Brooks’ Theorem states that if $G$ is a connected graph with maximum degree $\Delta$ at least 3, then $G$ can be colored with $\Delta$ colors. This result has been generalized to list-coloring and more general contexts. The square $G^2$ of a graph $G$ is formed from $G$ by adding an edge between each pair of vertices at distance two. When $G$ has maximum degree $\Delta$, it is easy to show that $G^2$ has maximum degree at most $\Delta^2$; so Brooks’ Theorem implies that $G^2$ can be colored with $\Delta^2$ colors.

Cranston and Kim conjectured that we can improve this upper bound by at least 1. Specifically, they conjectured that $\chi^\ell(G^2) \leq \Delta^2 - 1$ unless $G$ is a Moore graph (here $\chi^\ell$ denotes the list chromatic number). We prove their conjecture and survey some harder conjectures about coloring squares of graphs.

This is joint work with Landon Rabern.