SECTION 3.4 Exercises

Eigenvalues and Eigenvectors

In Exercises 1–4, verify that λ_i is an eigenvalue of A and that $\mathbf{x}_i$ is a corresponding eigenvector.

1. $A = \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$; $\lambda_1 = 1$, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$;

$\lambda_2 = -3$, $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

2. $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$; $\lambda_1 = 5$, $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$;

$\lambda_2 = 1$, $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$; $\lambda_1 = 2$, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$;

$\lambda_2 = 0$, $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$; $\lambda_3 = 1$, $\mathbf{x}_3 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$; $\lambda_1 = 1$, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$;

$\lambda_2 = 2$, $\mathbf{x}_2 = \begin{bmatrix} -7 \\ 4 \\ 1 \end{bmatrix}$

In Exercises 5–14, find (a) the characteristic equation, (b) the eigenvalues, and (c) the corresponding eigenvectors of the matrix.

5. $\begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$

9. $\begin{bmatrix} -2 & 4 \\ 2 & 5 \end{bmatrix}$

10. $\begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}$

11. $\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$

12. $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$

In Exercises 15–24, use a graphing utility or computer software program with matrix capabilities to find the eigenvalues of the matrix. Then find the corresponding eigenvectors.

15. $\begin{bmatrix} 2 & 5 \\ -1 & -4 \end{bmatrix}$

16. $\begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix}$

17. $\begin{bmatrix} 4 & -2 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

18. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & -2 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 0 \\ 0 & -2 & -2 \end{bmatrix}$

20. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -2 & 2 \end{bmatrix}$

21. $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & -1 & 3 & 1 \end{bmatrix}$

23. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

24. $\begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

True or False? In Exercises 25 and 26, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

25. (a) If $\mathbf{x}$ is an eigenvector corresponding to a given eigenvalue λ, then any multiple of $\mathbf{x}$ is also an eigenvector corresponding to that same λ.

(b) If $\lambda = a$ is an eigenvalue of the matrix A, then $\lambda = a$ is a solution of the characteristic equation $\lambda I - A = 0$.

26. (a) The characteristic equation of the matrix $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ yields eigenvalues $\lambda_1 = \lambda_2 = 1$.

(b) The matrix $A = \begin{bmatrix} 4 & -2 \\ -1 & 0 \end{bmatrix}$ has irrational eigenvalues $\lambda_1 = 2 + \sqrt{6}$ and $\lambda_2 = 2 - \sqrt{6}$.

57. No; in general, $P^{-1}AP \neq A$. For example, let

$$P = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix},\ P^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix},$$

and $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$.

Then you have

$$P^{-1}AP = \begin{bmatrix} -27 & -49 \\ 16 & 29 \end{bmatrix} \neq A.$$

The equation $|P^{-1}AP| = |A|$ is true in general because

$$|P^{-1}AP| = |P^{-1}||A||P| = |P^{-1}||P||A| = \frac{1}{|P|}|P||A| = |A|.$$

59. (a) False. See Theorem 3.6, page 144.
(b) True. See Theorem 3.8, page 146.
(c) True. See "Equivalent Conditions for a Nonsingular Matrix," parts 1 and 2, page 147.

61. Proof **63.** Orthogonal **65.** Not orthogonal

67. Orthogonal **69.** Proof

71. (a) $\begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$ (c) 1

A is orthogonal.

73. Proof

Section 3.4 *(page 157)*

1. $\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix};$

$\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix} = -3\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = 1\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

5. (a) $\lambda^2 - \lambda - 2 = 0$ (b) $2, -1$
(c) $\lambda = 2$: $\mathbf{x} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$; $\lambda = -1$: $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

7. (a) $\lambda^2 - 2\lambda - 3 = 0$ (b) $3, -1$
(c) $\lambda = 3$: $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\lambda = -1$: $\mathbf{x} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

9. (a) $\lambda^2 - 3\lambda - 18 = 0$ (b) $6, -3$
(c) $\lambda = 6$: $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; $\lambda = -3$: $\mathbf{x} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

11. (a) $\lambda^3 - 3\lambda^2 - 4\lambda + 12 = 0$ (b) $2, 3, -2$
(c) $\lambda = 2$: $\mathbf{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$; $\lambda = 3$: $\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$;
$\lambda = -2$: $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$

13. (a) $\lambda^3 - 3\lambda^2 - \lambda + 3 = 0$ (b) $-1, 1, 3$
(c) $\lambda = 1$: $\mathbf{x} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$; $\lambda = -1$: $\mathbf{x} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$
$\lambda = 3$: $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

15. Eigenvalues: $\lambda = -3, 1$
Eigenvectors: $\lambda = -3$: $\mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
$\lambda = 1$: $\mathbf{x} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$

17. Eigenvalues: $\lambda = 1, 2, 3$
Eigenvectors:
$\lambda = 1$: $\mathbf{x} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$; $\lambda = 2$: $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$; $\lambda = 3$: $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

19. Eigenvalues: $\lambda = 1, -2$

Eigenvectors:

$$\lambda = 1: \mathbf{x} = \begin{bmatrix}1\\0\\0\end{bmatrix}; \lambda = -2: \mathbf{x} = \begin{bmatrix}1\\0\\3\end{bmatrix}$$

21. Eigenvalues: $\lambda = -3, -1, 3, 5$

Eigenvectors:

$$\lambda = -3: \mathbf{x} = \begin{bmatrix}0\\0\\-1\\1\end{bmatrix}; \lambda = -1: \mathbf{x} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix};$$

$$\lambda = 3: \mathbf{x} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}; \lambda = 5: \mathbf{x} = \begin{bmatrix}0\\0\\5\\3\end{bmatrix}$$

23. Eigenvalues: $\lambda = -2, -1, 1, 3$

Eigenvectors:

$$\lambda = -2: \mathbf{x} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}; \lambda = -1: \mathbf{x} = \begin{bmatrix}1\\0\\-2\\6\end{bmatrix}$$

$$\lambda = 1: \mathbf{x} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}; \lambda = 3: \mathbf{x} = \begin{bmatrix}1\\0\\2\\2\end{bmatrix}$$

25. (a) False. If $\mathbf{x}$ is an eigenvector corresponding to λ, then any *nonzero* multiple of $\mathbf{x}$ is also an eigenvector corresponding to λ. See page 153, first paragraph.

(b) False. If $\lambda = a$ is an eigenvalue of the matrix A, then $\lambda = a$ is a solution of the characteristic equation $|\lambda I - A| = 0$. See page 153, second paragraph.

Section 3.5 *(page 168)*

1. $\text{adj}(A) = \begin{bmatrix}4 & -2\\-3 & 1\end{bmatrix}, A^{-1} = \begin{bmatrix}-2 & 1\\\frac{3}{2} & -\frac{1}{2}\end{bmatrix}$

3. $\text{adj}(A) = \begin{bmatrix}0 & 0 & 0\\0 & -12 & -6\\0 & 4 & 2\end{bmatrix}$, A^{-1} does not exist.

5. $\text{adj}(A) = \begin{bmatrix}-7 & -12 & 13\\2 & 3 & -5\\2 & 3 & -2\end{bmatrix}, A^{-1} = \begin{bmatrix}\frac{7}{3} & 4 & -\frac{13}{3}\\-\frac{2}{3} & -1 & \frac{5}{3}\\-\frac{2}{3} & -1 & \frac{2}{3}\end{bmatrix}$

7. $\text{adj}(A) = \begin{bmatrix}7 & 1 & 9 & -13\\7 & 1 & 0 & -4\\-4 & 2 & -9 & 10\\2 & -1 & 9 & -5\end{bmatrix},$

$$A^{-1} = \begin{bmatrix}\frac{7}{9} & \frac{1}{9} & 1 & -\frac{13}{9}\\\frac{7}{9} & \frac{1}{9} & 0 & -\frac{4}{9}\\-\frac{4}{9} & \frac{2}{9} & -1 & \frac{10}{9}\\\frac{2}{9} & -\frac{1}{9} & 1 & -\frac{5}{9}\end{bmatrix}$$

9. Proof

11. Proof

13. $|\text{adj}(A)| = \begin{vmatrix}-2 & 0\\-1 & 1\end{vmatrix} = -2,$

$|A|^{2-1} = \begin{vmatrix}1 & 0\\1 & -2\end{vmatrix}^{2-1} = -2$

15. Proof

17. $x_1 = 1$, $x_2 = 2$

19. $x_1 = 2$, $x_2 = -2$

21. $x_1 = \frac{3}{4}$, $x_2 = -\frac{1}{2}$

23. Cramer's Rule does not apply because the coefficient matrix has a determinant of zero.

25. Cramer's Rule does not apply because the coefficient matrix has a determinant of zero.

27. $x_1 = 1$, $x_2 = 1$, $x_3 = 2$

29. $x_1 = 1$, $x_2 = \frac{1}{2}$, $x_3 = \frac{3}{2}$

31. $x_1 = 0$, $x_2 = -\frac{1}{2}$, $x_3 = \frac{1}{2}$

33. Cramer's Rule does not apply because the coefficient matrix has a determinant of zero.

35. $x_1 = -4$

37. $x_1 = -1$

39. $x_1 = -7$

41. $x_1 = 5$

43. $x = \dfrac{4k - 3}{2k - 1}, y = \dfrac{4k - 1}{2k - 1}$

The system will be inconsistent if $k = \frac{1}{2}$.

45. 3

47. 3

49. Collinear

51. Not collinear

53. $3y - 4x = 0$

55. $x = -2$