 labeled 4, together with an independent 2-set from among the two leftmost and two rightmost vertices (by planarity, they cannot all four be pairwise adjacent).

The only remaining possibility is that the bottommost vertex labeled 2 is identified with the bottommost vertex labeled 3 (since the two topmost vertices labeled 4 are adjacent). If we are in Figure 11(B), then the vertex labeled $4b$ is a 5-vertex; since it shares two neighbors with u_3, another 5-vertex, we contradict Lemma 15. Hence, we must be in Figure 11(C). Now our independent 4-set consists of the two neighbors of u_1 labeled 4b and 4c, together with an indpendent 2-set from among the four topmost vertices (again, by planarity, they cannot all be pairwise adjacent). ☐

Lemma 29. *Suppose that a minimal G contains a 7-vertex v with no 5-neighbor. Now v cannot have at least five 6-neighbors, each of which has a 5-neighbor.*

Proof. Suppose to the contrary. Denote the neighbors of v in clockwise order by $u_1, \ldots, u_7$.

Case 1: Vertices u_1, u_2, u_3, u_4 are 6-vertices, each with a 5-neighbor.

First, suppose that u_2 and u_3 have a common 5-neighbor, w_2. Consider the 5-neighbor w_1 of u_1. By Lemma 15, it cannot be common with u_2; similarly, the 5-neighbor w_4 of u_4 cannot be common with u_3. (We must have w_1 and w_4 distinct, since otherwise we apply Lemma 27 to $\{u_1, u_4, w_2\}$. Also, we must have w_1 and w_4 each distinct from w_2, since G has no separating 3-cycles.)

First, suppose that w_1 has two common neighbors with u_2. If $w_1 \not\leftrightarrow u_4$, then we apply Lemma 25 to $\{w_1, u_2, u_4\}$; so assume $w_1 \leftrightarrow u_4$. Now let $J = \{u_1, u_4, w_2\}$. Clearly, J is an independent 3-set. Also $|N(J)| \leq 6 + 6 + 5 - 4 = 13$, so we are done by Lemma 27. So w_1 cannot have two common neighbors with u_2. Similarly, w_4 cannot have two common neighbors with u_3. Hence, $w_1 \leftrightarrow u_7$ and also $w_4 \leftrightarrow u_5$. Now we must have $w_1 \leftrightarrow w_4$; otherwise we apply Lemma 28 to $\{v, w_1, w_4\}$. Similarly, we must have $w_1 \leftrightarrow w_2$ and $w_2 \leftrightarrow w_4$; these edges cut off w_4 from u_1, so $u_1 \not\leftrightarrow w_4$. Since u_1 and w_4 are non-adjacent, but have a 5-neighbor in common, they must have two neighbors in common. So we apply Lemma 25 to $\{u_1, u_3, w_4\}$. Hence, we conclude that the common neighbor of u_2 and u_3 is not a 5-neighbor.

Since u_1 and u_3 are 6^--vertices, by Lemma 23, vertex u_2 cannot have another 6^--vertex that is nonadjacent to u_1 and u_3. Thus, the common neighbor w_1 of u_1 and u_2 is a 5-vertex; similarly, the common neighbor w_4 of u_3 and u_4 is a 5-vertex. We must have $w_1 \leftrightarrow w_4$, for otherwise we apply Lemma 28. We may assume that u_6 is a 6-vertex. If not, then v's five 6-neighbors, each with a 5-neighbor, are *successive*; so, by symmetry, we are in the case above, where u_2 and u_3 have a common 5-neighbor.

By planarity and symmetry, either $u_1 \not\leftrightarrow w_4$ or else $u_4 \not\leftrightarrow w_1$; assume the former. Since u_1 and w_4 share a 5-neighbor (and are non-adjacent), they have two common neighbors. Now if $u_6 \not\leftrightarrow w_4$, then we apply Lemma 25 to $\{u_1, u_6, w_4\}$. Hence, assume $u_6 \leftrightarrow w_4$. This implies that $u_4 \not\leftrightarrow w_1$. Now, the same argument implies that $u_6 \leftrightarrow w_1$. Now let $J = \{u_1, u_4, u_6\}$. Lemma 12 gives $12 \le |N(J)| \le 6+6+6-6 = 12$. Thus the vertices of J have no additional pairwise common neighbors. Hence, we have an independent 2-set M_1 in $N(u_1) \setminus (N(u_4) \cup N(u_6))$. Similarly, we have an independent 2-set M_4 in $N(u_4) \setminus (N(u_1) \cup N(u_6))$. Now we apply Lemma 16 with $J = \{u_1, u_4, u_6\}$ and $S_1 = M_1 \cup \{u_1\}$ and $S_2 = M_4 \cup \{u_4\}$. In each case, we have $\alpha(G[S_i \cup J]) \ge |M_i \cup \{u_{5-i}, u_6\}| = 4$. This implies that $|N(J)| \ge 13$, a contradiction. Hence, v cannot have four successive 6-neighbors, each with a 5-neighbor.

Case 2: Vertices u_1, u_2, u_3, u_5, u_6 are 6-vertices, each with a 5-neighbor.

Suppose that the common neighbor w_5 of u_5 and u_6 is a 5-vertex. By symmetry (between u_1 and u_3) and Lemma 23, assume that the common neighbor w_2 of u_2 and u_3 is a 5-vertex. If $w_2 \not\leftrightarrow w_5$, then we apply Lemma 28; so assume that $w_2 \leftrightarrow w_5$. If $u_6 \not\leftrightarrow w_2$, then apply Lemma 25 to $\{u_6, u_1, w_2\}$; note that u_6 and w_2 have two common neighbors, since they have a common 5-neighbor. So assume that $u_6 \leftrightarrow w_2$. Similarly, we assume that $u_3 \leftrightarrow w_5$, since otherwise we apply Lemma 25 to $\{u_3, u_1, w_5\}$. Now consider the 5-neighbor w_1 of u_1. By Lemma 15, it cannot be a common neighbor of u_2 (because of w_2). If it is a common neighbor of u_7, then we apply Lemma 28 to $\{w_1, w_5, v\}$; note that $w_1 \not\leftrightarrow w_5$, since they are cut off by edge w_2u_6. Hence, w_1 is neither a common neighbor of u_7 nor of u_2. Now we apply Lemma 25 to $\{u_2, w_1, w_5\}$. Thus, we conclude that the common neighbor of u_5 and u_6 is not a 5-vertex.

Let x denote the common neighbor of u_5 and u_6; as shown in the previous paragraph, x must be a 6^+-vertex. Suppose that the 5-neighbor w_5 of u_5 is also a neighbor of x. If $w_5 \not\leftrightarrow u_1$, then we apply Lemma 25 to $\{u_6, u_1, w_5\}$; so assume that $w_5 \leftrightarrow u_1$. Now if the 5-neighbor w_6 of u_6 is also adjacent to x, then we apply Lemma 25 to $\{u_5, w_6, u_3\}$; we must have $w_6 \not\leftrightarrow u_3$ due to edge w_5u_1. So, by symmetry (between u_5 and u_6), we may assume that $w_5 \leftrightarrow u_4$. Now, by Lemma 23, the 5-neighbor w_2 of u_2 has a common neighbor with either u_1 or u_3. In either case, we must have $w_2 \leftrightarrow w_5$; otherwise, we apply Lemma 28 to $\{v, w_2, w_5\}$. If $w_6 \leftrightarrow u_7$, then $w_6 \leftrightarrow w_2$ and $w_6 \leftrightarrow w_5$; otherwise, we apply Lemma 28 to $\{v, w_6, w_2\}$ or $\{v, w_6, w_5\}$. Now we apply Lemma 25 to $\{u_5, u_3, w_6\}$. So instead $w_6 \not\leftrightarrow u_7$. Finally, we apply Lemma 25 to $\{u_5, w_6, u_3\}$. This completes the proof. □

3.2. **Discharging.**

Theorem 2. *Every planar graph G has independence ratio at least $\frac{3}{13}$.*

Proof. We will use discharging with initial charge $ch(v) = d(v) - 6$. We use the following discharging rules to guarantee that each vertex finishes with nonnegative charge.

(R1) Each 6-vertex gives $\frac{1}{2}$ to each 5-neighbor unless either they share a common 6-neighbor and no common 5-neighbor or else the 5-neighbor receives charge from at least four vertices; in either of these cases, the 6-vertex gives the 5-neighbor $\frac{1}{4}$.
(R2) Each 8^+-vertex v gives gives $\frac{1}{4} + \frac{h_w}{8}$ to each 6^--neighbor w where h_w is the number of 7^+-vertices in $N(v) \cap N(w)$.
(R3) Each 7-vertex gives $\frac{1}{2}$ to each isolated 5-neighbor; gives 0 to each crowded 5-neighbor; gives $\frac{1}{4}$ to each other 5-neighbor; and gives $\frac{1}{4}$ to each 6-neighbor unless neither the 7-vertex nor the 6-vertex has a 5-neighbor.
(R4) After applying (R1)–(R3), each 5-vertex with positive charge splits it equally among its 6-neighbors that gave it $\frac{1}{2}$.
(R5) After applying (R1)–(R4), each 6-vertex with positive charge splits it equally among its 6-neighbors with negative charge.

$\mathbf{d(v) \geq 8}$: We will show that v gives away charge at most $\frac{d(v)}{4}$. To see that it does, let v first give charge $\frac{1}{4}$ to each neighbor. Now let each 6^--neighbor w take $\frac{1}{8}$ from each 7^+-vertex in $N(v) \cap N(w)$. Since $G[N(v)]$ is a cycle, each 7^+-neighbor gives away at most the $\frac{1}{4}$ it got from v. Each neighbor of v has received at least as much charge as by rule (R2) and v has given away charge $\frac{d(v)}{4}$. Now $d(v) \geq 8$ implies $\frac{d(v)}{4} \leq d(v) - 6$, so $\text{ch}^*(v) \geq 0$.

$\mathbf{d(v) = 7}$: Let $u_1, \ldots, u_7$ denote the neighbors of v in clockwise order. First suppose that v has an isolated 5-neighbor. Now the subgraph induced by the remaining 6^--neighbors must have independence number at most 1, by Lemma 26. Hence v gives away charge at most either $\frac{1}{2} + \frac{1}{2}$ or $\frac{1}{2} + 2(\frac{1}{4})$; in either case, $\text{ch}^*(v) \geq 0$. Assume instead that v has no isolated 5-neighbor. Suppose first that v has a (non-isolated) 5-neighbor. Now v has at most five total 6^--neighbors, again by Lemma 26. If v has at most four 6^- neighbors, then, since each 6^--neighbor receives charge at most $\frac{1}{4}$, v gives away at most $4(\frac{1}{4})$, so $\text{ch}^*(v) \geq 0$. By Lemma 26, if v has exactly five 6^--neighbors, then one is a crowded 5-neighbor, which receives no charge from v. So, again, v gives away charge at most $4(\frac{1}{4})$, so $\text{ch}^*(v) \geq 0$.

Finally, suppose that v has only 6^+-neighbors. By Lemma 29, v gives charge to at most four 6-neighbors, so $\text{ch}^*(v) \geq 0$.

$\mathbf{d(v) = 5}$: We must show that v receives total charge at least 1. Let $u_1, \ldots, u_5$ be the neighbors of v. First suppose that v has five 6^+-neighbors. It will receive charge at least $4(\frac{1}{4})$ unless exactly two of these are 7-vertices for which v is a crowded 5-neighbor. However, in this case the other three neighbors are all 6-neighbors, so v receives $2(\frac{1}{4}) + (\frac{1}{2})$. Now suppose that v has exactly four 6^+-neighbors, say $u_1, \ldots, u_4$. If v receives charge from each, then v receives at least $4(\frac{1}{4})$; so suppose that v receives charge from at most three neighbors. In total, v receives charge at least $\frac{1}{2}$ from u_1 and u_2: at least $2(\frac{1}{4})$ if u_1 is not a 6-vertex and at least $\frac{1}{2} + 0$ if u_1 is a 6-vertex. Similarly, v receives at least $\frac{1}{2}$ in total from u_3 and u_4; so, v receives total charge at least $2(\frac{1}{2})$. Now suppose that v has exactly three 6^+-neighbors, say u_1, u_2, u_3. If u_1 and u_3 are both 6-vertices, then v receives charge $\frac{1}{2}$ from each. If both are 7^+-vertices, then v receives charge $\frac{1}{4}$ from each and charge $\frac{1}{2}$ from u_2. So assume that exactly one of u_1 and u_3 is a 6-vertex, say u_1. Now v receives charge $\frac{1}{2}$ from u_1 and charge $\frac{1}{4}$ from each of u_2 and u_3, for a total of $\frac{1}{2} + 2(\frac{1}{4})$.

$\mathbf{d(v) = 6}$: Note that (R5) will never cause a 6-vertex to have negative charge. Thus, in showing that a 6-vertex has nonnegative charge, we need not consider it.

Clearly, a 6-vertex with no 5-neighbor finishes (R1)–(R3) with nonnegative charge. Suppose that v is a 6-vertex with exactly one 5-neighbor. We will show that v finishes (R1)–(R3) with charge at least $\frac{1}{4}$. Let $u_1, \ldots, u_6$ denote the neighbors of v and assume that u_1 is the only 5-vertex. By Lemma 23, at least one of u_1, u_3, u_5 is a 7^+-vertex, so it gives v charge $\frac{1}{4}$. If one of u_6 and u_2 is a 6-vertex, then v gives charge only $\frac{1}{4}$ to u_1, finishing with charge at least $2(\frac{1}{4}) - \frac{1}{4}$. Otherwise, v receives charge at least $\frac{1}{4}$ from each of u_6 and u_2, so finishes with charge at least $3(\frac{1}{4}) - \frac{1}{2}$. Similarly, if v has no 5-neighbor and at least one 8^+-neighbor, then v finishes (R1)–(R3) with charge at least $\frac{1}{4}$.

Now suppose that v has at least two 5-neighbors. By Lemma 15, At most one of u_1, u_3, u_5 can be a 5-vertex. Similarly, for u_2, u_4, u_6; hence, assume that v has exactly two 5-neighbors. These 5-neighbors can either be "across", say u_1 and u_4, or "adjacent", say u_1 and u_2.

Suppose that v has 5-neighbors u_1 and u_4. Note that all of its remaining neighbors must be 6^+-vertices. At least one of u_1, u_3, u_5 must be a 7^+-vertex; similarly for u_2, u_4, u_6. Now we show that the total net charge that v gives to u_3, u_4, u_5 is 0. Similarly, the total net charge that v gives to u_6, u_1, u_2 is 0. If both u_3 and u_5 are 7^+-vertices, then v gets $\frac{1}{4}$ from each and gives $\frac{1}{2}$ to u_4. Otherwise, one of u_3 and u_5 is a 6-vertex and the other is a 7^+-vertex; now v gets $\frac{1}{4}$ from the 7^+-vertex and gives only $\frac{1}{4}$ to u_4. The same is true for u_6, u_1, u_2. Thus, v finishes with charge 0.

Suppose instead that v has 5-neighbors u_1 and u_2. By Lemmas 23 and 24 either both of u_3 and u_5 are 7^+-vertices or one is a 6-vertex and the other an 8^+-vertex. The same holds for u_4 and u_6. Let $w_1, \ldots, w_5$ be the common neighbors of successive pairs of vertices in the list $u_6, u_1, u_2, u_3, u_4, u_5$. Consider the possible degrees for u_3, u_4, u_5, u_6. Up to symmetry, they are (i) $7^+, 7^+, 7^+, 7^+$, (ii) $7^+, 8^+, 7^+, 6$, (iii) $7^+, 6, 7^+, 8^+$, (iv) $8^+, 6, 6, 8^+$, (v) $8^+, 8^+, 6, 6$, and (vi) $6, 8^+, 8^+, 6$.

In **Case (i)**, v receives charge at least $4(\frac{1}{4})$, so $\text{ch}^*(v) \geq 0$. In **Case (ii)**, v receives charge at least $\frac{1}{4} + (\frac{1}{4} + \frac{1}{8} + \frac{1}{8}) + \frac{1}{4}$, so $\text{ch}^*(v) \geq 0$. In **Case (iii)**, v receives charge at least $(\frac{1}{4} + \frac{1}{8}) + \frac{1}{4} + \frac{1}{4} = \frac{7}{8}$. If w_2 is a 6^+-vertex, then v gives only $\frac{1}{4}$ to u_2, so $\text{ch}^*(v) \geq 0$. So suppose that w_2 is a 5-vertex. Recall that w_3 is a 6^+-vertex by Lemma 15. Now in each case v get charge at least $\frac{1}{8}$ back from u_2. If w_3 is a 6-vertex, then u_3 receives charge $2(\frac{1}{2}) + \frac{1}{4}$ and sends back $\frac{1}{8}$ to each of v and w_3. Otherwise, w_3 is a 7^+-vertex, so u_3 sends v charge at least $\frac{3}{8}$, and v gets back at least $\frac{1}{8}$. Thus, in each instance of Case (iii), we have $\text{ch}^*(v) \geq 0$. So we are in Cases (iv), (v), or (vi).

Case (iv): $8^+, 6, 6, 8^+$. If w_2 is a 6^+-vertex, then both u_1 and u_2 are sent charge by four vertices and hence v gives away at most $\frac{1}{2}$. Since v gets at least $\frac{1}{2}$ from u_3 and u_6, we have $\text{ch}^*(v) \geq 0$. Hence, we assume that w_2 is a 5-vertex.

Now if w_1 is a 6-vertex, then u_1 receives charge $\frac{5}{4}$, so gives back $\frac{1}{8}$ to v. If instead w_1 is a 7^+-vertex, then u_1 receives charge at least $\frac{3}{4}$ from v and w_1 together and then charge at least $\frac{1}{4} + \frac{1}{8}$ from u_6 for a total of $\frac{9}{8}$. Since u_1 has only one 6-neighbor, it gives the extra $\frac{1}{8}$

back to v by (R4). The same holds for u_2, so v gets $\frac{1}{8}$ back from each of u_1 and u_2; so v gets charge at least $\frac{3}{4}$.

Suppose that u_4 has at least two 5-neighbors. Now one of them, call it x, is a common neighbor with either u_3 or u_5, so we can apply Lemma 25 to $\{v, w_2, x\}$ (again $x \not\leftrightarrow w_2$, since w_2 has two other 5-neighbors; x cannot be identified with one of these other 5-neighbors, since G has no separating 3-cycle). Similarly, u_5 has at most one 5-neighbor. Hence, by our argument above, both u_4 and u_5 finish (R1)–(R3) with charge at least $\frac{1}{4}$. Now we show that u_4 has at most three 6-neighbors; similarly for u_5.

Suppose w_4 is a 6-vertex. We can apply Lemma 25 to v, w_2, w_4 unless $w_2 \leftrightarrow w_4$. In that case, we apply Lemma 27 to u_1, u_4, w_4. Thus, w_4 is a 7^+-vertex. If w_5 is a 6-vertex, then we can apply Lemma 25 to v, w_2, w_5 unless $w_2 \leftrightarrow w_5$, so assume this. Now if the final neighbor of u_4, call it y, is a 6-vertex, then we apply Lemma 25 to u_5, y, u_1; we must have $y \not\leftrightarrow u_1$, since $w_5 \leftrightarrow w_2$. Thus, we conclude that u_4 has at most two 6-neighbors other than u_5, so at most two 6-neighbors that finish (R1)–(R3) with negative charge. An analogous argument holds for u_5. Hence v gets at least $\frac{1}{8}$ from each of u_4 and u_5 from (R5), for a total of $\frac{3}{4} + \frac{1}{4}$ as needed.

Case (v): $8^+, 8^+, 6, 6$. Note that v receives charge at least $2(\frac{3}{8}) = \frac{3}{4}$ from u_5 and u_6. If w_2 is a 6^+-vertex, then u_1 receives charge from four neighbors, so v gives away charge at most $\frac{1}{4} + \frac{1}{2}$. Thus $\text{ch}^*(v) \geq 0$. So assume w_2 is a 5-vertex. First, we show that v gets back at least $\frac{1}{8}$ from u_1. If $d(w_1) = 6$, then u_1 gets charge $\frac{1}{2} + \frac{1}{4} + \frac{1}{2} = \frac{5}{4}$, so returns charge $\frac{1}{8}$ to each of v and w_1. Otherwise $d(w_1) \geq 7$, so u_6 sends charge $\frac{3}{8}$ to u_1, and u_1 returns at least $\frac{1}{8}$ to v. Thus, v gets charge at least $\frac{7}{8}$.

If w_3 is a 6-vertex, then v gets back charge $\frac{1}{8}$ from u_2, so $\text{ch}^*(v) \geq 0$. Instead, assume w_3 is a 7^+-vertex. Now we show that v gets charge at least $\frac{1}{8}$ from u_3 by (R5). Let y be the neighbor of u_3 other than v, u_2, w_3, w_4, u_4. By Lemma 24, y is an 8^+-vertex. If w_4 is a 5-vertex, then we apply Lemma 25 to v, w_2, w_4 to get a contradiction (w_4 cannot be adjacent to w_2, since w_2 already has two other 5-neighbors, and w_4 cannot be identified with u_1 or w_2, since G has no separating 3-cycles). Hence w_4 is a 6^+-vertex. So u_3 receives charge at least $\frac{1}{4}$ from w_3 and at least $\frac{1}{4} + \frac{1}{8}$ from y. After u_3 gives charge $\frac{1}{4}$ to u_2, it has charge at least $\frac{3}{8}$. So, by (R5), it gives each of its at most three 6-neighbors charge at least $\frac{1}{3}(\frac{3}{8}) = \frac{1}{8}$. Thus, $\text{ch}^*(v) \geq 0$.

Case (vi): $6, 8^+, 8^+, 6$. First suppose that w_2 is a 6^+-vertex. Note that v gets charge at least $2(\frac{3}{8})$ from u_4 and u_5, so it suffices to show that v gives net charge at most $\frac{3}{8}$ to each of u_1 and u_2. We consider u_1; the case for u_2 is symmetric. If w_1 gives charge to u_1, then u_1 receives charge from four neighbors, so it gets charge only $\frac{1}{4}$ from v. Suppose instead that w_1 is a 7-vertex and w_2 is a 6-vertex. Now u_1 gets charge $\frac{1}{2} + \frac{1}{4} + \frac{1}{2} = \frac{5}{4}$, so returns charge $\frac{1}{8}$ to each of v and w_2. Thus $\text{ch}^*(v) \geq 0$. So instead, assume that w_2 is a 5-vertex.

If w_1 and w_3 are 6-vertices, then v gets back $\frac{1}{8}$ from each of u_1 and u_2, since each receives $\frac{1}{2} + \frac{1}{4} + \frac{1}{2}$ and returns $\frac{1}{8}$ to each vertex that gave it $\frac{1}{2}$. Since v gets $2(\frac{3}{8})$ from u_4 and u_5, we have $\text{ch}^*(v) \geq 0$. So assume, by symmetry, that w_3 is a 7^+-vertex. If $w_4 \leftrightarrow w_2$, then we apply Lemma 14 to w_2 and u_2; so $w_4 \not\leftrightarrow w_2$. If w_4 is a 6^--vertex, then we apply Lemma 25 to v, w_2, w_4 to get a contradiction (as above, w_4 cannot be identified with u_1 or w_2, since G has no separating 3-cycle). Thus, w_4 is a 7^+-vertex. So u_3 has at least three 7^+-neighbors and at most two 6-neighbors. Thus, after u_3 gives charge $\frac{1}{4}$ to u_2, by (R5) it gives charge $\frac{1}{2}(\frac{1}{2}) = \frac{1}{4}$ to v. Thus, $\mathrm{ch}^*(v) \geq 0$. □

Acknowledgments

As we mentioned in the introduction, the ideas in this paper come largely from Albertson's proof [1] that planar graphs have independence ratio at least $\frac{2}{9}$. In fact, many of the reducible configurations that we use here are special cases of the reducible configurations in that proof. We very much like that paper, and so it was a pleasure to be able to extend Albertson's work. It seems that the part of his own proof that Albertson was least pleased with was verifying "unavoidability", i.e., showing that every planar graph contains a reducible configuration. In the introduction to [1], he wrote: "Finally Section 4 is devoted to a massive, ugly edge counting which demonstrates that every planar triangulation of the plane must contain some forbidden subgraph." In the appendix that follows, we give a short proof of this same unavoidability statement, via discharging. We think Mike might have liked it.

The first author thanks his Lord and Savior, Jesus Christ.

Appendix

Here we give a short discharging proof that every planar triangulation with minimum degree 5 and no separating 3-cycle must contain a certain configuration, which Albertson showed could not appear in a minimal planar graph with independence ratio less than $\frac{2}{9}$. (In fact, finding this proof helped encourage us to begin work on the present paper.)

Lemma A. *Let u and v be adjacent vertices, such that uvw and uvx are 3-faces and $d(w) = 5$ and $d(x) \leq 6$; call this configuration H. If G is a plane triangulation with minimum degree 5 and no separating 3-cycle, then G contains a copy of H.*

(A) Adjacent vertices u and v, with nonadjacent common 5-neighbors w and x.

(B) Adjacent vertices u and v, with nonadjacent common neighbors w and x, of degree 5 and 6.

Figure 12. The two instances of configuration H.

Proof. Assume that G has minimum degree 5 and no separating 3-cycle, but also has no copy of H. This assumption leads to a contradiction, which implies the result. An immediate consequence of this assumption (by Pigeonhole) is that the number of 5-neighbors of each vertex v is at most $\frac{d(v)}{2}$. Below, when we verify that each vertex finishes with nonnegative charge, we consider both the degree of v and its number of 5-neighbors. We write (a, b)-*vertex* to denote a vertex of degree a that has b 5-neighbors.

We assign to each vertex v a charge $\mathrm{ch}(v)$, where $\mathrm{ch}(v) = d(v)-6$. Note that $\sum_{v\in V}\mathrm{ch}(v) = 2|E(G)|-6|V(G)|$. Since G is a plane triangulation, Euler's formula implies that $2|E(G)| - 6|V(G)| = -12$. Now we redistribute the charge, without changing the sum, so that each vertex finishes with nonnegative charge. This redistribution is called *discharging*, and we write $\mathrm{ch}^*(v)$ to denote the charge at each vertex v after discharging. Since each vertex finishes with nonnegative charge, we get the obvious contradiction $-12 = \sum_{v\in V}\mathrm{ch}(v) = \sum_{v\in V}\mathrm{ch}^*(v) \geq 0$. We redistribute the charge via the following three discharging rules, which we apply simultaneously everywhere they are applicable.

(R1) Each 7^+-vertex gives charge $\frac{1}{3}$ to each 5-neighbor.
(R2) Each 7^+-vertex gives charge $\frac{1}{7}$ to each 6-neighbor that has at least one 5-neighbor.
(R3) Each 6-vertex gives charge $\frac{2}{7}$ to each 5-neighbor.

We now verify that after discharging, each vertex v has nonnegative charge.

$\mathbf{d(v) = 5}$: Note that each $(5,0)$-vertex has five 6^+-neighbors; each $(5,1)$-vertex has four 6^+-neighbors, at least two of which are 7^+-neighbors; and each $(5,2)$-vertex has three 7^+-neighbors. Thus, if v is a $(5,0)$-vertex: $\mathrm{ch}^*(v) \geq -1 + 5\left(\frac{2}{7}\right) > 0$; if v is a $(5,1)$-vertex: $\mathrm{ch}^*(v) \geq -1 + 2\left(\frac{1}{3}\right) + 2\left(\frac{2}{7}\right) > 0$; and if v is a $(5,2)$-vertex: $\mathrm{ch}^*(v) = -1 + 3\left(\frac{1}{3}\right) = 0$;

$\mathbf{d(v) = 6}$: Note that each $(6,1)$-vertex has at least two 7^+-neighbors; and each $(6,2)$-vertex has four 7^+-neighbors. Thus, if v is a $(6,0)$-vertex: $\mathrm{ch}^*(v) = \mathrm{ch}(v) = 0$; if v is a $(6,1)$-vertex: $\mathrm{ch}^*(v) \geq 0 + 2\left(\frac{1}{7}\right) - \left(\frac{2}{7}\right) = 0$; and if v is a $(6,2)$-vertex: $\mathrm{ch}^*(v) = 0 + 4\left(\frac{1}{7}\right) - 2\left(\frac{2}{7}\right) = 0$.

$\mathbf{d(v) = 7}$: Note that each $(7,1)$-vertex has six 6^+-neighbors, at least two of which are 7^+-vertices; each $(7,2)$-vertex has five 6^+-neighbors, at least three of which are 7^+-vertices; and each $(7,3)$-vertex has four 7^+-neighbors. Thus, if v is a $(7,0)$-vertex, then $\mathrm{ch}^*(v) \geq 1 - 7\left(\frac{1}{7}\right) = 0$; if v is a $(7,1)$-vertex, then $\mathrm{ch}^*(v) \geq 1 - 1\left(\frac{1}{3}\right) - 4\left(\frac{1}{7}\right) > 0$; if v is a $(7,2)$-vertex, then $\mathrm{ch}^*(v) \geq 1 - 2\left(\frac{1}{3}\right) - 2\left(\frac{1}{7}\right) > 0$; and if v is a $(7,3)$-vertex, then $\mathrm{ch}^*(v) = 1 - 3\left(\frac{1}{3}\right) = 0$.

$\mathbf{d(v) = 8}$: v has at most four 5-neighbors, and gives each of these charge $\frac{1}{3}$; v gives each other neighbor charge at most $\frac{1}{7}$. Thus $\mathrm{ch}^*(v) \geq 8 - 6 - 4(\frac{1}{3}) - 4(\frac{1}{7}) > 0$.

$\mathbf{d(v) \geq 9}$: v gives each neighbor charge at most $\frac{1}{3}$, so $\mathrm{ch}^*(v) \geq d(v) - 6 - d(v)(\frac{1}{3}) = \frac{2}{3}(d(v) - 9) \geq 0$.

Thus $-12 = \sum_{v\in V}\mathrm{ch}(v) = \sum_{v\in V}\mathrm{ch}^*(v) \geq 0$. This contradiction implies the result. □

References

[1] M. O. Albertson. A lower bound for the independence number of a planar graph. *J. Combinatorial Theory Ser. B*, 20(1):84–93, 1976.

[2] C. Berge. *Graphes et hypergraphes*. Dunod, Paris, 1970. Monographies Universitaires de Mathématiques, No. 37.

[3] G. Birkhoff. The reducibility of maps. *American Journal of Mathematics*, pages 115–128, 1913.

[4] R. Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, third edition, 2005.

[5] P. Hell. An introduction to the category of graphs. In *Topics in graph theory (New York, 1977)*, volume 328 of *Ann. New York Acad. Sci.*, pages 120–136. New York Acad. Sci., New York, 1979.

[6] A. J. W. Hilton, R. Rado, and S. H. Scott. A (<5)-colour theorem for planar graphs. *Bull. London Math. Soc.*, 5:302–306, 1973.

[7] P. C. Kainen. A generalization of the 5-color theorem. *Proc. Amer. Math. Soc.*, 45:450–453, 1974.

[8] L. Lovász. Kneser's conjecture, chromatic number, and homotopy. *J. Combin. Theory Ser. A*, 25(3):319–324, 1978.

[9] R. Naserasr. K_5-free bound for the class of planar graphs. *European J. Combin.*, 27(7):1155–1158, 2006.

[10] J. Nešetřil and P. Ossona de Mendez. Folding. *J. Combin. Theory Ser. B*, 96(5):730–739, 2006.

[11] N. Sauer. Hedetniemi's Conjecture—A Survey. *Discrete Math.*, 229(1-3):261–292, 2001. Combinatorics, graph theory, algorithms and applications.

[12] E. R. Scheinerman and D. H. Ullman. *Fractional Graph Theory*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., New York, 1997. A rational approach to the theory of graphs, With a foreword by Claude Berge, A Wiley-Interscience Publication.

[13] K. Wagner. Über eine Eigenschaft der ebenen Komplexe. *Math. Ann.*, 114(1):570–590, 1937.