Lecture

CHAPTER 9. Vectors and the Geometry of Space

This chapter introduces you to a new "mathematical object" called a VECTOR, which actually will not be new to many of you.

VECTORS are connected with FUNCTIONS OF MORE THAN ONE VARIABLE, e.g.,

$f(x,y) = xy^2 e^{-y}$: real-valued function of two variables

$f(x,y) = (\sin xy,\ x^2)$: vector-valued function of two variables

$f(x,y,z) = (x, y, z)$: vector-valued function of three variables

You will learn how to take derivatives and evaluate integrals of these functions through the use of VECTORS.

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Sections 9.1 & 9.2. Three-Dimensional Coordinate Systems & Vectors.

There are three different ways to introduce VECTORS:

1) As physical quantities with magnitude and direction

2) As points in space or in some coordinate system

3) As a combination of a set of special vectors known as standard basis vectors.

We will begin with the "physical approach" in introducing vectors.

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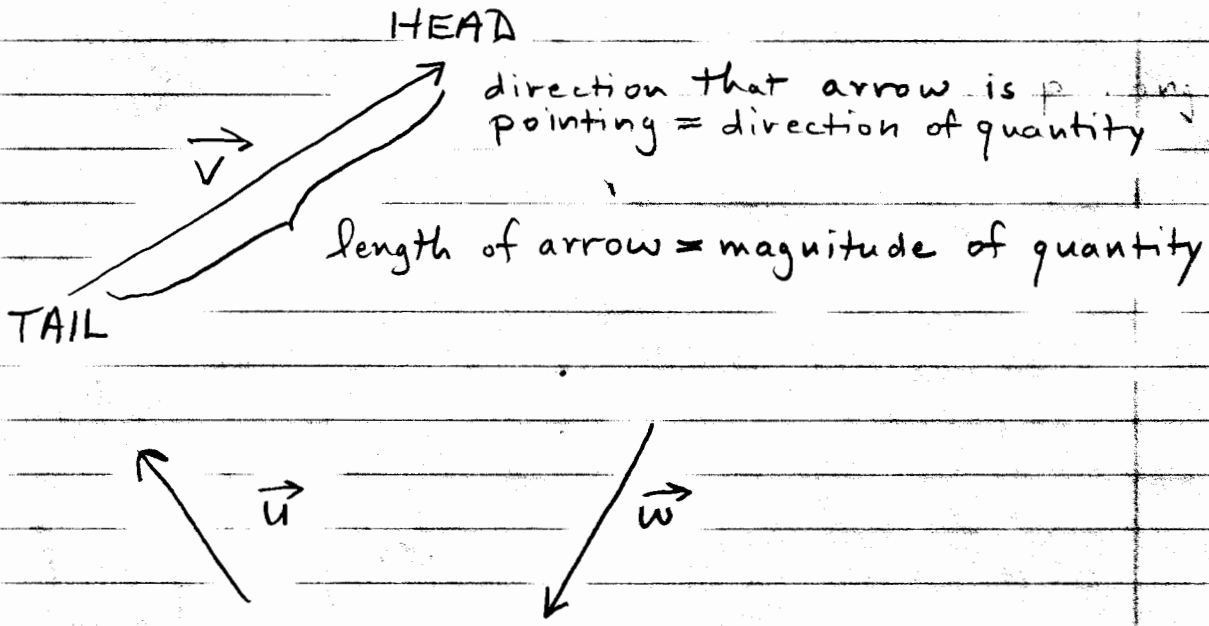
SCALAR QUANTITY = a quantity that is completely defined by a single real number or magnitude

E.g., MASS of an object = 1 kilogram
SPEED of an object = 60 miles / hour
TEMPERATURE of an object = 37° Celsius
WORK DONE on an object = 1000 foot-pounds

VECTOR QUANTITY = a quantity that is defined not only by magnitude but by direction

E.g., MOMENTUM (= mass x velocity)
VELOCITY (= speed with a direction)
ACCELERATION (= dv/dt)
FORCE (= mass x acceleration)

One represents a VECTOR QUANTITY, or simply, VECTOR, by an arrow:



Notation.

1. A vector is denoted by either an arrow over a variable, like this

$$\vec{x},$$

or a variable in boldface type, like this

$$\mathbf{x}.$$

2. $\|\vec{v}\|$ = length of the vector $\vec{v}$.

$\|\vec{v}\|$ is also called the norm of $\vec{v}$.

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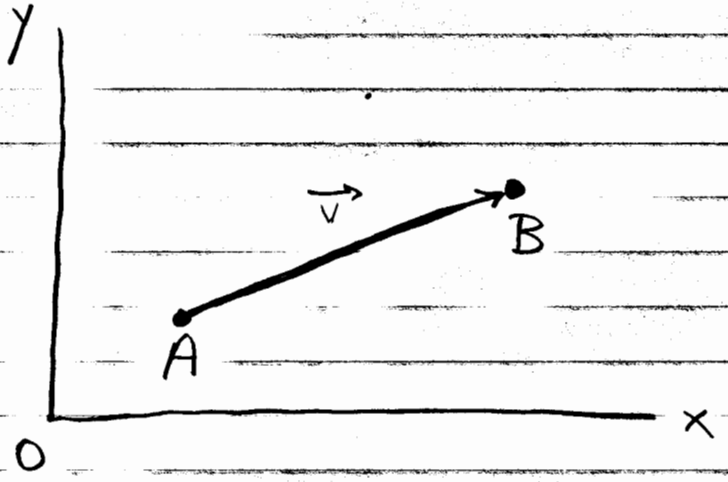
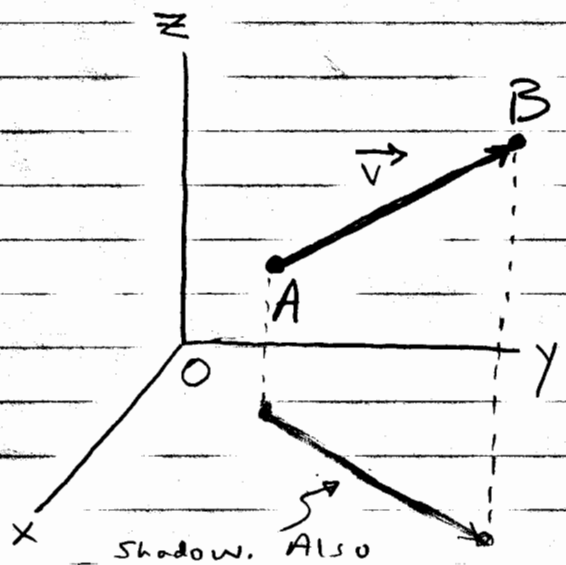
In order to measure the LENGTH and DIRECTION of a vector, we can place the vector in the context of the Cartesian plane:



or in the context of three-dimensional space with the x, y, z-axes:



Shadow. Also known as PROJECTION ONTO THE x,y-PLANE.

NOTE: The text denotes the length of $\vec{v}$ by

$$|AB|$$

instead of by

$$\|\vec{v}\|.$$

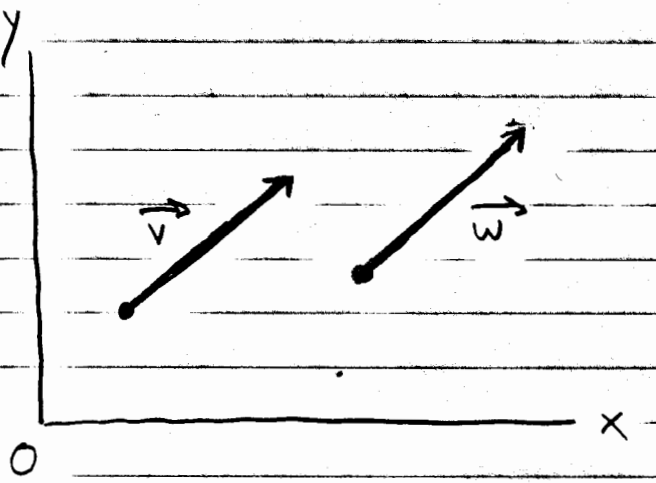
We will use either notation, and which notation we use will depend on which is more convenient to use for the particular problem at hand.

Properties.

1. $\vec{v} = \vec{w}$ if (1) $\|\vec{v}\| = \|\vec{w}\|$ (i.e., $\vec{v}$ and $\vec{w}$ have the same length)

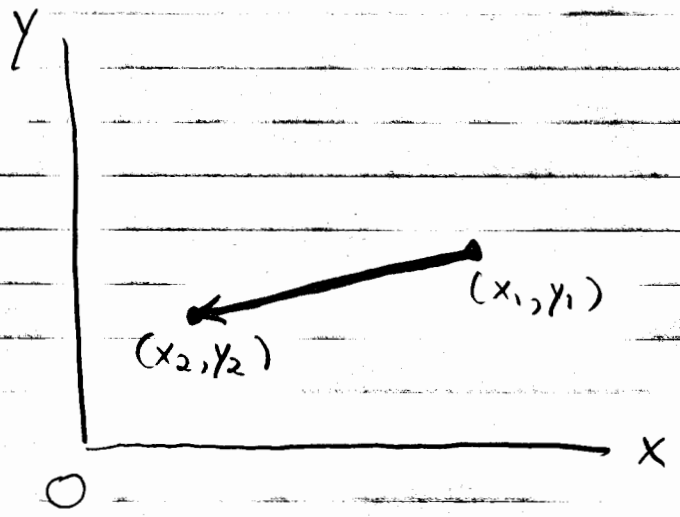
   (2) $\vec{v} \parallel \vec{w}$ (i.e., $\vec{v}$ and $\vec{w}$ are parallel to each other or have the same direction)

E.g.,



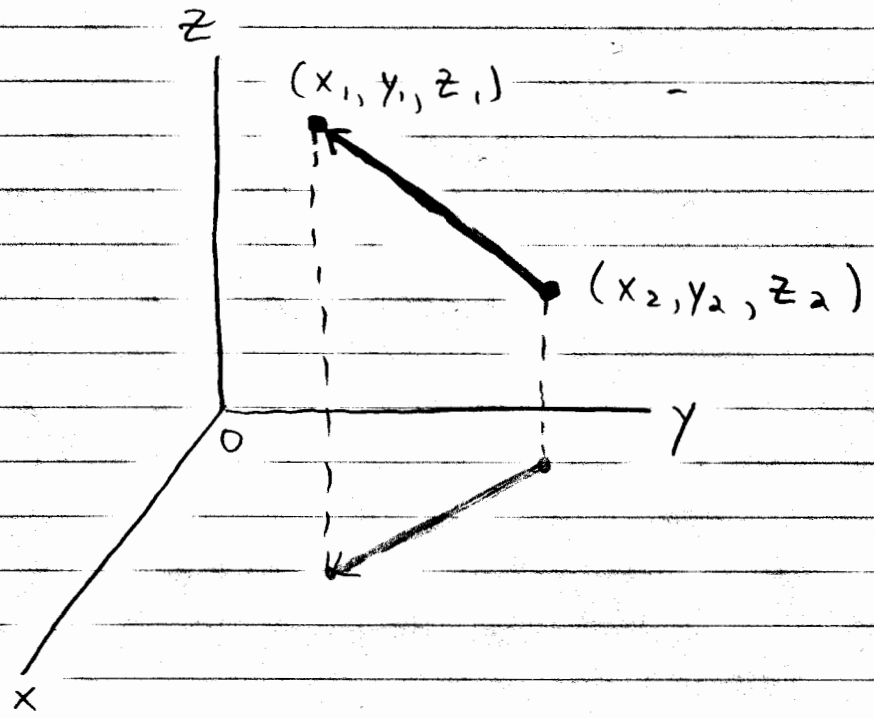
$\vec{v} = \vec{w}$ (they do not have to be in the exact same location in the plane or in space)

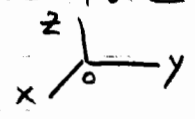
2. $\|\vec{v}\| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$



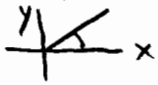
or

$$\|\vec{v}\| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$$



Definition. If a vector $\vec{v}$ is moved within a coordinate system, (like the 3D rectangular coordinate system 

) such that its direction remains the same and its tail is placed at the origin, then $\vec{v}$ in its new position is called a position vector.

Remark. Relocating a vector in a coordinate system so it is a position vector is like relocating the angle of a triangle so it is in standard position in the Cartesian plane 

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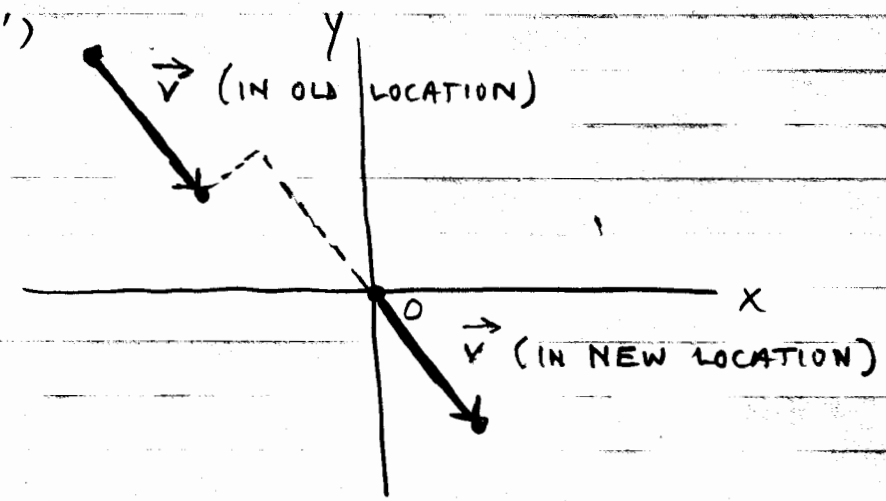
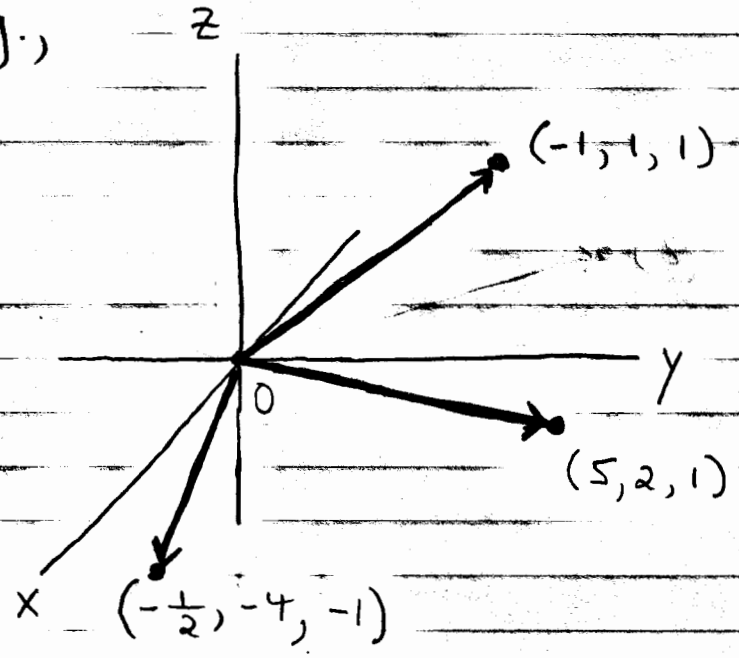
E.g.,



A <u>position vector</u> can then be associated with <u>every point</u> in two- or three-dimensional space

E.g.,



In fact, we may represent position vectors by these points. So, we write

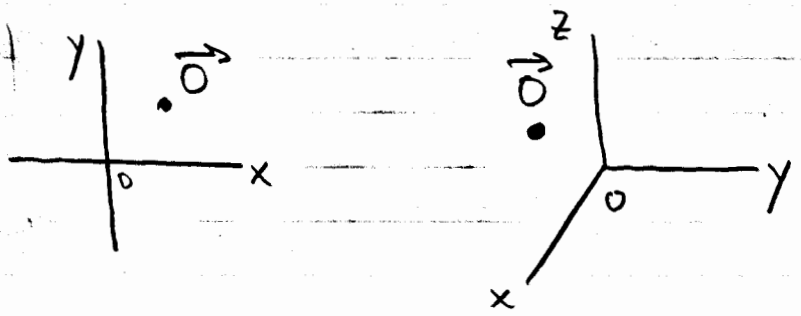
$$\vec{v} = \langle x, y, z \rangle$$

(where (x,y,z) = actual point, $\langle x,y,z \rangle$ = vector associated with point).

Defn. The zero vector, $\vec{0}$, is the vector with no length and no direction.

Remark. Essentially one may look at the zero vector as any point in the plane or any point in space:

E.g.,



However, as a position vector, the zero vector is the point $(0,0)$ or $(0,0,0)$:

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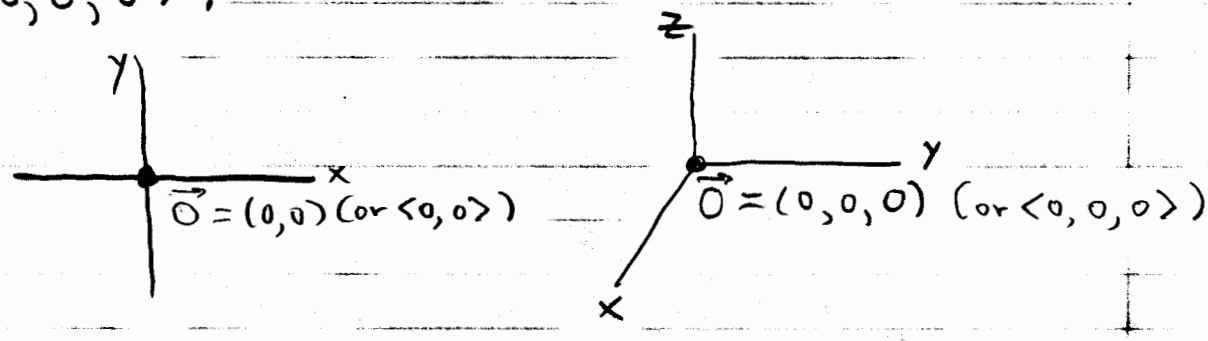

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We can perform two "familiar" operations on vectors (as well as two "unfamiliar" operations — SEE Sections 9.3 and 9.4):

- ADDITION (which includes subtraction)
- MULTIPLICATION BY A SCALAR (and not another vector — we will look at multiplication of vectors in Sections 9.3 and 9.4, in which the "dot product" and "cross product" are introduced)

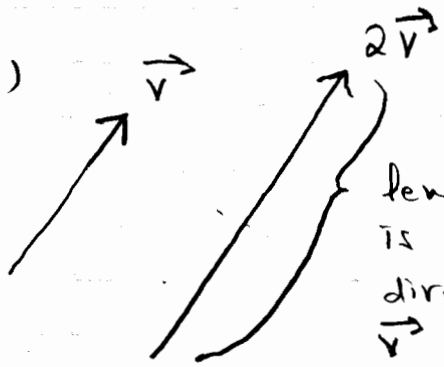
Examples of the above operations.

1. Multiplication by a scalar: $\alpha \vec{v}$

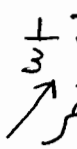
The scalars are often, but not always, denoted by lowercase Greek letters such as

α = alpha
β = beta
γ = gamma
δ = delta

E.g.,



length of $\vec{v}$ is doubled; direction of $\vec{v}$ stays the same and is unaffected

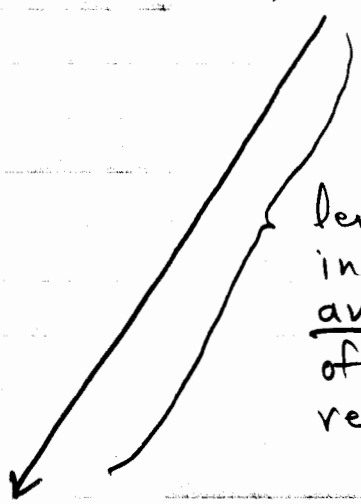


length of $\vec{v}$ is cut down to $\frac{1}{3}$ of its original length; direction stays the same

$(-1)\vec{v}$ $(= -\vec{v})$

length of $\vec{v}$ stays the same <u>but</u> direction of $\vec{v}$ is reversed

$(-5)\vec{v}$ $(= -5\vec{v})$



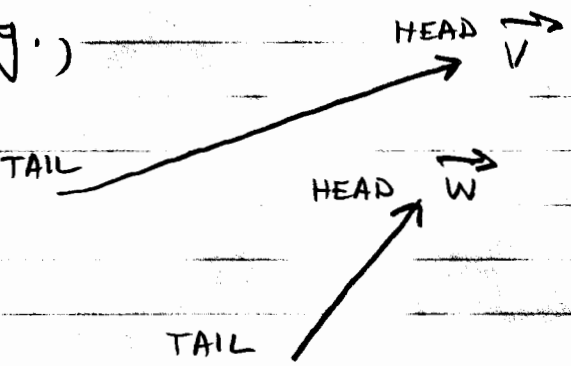
length of $\vec{v}$ is increased 5 times <u>and</u> direction of $\vec{v}$ is reversed

$0\vec{v}$ $(= \vec{0})$

$\vec{v}$ is turned into a point!

2. Addition: $\vec{v} + \vec{w}$

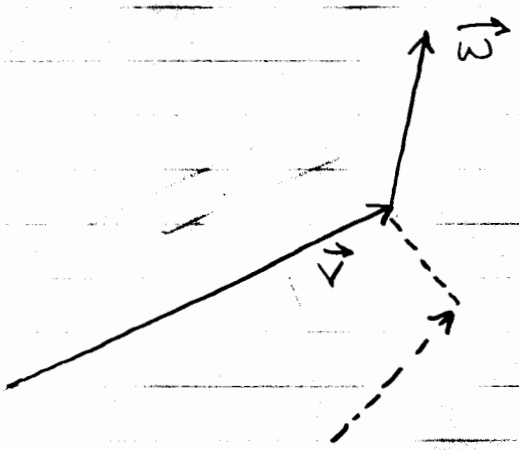
E.g.,



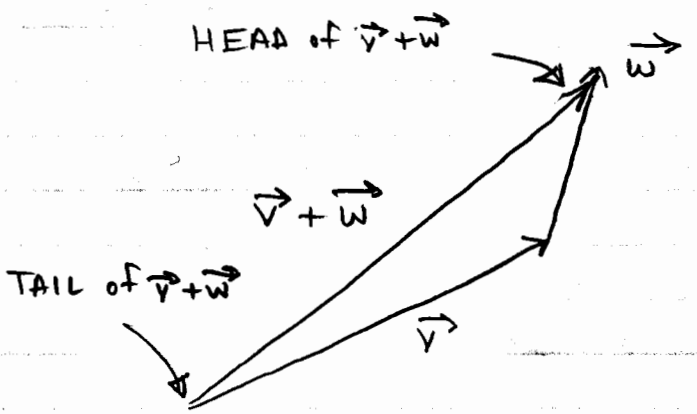
we will add $\vec{w}$ to $\vec{v}$

STEP 1. Do not move the first term, $\vec{v}$.

STEP 2. Move the second term, $\vec{w}$ (without changing its direction), so that its tail coincides with the head of the first term, $\vec{v}$.



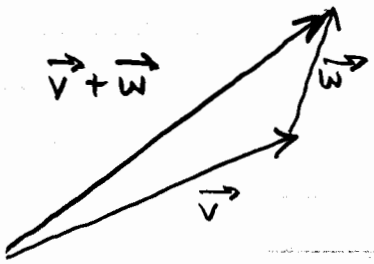
STEP 3. Then the sum, or resultant, $\vec{v} + \vec{w}$, is defined as the "arrow" drawn from the tail of $\vec{v}$ to the head of $\vec{w}$

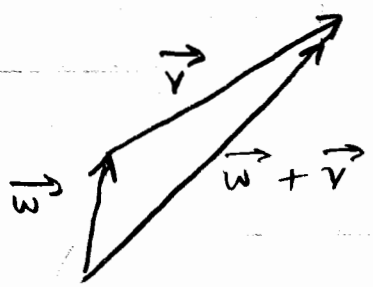


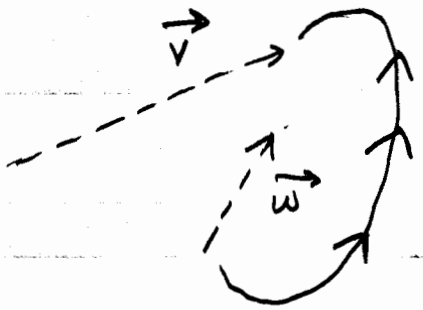
Remark. In actuality, the order in which you add $\vec{v}$ and $\vec{w}$ makes no difference. You will still end up with an "arrow" (or vector) with the same length and direction in either case (although this "arrow" or vector may be in a different position):

Addition of Vectors is "Commutative"!

E.g., $\vec{v} + \vec{w}$ $=\!=$ $\vec{w} + \vec{v}$

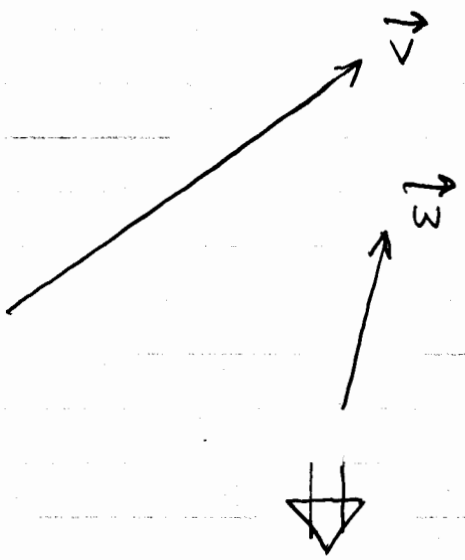


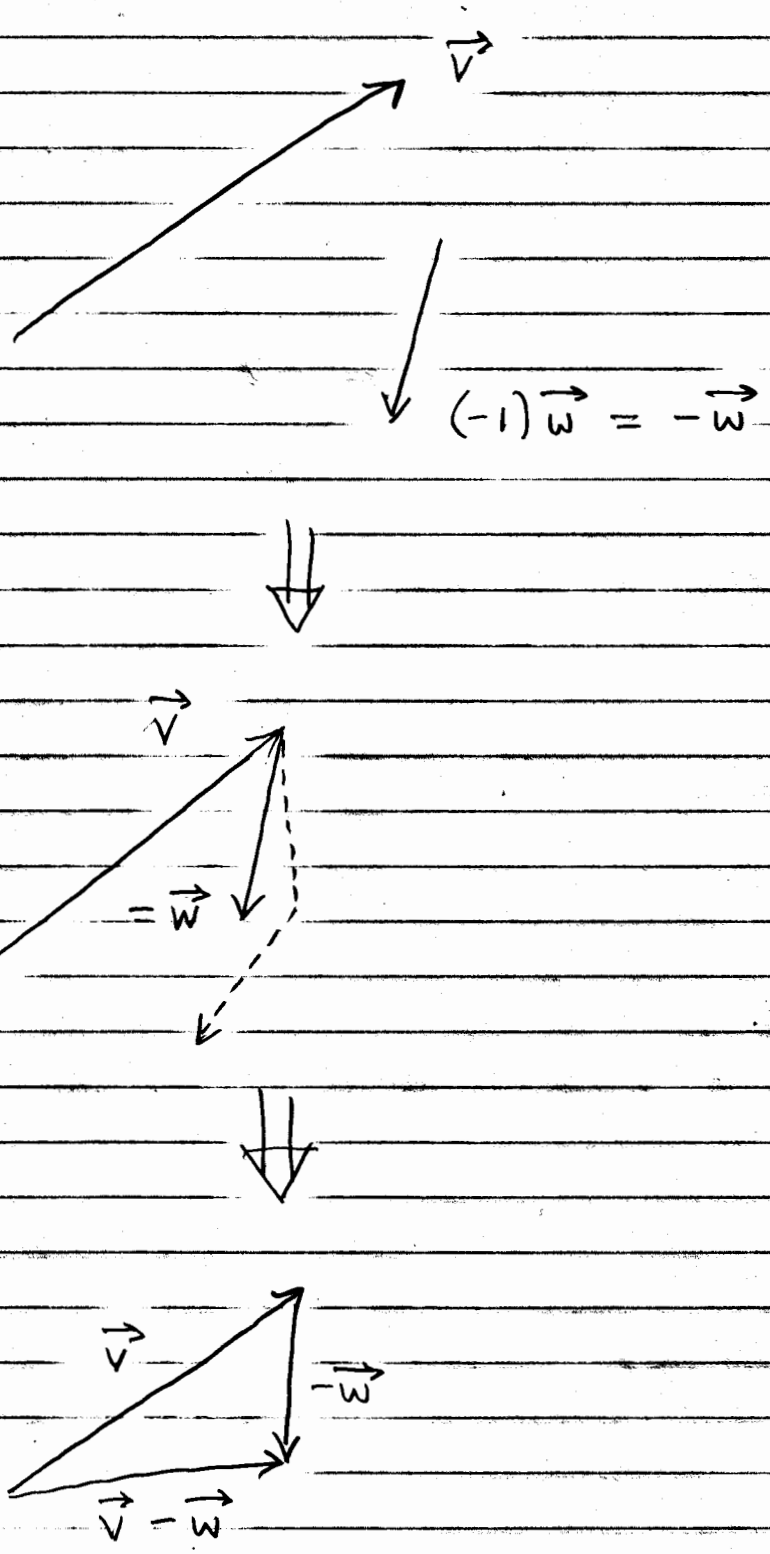




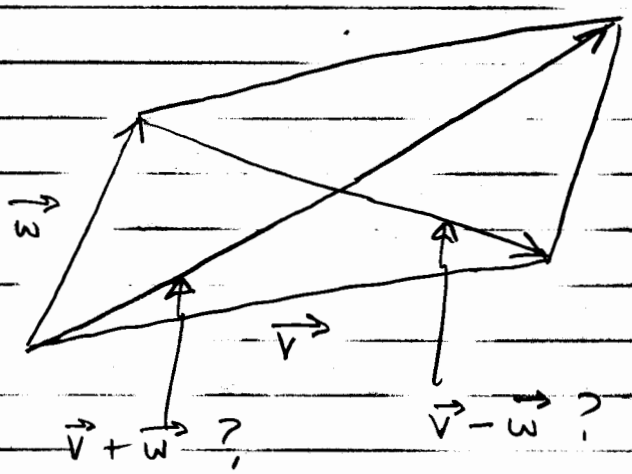
3, Subtraction: $\vec{v} - \vec{w}$ $\left(= \vec{v} + (-\vec{w}) \right.$
$\left. = \vec{v} + (-1)\vec{w} \right)$

E.g.,

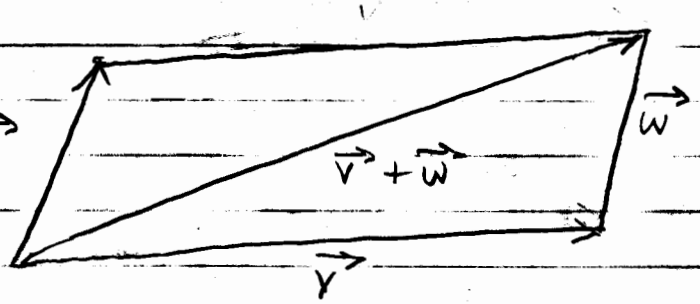



$\vec{v}$
$(-1)\vec{w} = -\vec{w}$
$\vec{v}$
$= \vec{w}$
$\vec{v}$
$-\vec{w}$
$\vec{v} - \vec{w}$

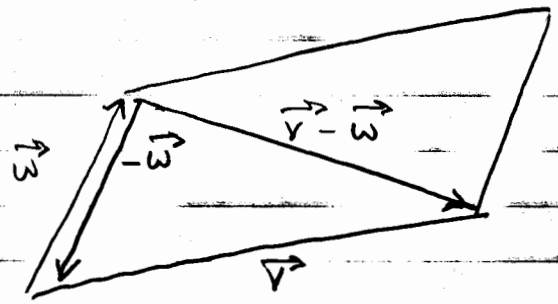
Example. Show that the diagonals of a parallelogram are equal to $\vec{v}-\vec{w}$ and $\vec{v}+\vec{w}$ if two of the parallelograms adjacent sides are considered to be the vectors $\vec{v}$ and $\vec{w}$:



A parallelogram by definition has its opposite sides parallel and equal in length. So, we have the following by VECTOR ADDITION and SUBTRACTION:



and



Parallel Vectors

Defn. Two vectors are parallel if they have the same or opposite direction (but, of course, not necessarily the same magnitude).

Because two parallel vectors, say, $\vec{v}$ and $\vec{w}$, differ only in magnitude (if that), then they have the following property:

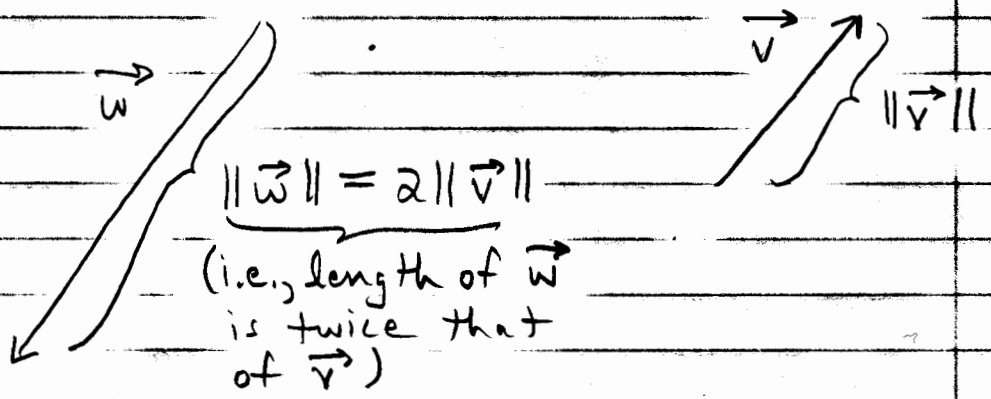
$$\vec{w} = \alpha \vec{v}$$

α a scalar (i.e., a real number)

We say that

$\vec{w}$ IS A SCALAR MULTIPLE OF $\vec{v}$

E.g.,



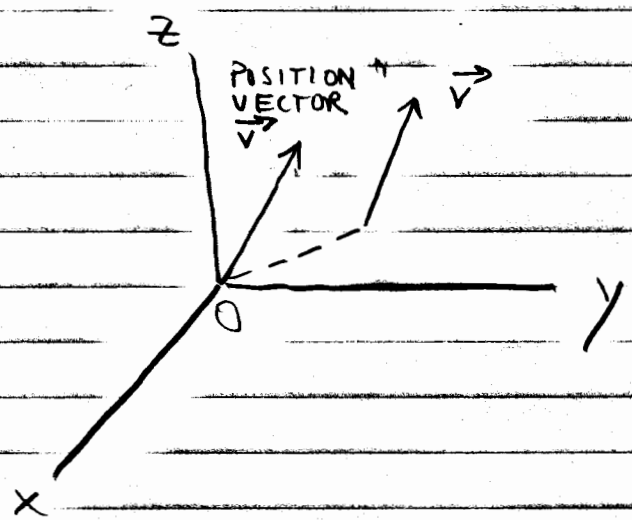
but

$\vec{w} = -2\vec{v}$

0

Component Representation of Vectors

Every vector $\vec{v}$ can be represented as a position vector, i.e., $\vec{v}$ translated to the origin or where the vector is moved so that its tail is at the origin but it still "points" in the same direction:



Once a vector is represented as a position vection (in a coordinate system), we can then, in turn, represent it numerically and as it "relates" to the coordinate axes.

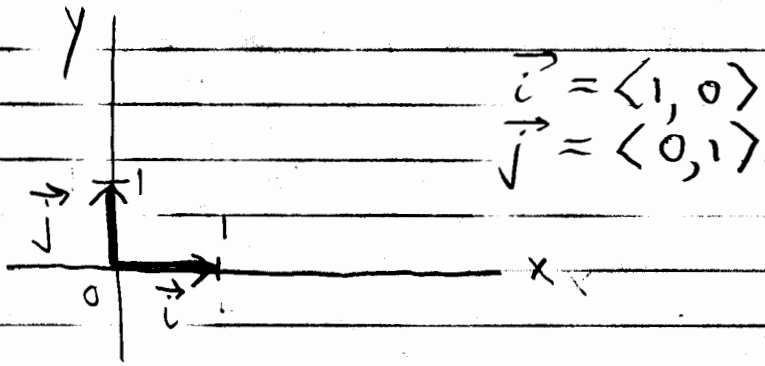
We first need to introduce some special vectors.

In the plane (the x,y-plane), let

$\vec{i}$ and $\vec{j}$

represent two vectors each of LENGTH 1 and in the DIRECTION OF THE x-AXIS and y-AXIS, respectively.

We will call $\vec{i}$ and $\vec{j}$ the UNIT VECTORS ALONG THE COORDINATE AXES:



$\vec{i} = \langle 1, 0 \rangle$

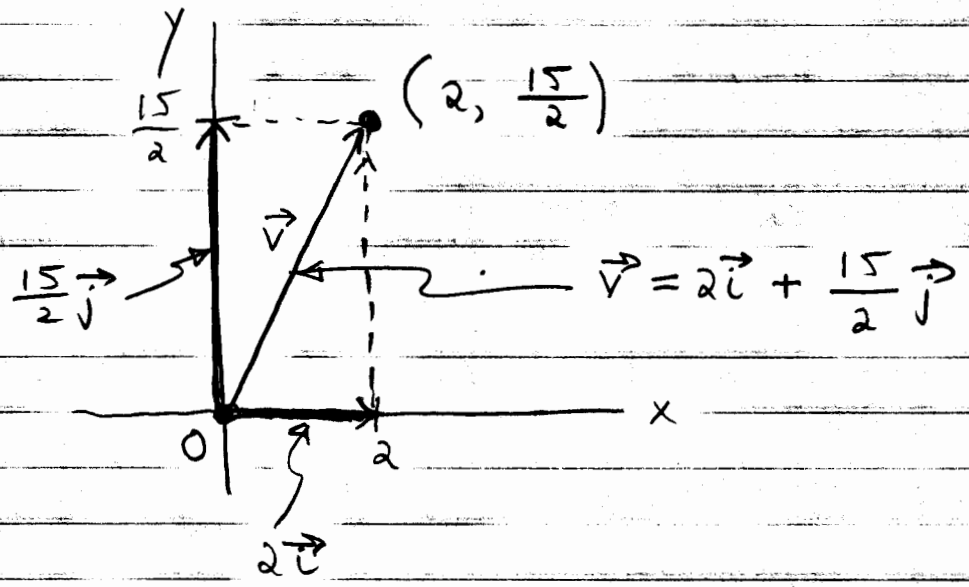
$\vec{j} = \langle 0, 1 \rangle$

Then using the

ADDITION and SCALAR MULTIPLICATION PROPERTIES

of vectors, we can represent any position vector $\vec{v}$ in terms of $\vec{i}$ and $\vec{j}$!

E.g.)

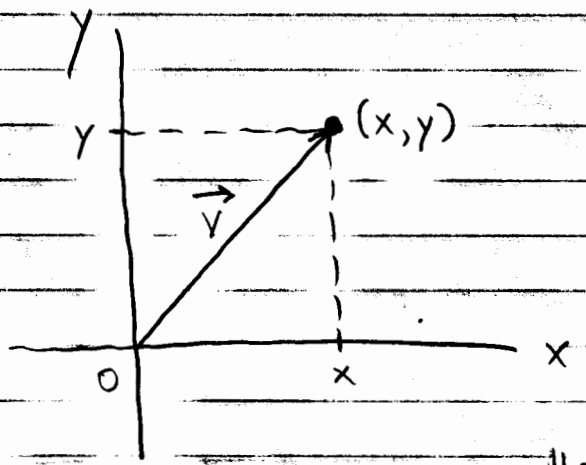


NOTE: The scalars multiplying $\vec{i}$ and $\vec{j}$ are the x- and y-coordinates of the point associated with $\vec{v}$.

OBSERVATION: By the PYTHAGOREAN THEOREM or DISTANCE FORMULA, we have

$$\|\vec{v}\| = \sqrt{(2)^2 + \left(\frac{15}{2}\right)^2} = \frac{\sqrt{241}}{2}$$

So, in general,



$\vec{v} = x\vec{i} + y\vec{j}$

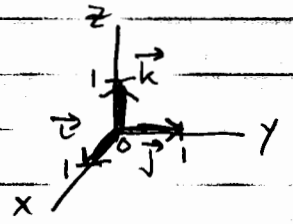
the COMPONENTS of $\vec{v}$

$$\|\vec{v}\| = \sqrt{x^2 + y^2}$$

In three dimensions, representing vectors in a componentwise fashion is similar, only now we must introduce a third unit vector,

$\vec{k} =$

which is positioned along the z-axis:

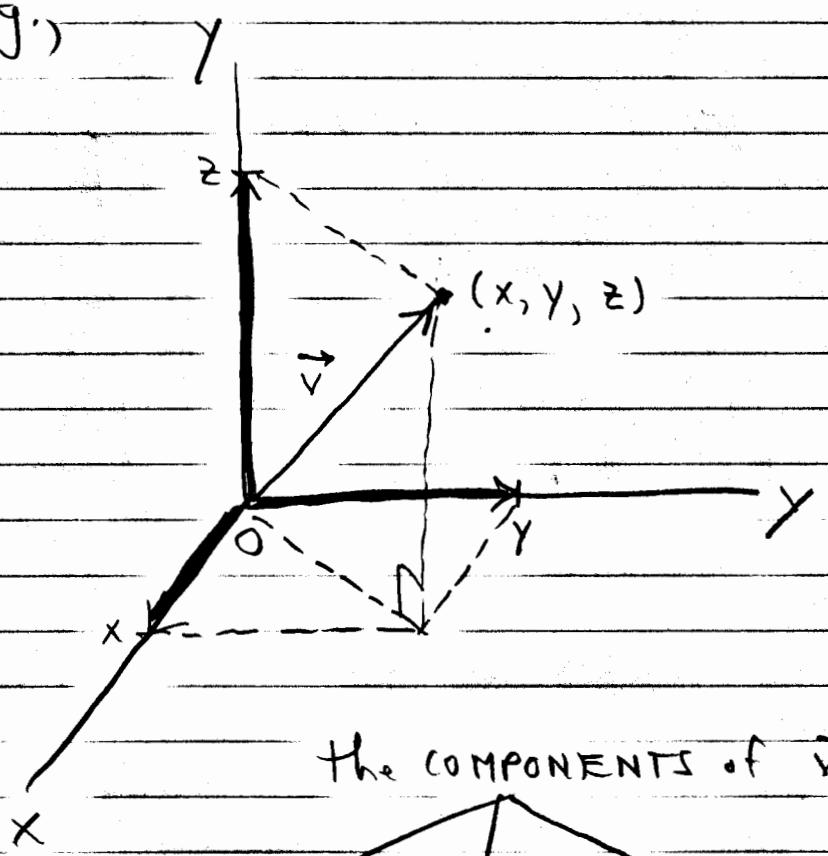


$\vec{i} = \langle 1, 0, 0 \rangle$

$\vec{j} = \langle 0, 1, 0 \rangle$

$\vec{k} = \langle 0, 0, 1 \rangle$

E.g.,

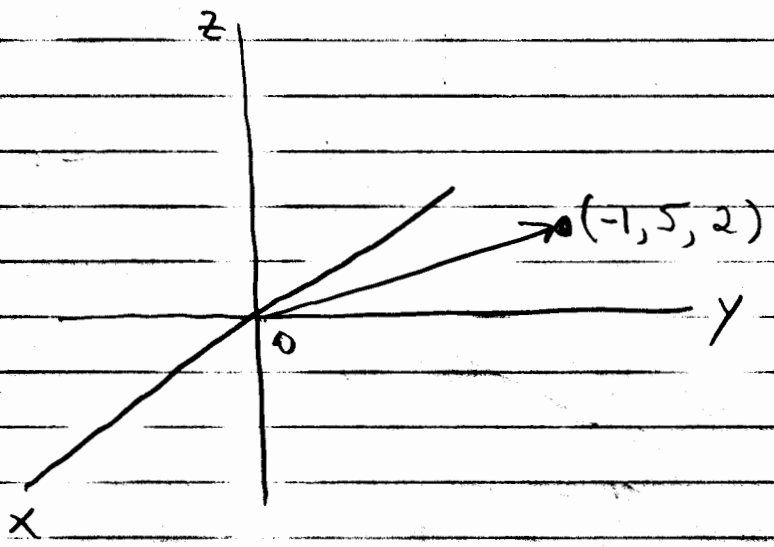


the COMPONENTS of $\vec{v}$

$$\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\|\vec{v}\| = \sqrt{x^2 + y^2 + z^2}$$

Example. Give a component representation of the position vector associated with the point $(-1, 5, 2)$. Also, find the length of the vector.



$$\boxed{\vec{v} = -\vec{i} + 5\vec{j} + 2\vec{k}}$$

$$\|\vec{v}\| = \sqrt{(-1)^2 + (5)^2 + (2)^2} = \boxed{\sqrt{30}}$$

Addition and Scalar Multiplication of Vectors Represented in Terms of Components

1. Addition:

$$\vec{v} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$$

$$\vec{w} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$$

$$\vec{v} + \vec{w} = (x_1\vec{i} + y_1\vec{j} + z_1\vec{k}) + (x_2\vec{i} + y_2\vec{j} + z_2\vec{k})$$

— COLLECT LIKE TERMS

$$= (x_1\vec{i} + x_2\vec{i}) + (y_1\vec{j} + y_2\vec{j}) + (z_1\vec{k} + z_2\vec{k})$$

— FACTOR OUT THE $\vec{i}$, $\vec{j}$, $\vec{k}$

$$= \boxed{(x_1 + x_2)\vec{i} + (y_1 + y_2)\vec{j} + (z_1 + z_2)\vec{k}}$$

2. Scalar multiplication:

$$\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$$

α = a scalar

$$\alpha\vec{v} = \alpha(x\vec{i} + y\vec{j} + z\vec{k})$$

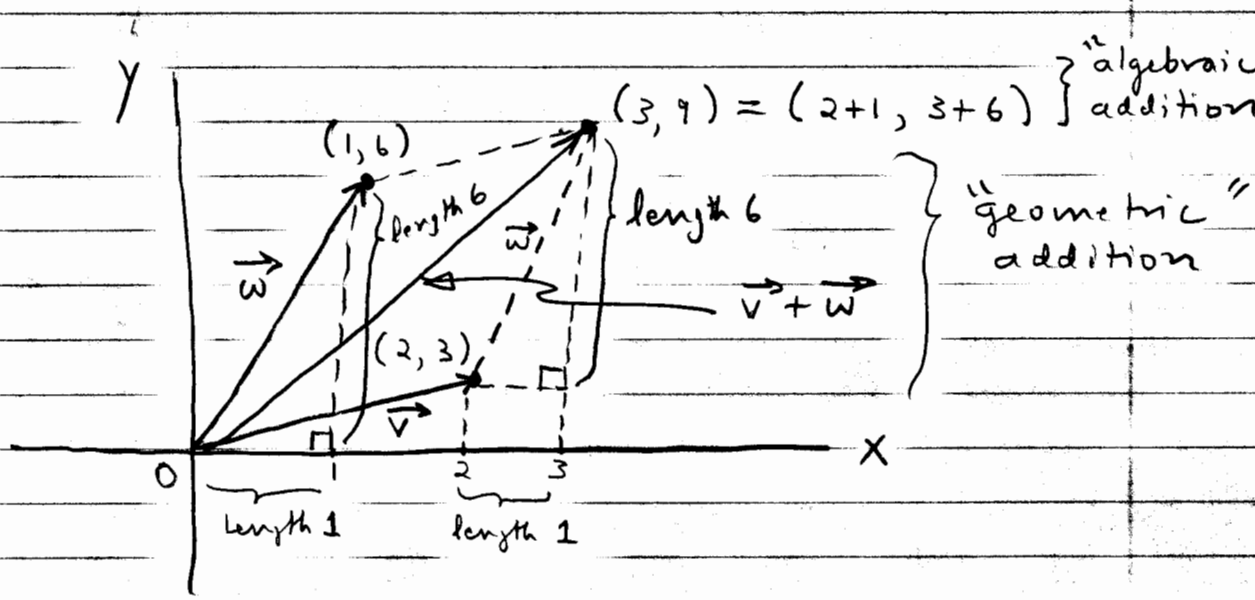
↓ – DISTRIBUTE THE α

$$= \alpha(x\vec{i}) + \alpha(y\vec{j}) + \alpha(z\vec{k})$$

↓ ASSOCIATIVE LAW OF MULTIPLICATION used here

$$= \boxed{(\alpha x)\vec{i} + (\alpha y)\vec{j} + (\alpha z)\vec{k}}$$

E.g., $\left.\begin{aligned}\vec{v} &= 2\vec{\imath} + 3\vec{\jmath} = \langle 2, 3\rangle \\ \vec{w} &= \vec{\imath} + 6\vec{\jmath} = \langle 1, 6\rangle\end{aligned}\right\}$ $\vec{v} + \vec{w} = ?$



$$\vec{v} + \vec{w} = (2\vec{\imath} + 3\vec{\jmath}) + (\vec{\imath} + 6\vec{\jmath})$$

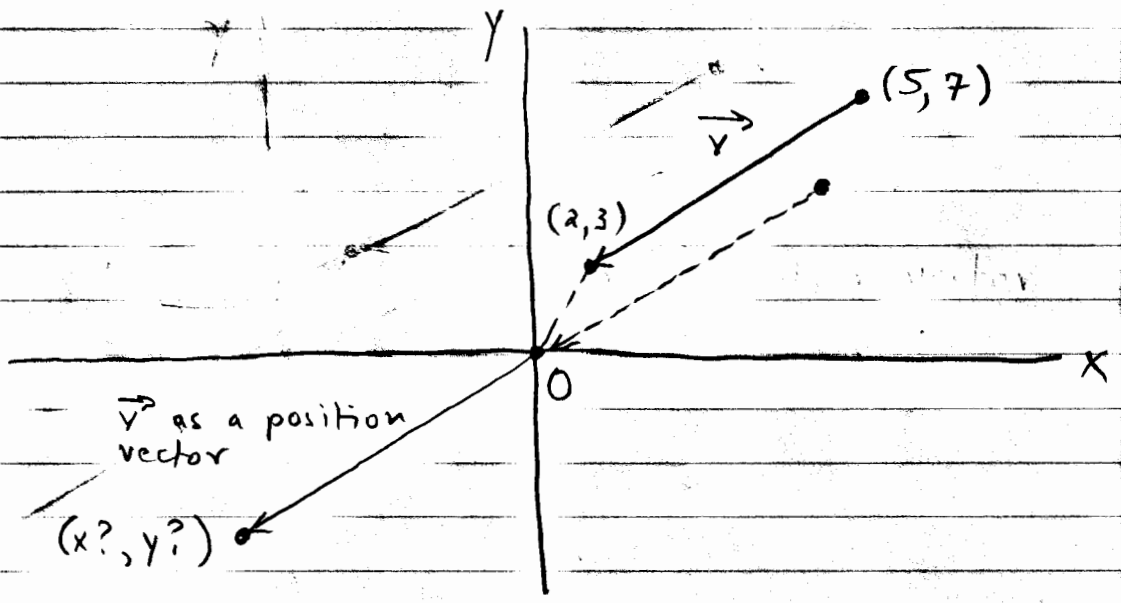
$$= (2\vec{\imath} + \vec{\imath}) + (3\vec{\jmath} + 6\vec{\jmath})$$

$$= \boxed{3\vec{\imath} + 9\vec{\jmath}}$$

$$= \boxed{\langle 3, 9\rangle}$$

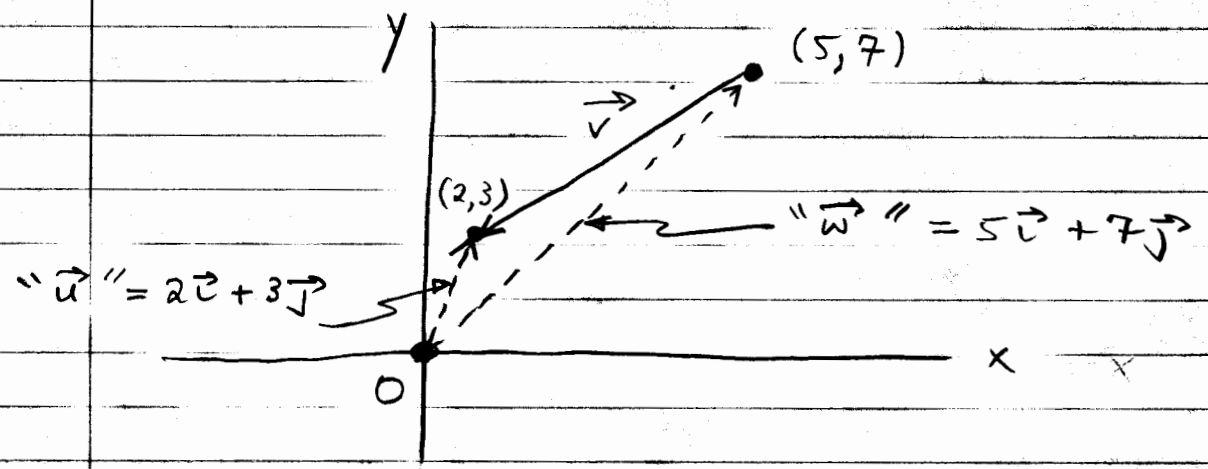
E.g., what are the components of the vector starting at the point (5, 7) and ending at the point (2, 3)?

What the question really is asking is, "What is the component representation of $\vec{v}$, starting at (5, 7) and ending at (2, 3), if $\vec{v}$ is moved to the origin and represented as a position vector?"



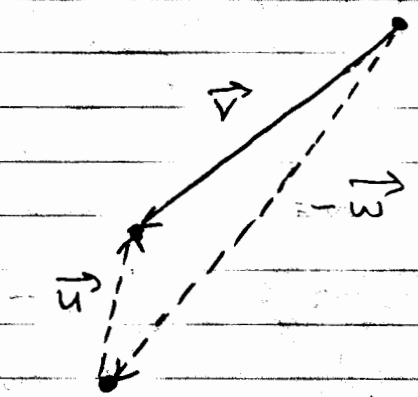
What we wish to do is to express $\vec{v}$ as the sum (or difference) of two other position vectors:

Let $\vec{u}$ and $\vec{v}$ be position vectors as defined below:



$\vec{v} = -\vec{w} + \vec{u} = \vec{u} + (-\vec{w}) = \vec{u} - \vec{w}$

where



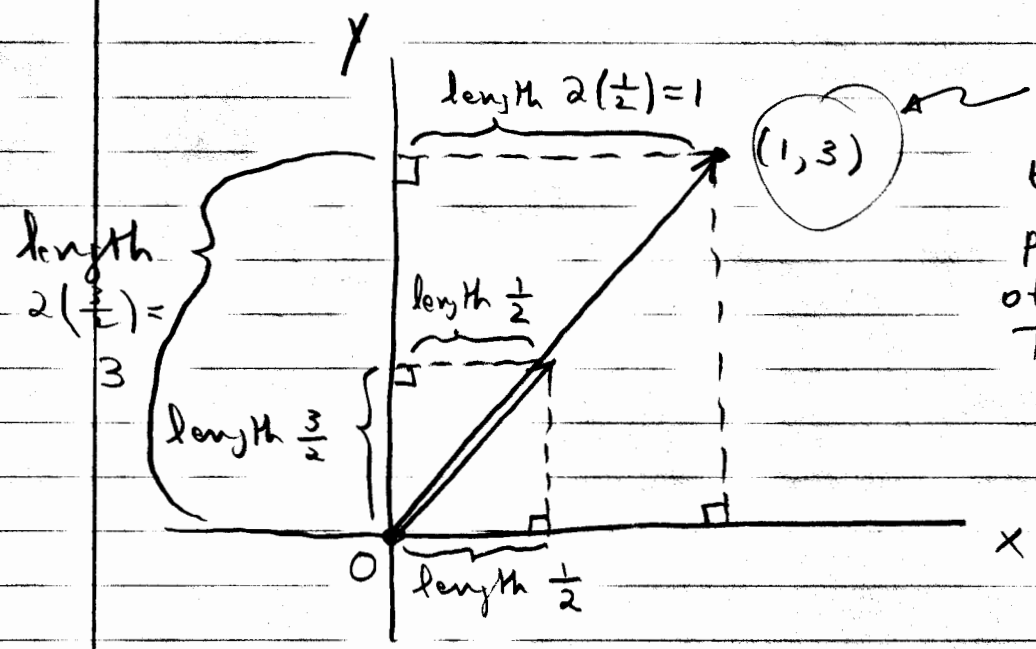
Therefore,

$$\vec{v} = \vec{u} - \vec{w} = (2\vec{i} + 3\vec{j}) - (5\vec{i} + 7\vec{j})$$

$$= (2\vec{i} - 5\vec{i}) + (3\vec{j} - 7\vec{j})$$

$$= \boxed{-3\vec{i} - 4\vec{j}}$$

E.g., $\vec{v} = \frac{1}{2}\vec{i} + \frac{3}{2}\vec{j}$, $\alpha = 2$ } $\alpha\vec{v} = ?$



$$2\vec{v} = 2\left(\frac{1}{2}\vec{i} + \frac{3}{2}\vec{j}\right) = 2\left(\frac{1}{2}\vec{i}\right) + 2\left(\frac{3}{2}\vec{j}\right)$$

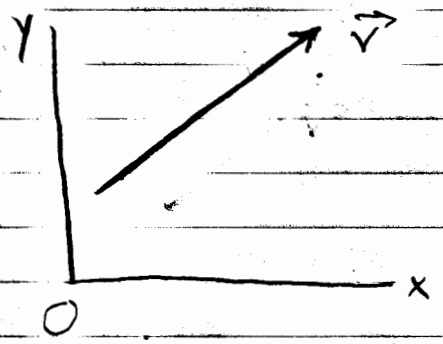
$$= \left(2\cdot\frac{1}{2}\right)\vec{i} + \left(2\cdot\frac{3}{2}\right)\vec{j}$$

$$= \boxed{\vec{i} + 3\vec{j}}$$

0

Resolving a Vector in Two Dimensions Into its Components

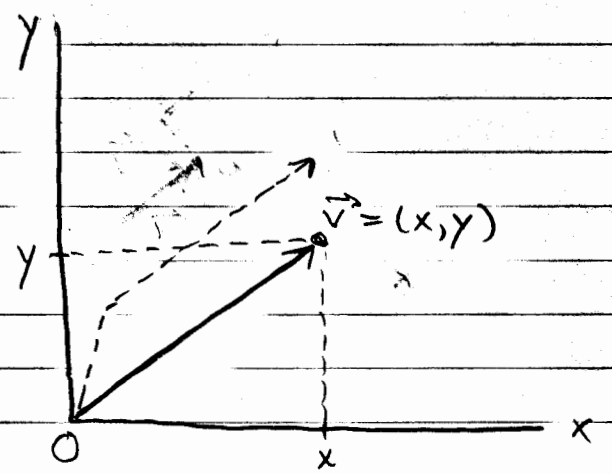
Let $\vec{v}$ be a vector in two dimensions



We can determine the components of $\vec{v}$, i.e., we can find x and y such that

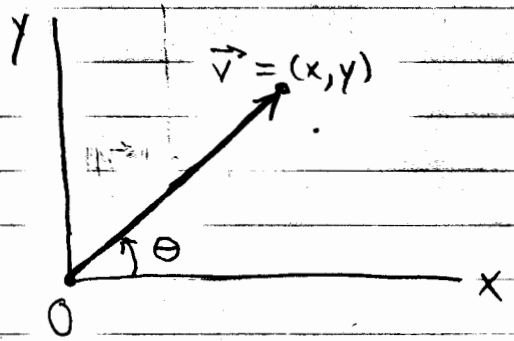
$$\vec{v} = x\vec{i} + y\vec{j},$$

if we are given the length of $\vec{v}$ and the direction of $\vec{v}$ where direction is given in terms of the angle that $\vec{v}$ makes with the x-axis when $\vec{v}$ is a position vector



So, given

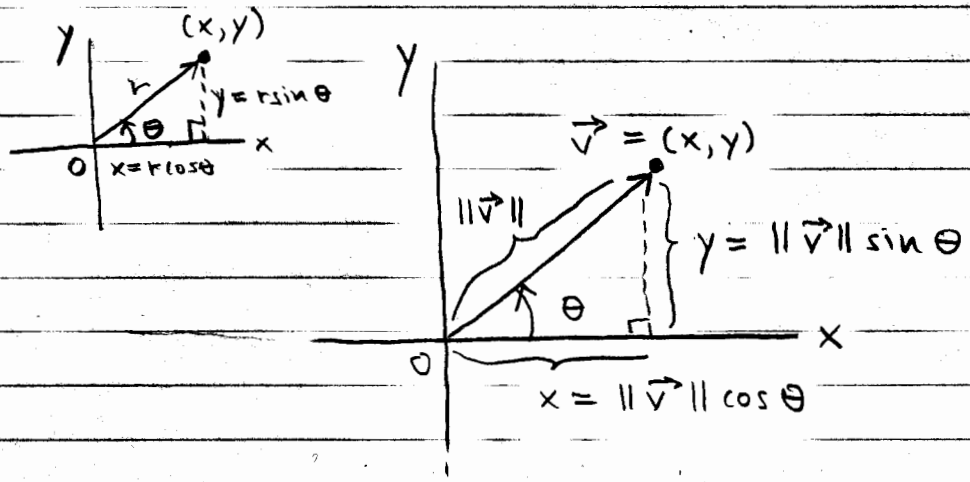
① $\|\vec{v}\|$ = length of $\vec{v}$
② θ = "direction" of $\vec{v}$ where



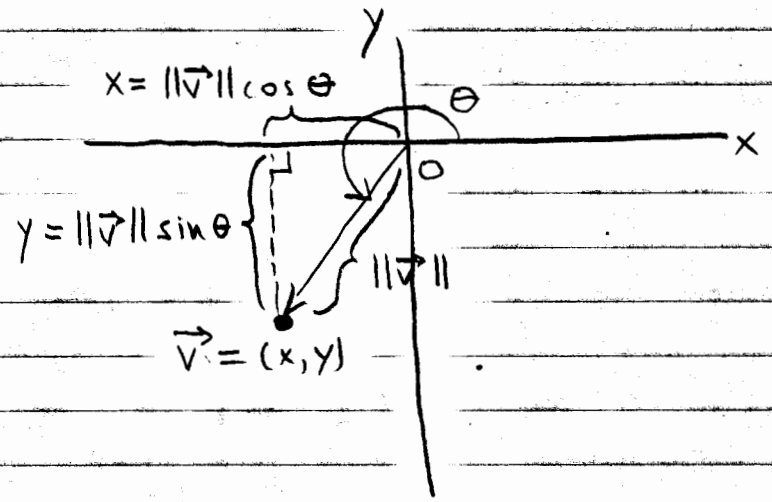
we can compute x and y in

$$\vec{v} = x\vec{i} + y\vec{j}$$

by using the following trigonometric properties:



or, if $\vec{v}$ were a position vector in the third quadrant, for example, one would have



etc.

E.g., resolve $\vec{v}$ into components if $\|\vec{v}\| = r = 5$ and $\theta = \pi$:

We have

$$\vec{v} = x\vec{i} + y\vec{j}$$

$$= (\|\vec{v}\| \cos\theta)\vec{i} + (\|\vec{v}\| \sin\theta)\vec{j}$$

$$= (5 \cos\pi)\vec{i} + (5 \sin\pi)\vec{j}$$

$$= [5(-1)]\vec{i} + (5 \cdot 0)\vec{j}$$

$$= \boxed{-5\vec{i} + 0 \cdot \vec{j}}$$

$$= -5\vec{i} + \vec{0}$$

$$= \boxed{-5\vec{i}}$$

Unit Vectors and Scaling Vectors So That They Have Length 1

Defn. A **unit vector** $\vec{v}$ is a vector whose length is 1, i.e.,

$$\|\vec{v}\| = 1.$$

Examples of unit vectors.

$$\vec{i}, \quad \vec{j}, \quad \vec{k}.$$

Application. A unit vector is useful in representing the common direction of a set of vectors which are parallel to each other.

Note. Any vector $\vec{v}$ can be converted to a unit vector by multiplying $\vec{v}$ by the scalar $\frac{1}{\|\vec{v}\|}$:

$\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$ $\quad (\vec{v} = (x, y, z))$

$\|\vec{v}\| = \sqrt{x^2 + y^2 + z^2}$

$$\frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{x^2+y^2+z^2}}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}}\vec{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\vec{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\vec{k}$$

Therefore,

$$\left\| \frac{1}{\|\vec{v}\|}\vec{v} \right\| = \sqrt{\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)^2 + \left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)^2}$$

$$= \sqrt{\frac{x^2}{x^2+y^2+z^2} + \frac{y^2}{x^2+y^2+z^2} + \frac{z^2}{x^2+y^2+z^2}}$$

$$= \sqrt{\frac{x^2+y^2+z^2}{x^2+y^2+z^2}}$$

$$= \sqrt{1}$$

$$= 1 \checkmark$$

E.g., find a unit vector $\vec{u}$, in the same direction as $\vec{v} = -3\vec{i} + 2\vec{j}$:

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{(-3)^2 + (2)^2}}(-3\vec{i} + 2\vec{j})$$

$$= \frac{1}{\sqrt{13}}(-3\vec{i} + 2\vec{j}) = \boxed{-\frac{3}{\sqrt{13}}\vec{i} + \frac{2}{\sqrt{13}}\vec{j}}$$

Check: $\|\vec{u}\| = \sqrt{\left(-\frac{3}{\sqrt{13}}\right)^2 + \left(\frac{2}{\sqrt{13}}\right)^2}$

$$= \sqrt{\frac{9}{13} + \frac{4}{13}}$$

$$= \sqrt{\frac{13}{13}}$$

$$= \sqrt{1}$$

$$= 1 \checkmark$$

EXAMPLES.

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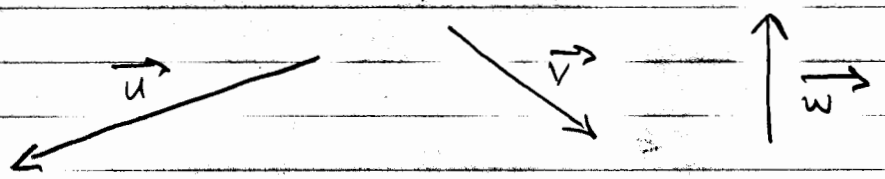
M **Exercises 9.2, pp. 659 – 660**

5. Copy the vectors in the figure and use them to draw the following vectors.

(a) $\vec{u} + \vec{v}$ (b) $\vec{u} - \vec{v}$

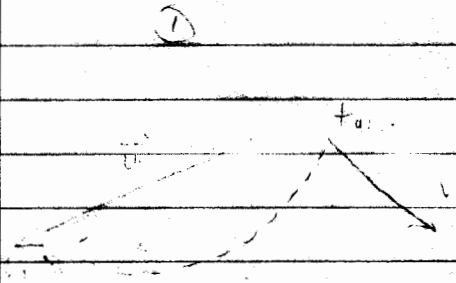
(c) $\vec{v} + \vec{w}$ (d) $\vec{w} + \vec{v} + \vec{u}$

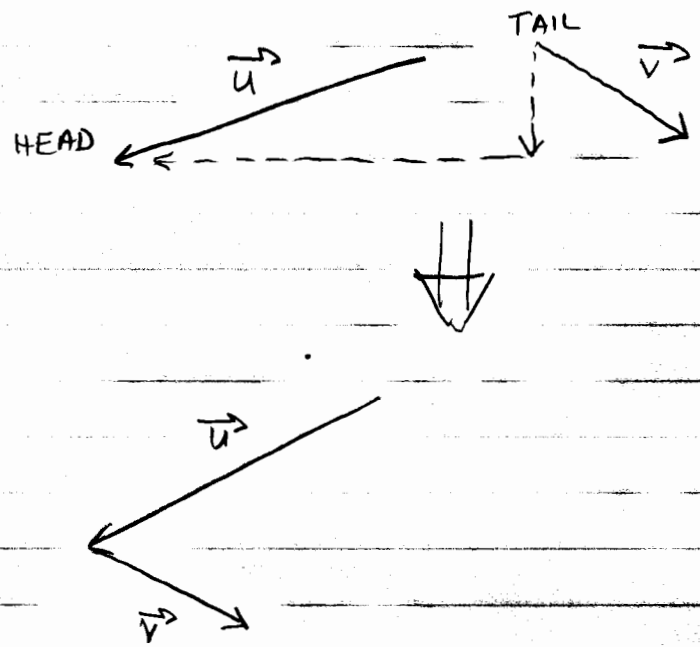


(a) $\vec{u} + \vec{v}$

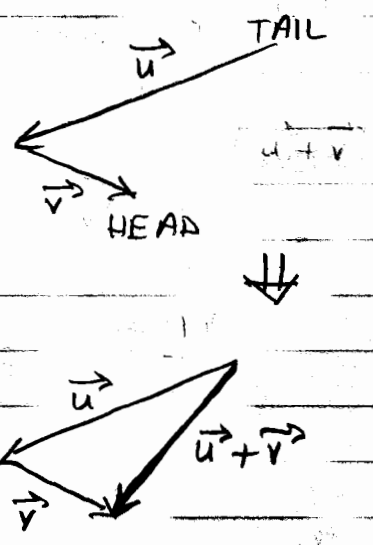
FIRST (VECTOR) SECOND (VECTOR)

VECTOR ADDITION: ① Keep FIRST fixed
② Move SECOND
ⓐ without changing its direction
ⓑ so that its TAIL is at the HEAD of the FIRST

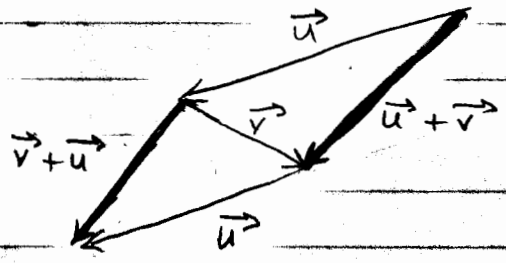




(3) Connect TAIL of FIRST with HEAD of SECOND to create the vector $\vec{u} + \vec{v}$



NOTE: $\vec{u}+\vec{v} = \vec{v}+\vec{u}$. The only difference between $\vec{u}+\vec{v}$ and $\vec{v}+\vec{u}$ is that they will be in two different locations.

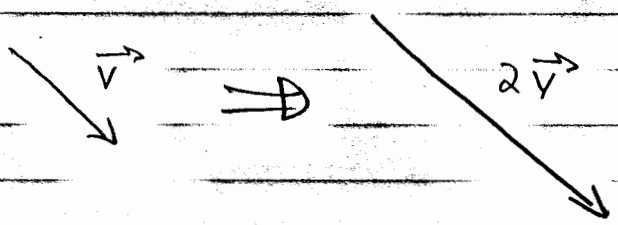


(b) $\vec{u}-\vec{v} \overset{\text{def}}{=} \vec{u}+(-\vec{v}) = \vec{u}+(-1)\vec{v}$

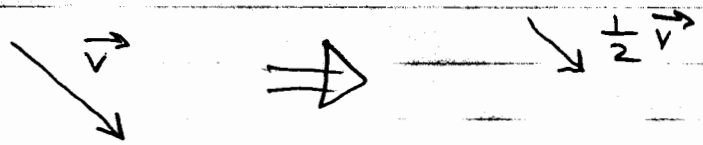
"INVERSE OF $\vec{v}$" — FIRST (VECTOR) — SCALAR — SECOND (VECTOR): "SCALAR MULTIPLE OF $\vec{v}$"

SCALAR MULTIPLICATION does one of four things to a vector $\vec{v}$

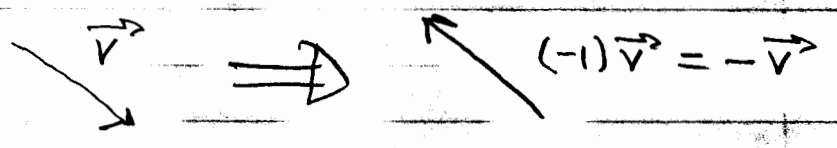
① $\alpha\vec{v}$ stretches $\vec{v}$ if $\alpha > 1$



② $\alpha\vec{v}$ shrinks $\vec{v}$ if $\alpha < 1$



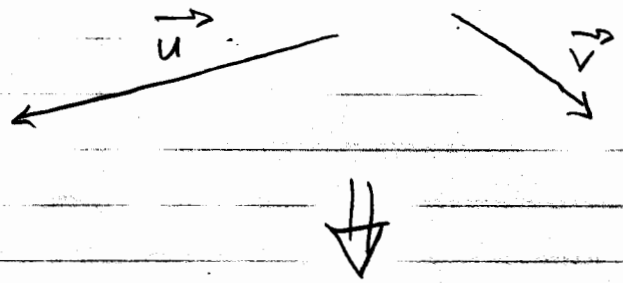
③ $\alpha\vec{v}$ reverses the direction of $\vec{v}$ (and expands or shrinks $\vec{v}$) if $\alpha < 0$

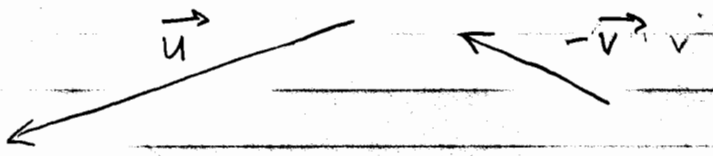


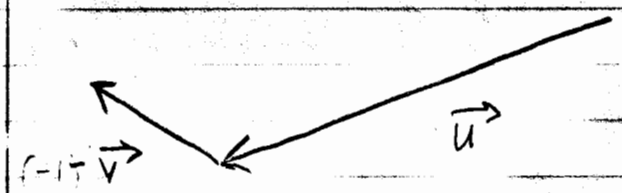
④ $\alpha\vec{v}$ does not change $\vec{v}$ if $\alpha = 1$

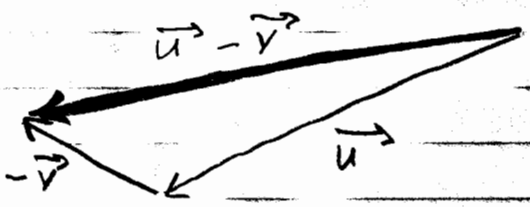
So, with $\vec{u} - \vec{v} = \vec{u} + (-1)\vec{v}$

FIRST SECOND



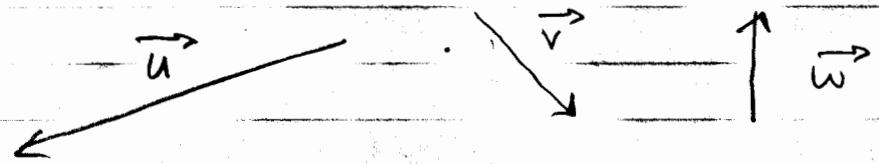

$\vec{u}$
$-\vec{v}$

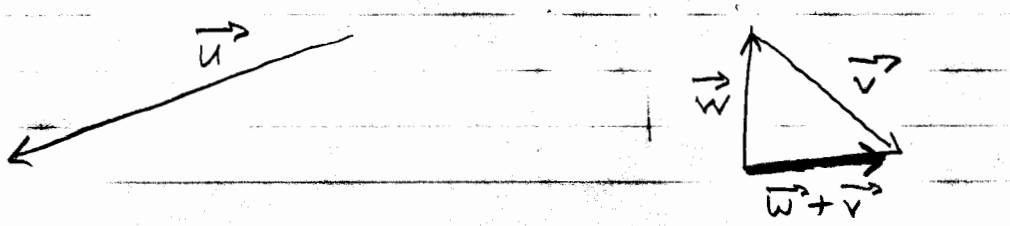

$\vec{u}$

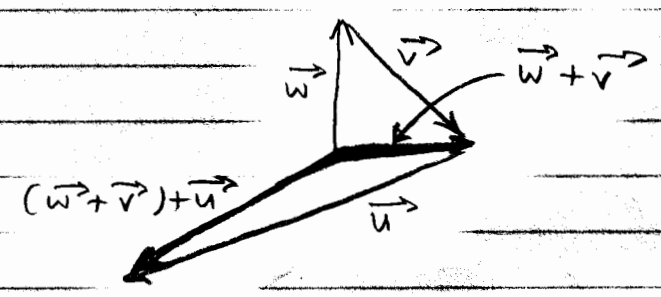

$\vec{u} - \vec{v}$
$-\vec{v}$
$\vec{u}$

(d) $\vec{w} + \vec{v} + \vec{u} = (\vec{w} + \vec{v}) + \vec{u}$

FIRST SECOND — NEW FIRST — NEW SECOND



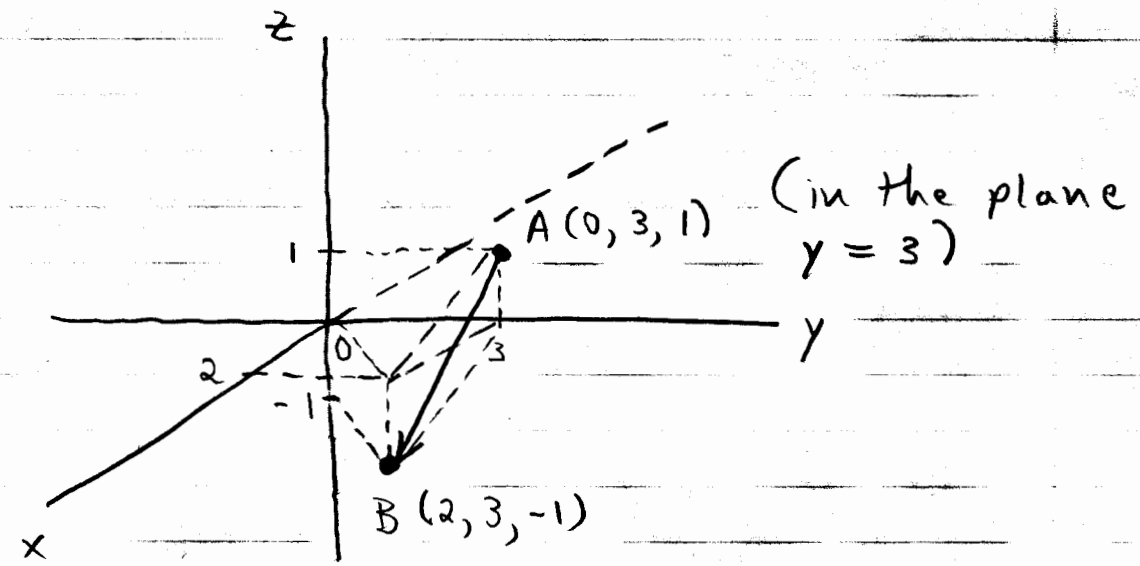




9. Find a vector $\vec{a}$ with representation given by the directed line segment $\overrightarrow{AB}$. Draw $\overrightarrow{AB}$ and the equivalent representation starting at the origin:

$A(0, 3, 1)$, $B(2, 3, -1)$.

$\overrightarrow{AB}$ is the DISPLACEMENT VECTOR, of, say, a particle moving from the point $A(0,3,1)$ to the point $B(2,3,-1)$:



Let $\vec{a}$ represent $\overrightarrow{AB}$ as a position vector, i.e. $\vec{a}$ is $\overrightarrow{AB}$ moved so that A, or the tail of $\overrightarrow{AB}$ is at the

M

origin.

NOTE: We can associate 2 vectors with the points $A(0,3,1)$ and $B(2,3,-1)$, respectively:

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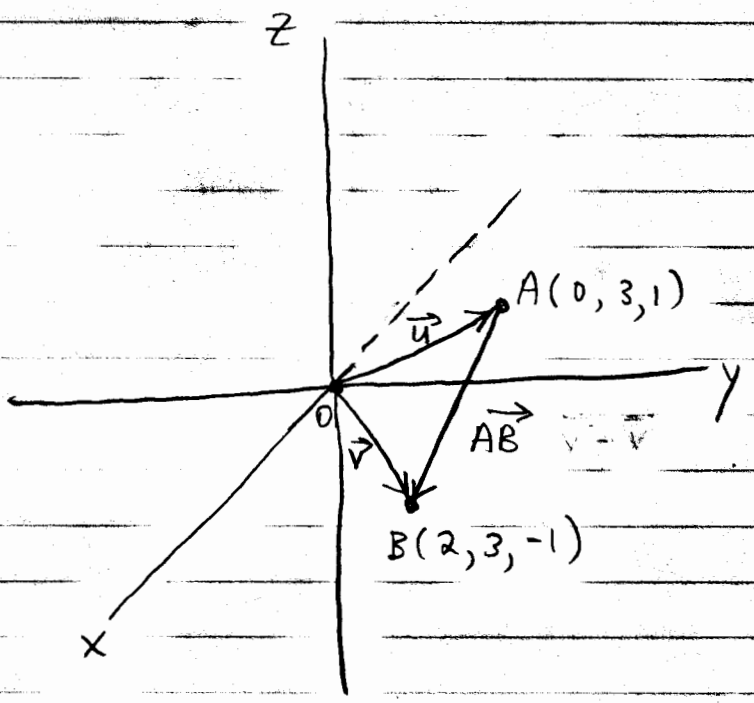
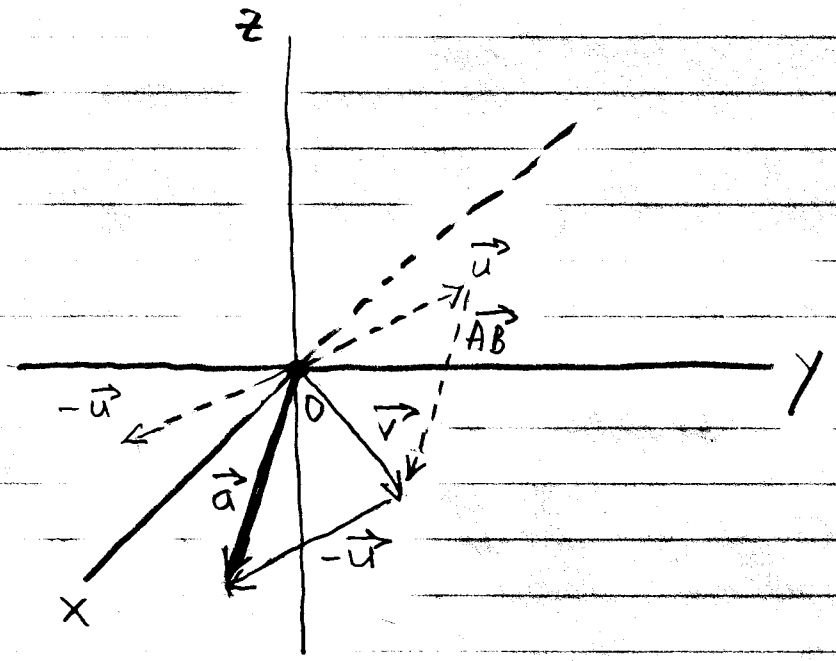
$$\vec{u} = \langle 0,3,1 \rangle \text{ and } \vec{v} = \langle 2,3,-1 \rangle$$



Then observe that

$\vec{a} = \vec{AB}$ moved to the origin as a POSITION VECTOR

$= \vec{v} - \vec{u}$:

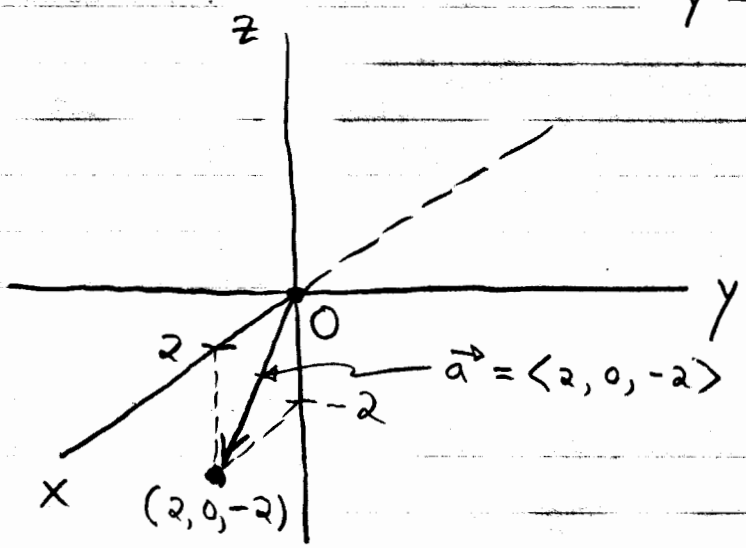

z
y
x
O
$\vec{u}$
$\overrightarrow{AB}$
$-\vec{u}$
$\vec{v}$
$\vec{a}$
$-\vec{u}$

Therefore, we have

$$\vec{a} = \langle 2, 3, -1 \rangle - \langle 0, 3, 1 \rangle$$

$$= \langle 2-0, 3-3, -1-1 \rangle$$

$$= \boxed{\langle 2, 0, -2 \rangle}$$ (in the x,z-plane OR in the plane $y = 0$)



11. Find the sum of the vectors

$\langle 3, -1 \rangle$, $\langle -2, 4 \rangle$

and illustrate geometrically.

Let $\vec{u} = \langle 3, -1 \rangle$ and $\vec{v} = \langle -2, 4 \rangle$.

Then $\vec{u} + \vec{v}$ can be found in 2 different ways.

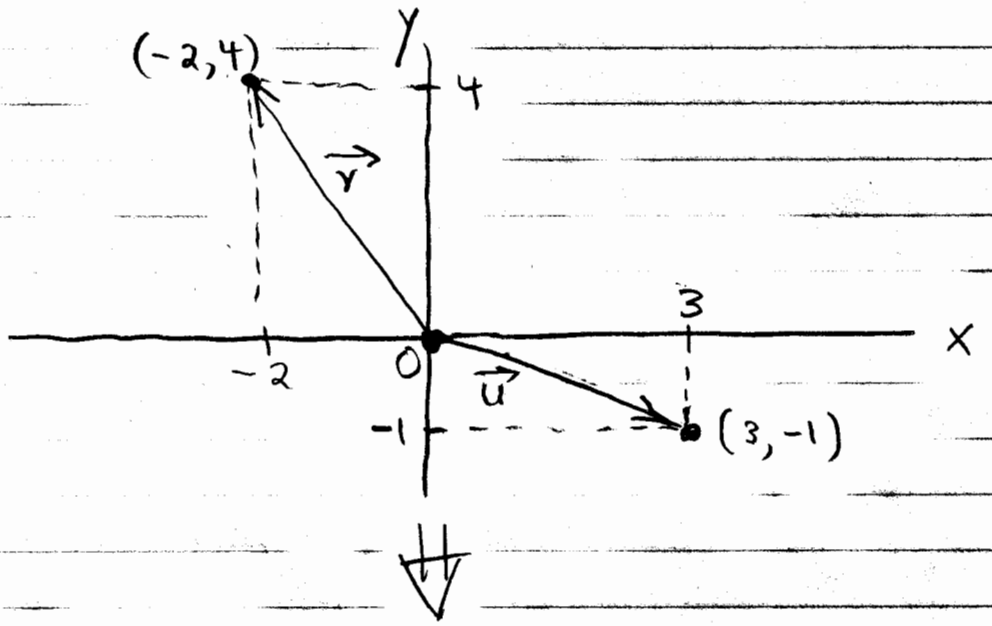
ALGEBRAICALLY:

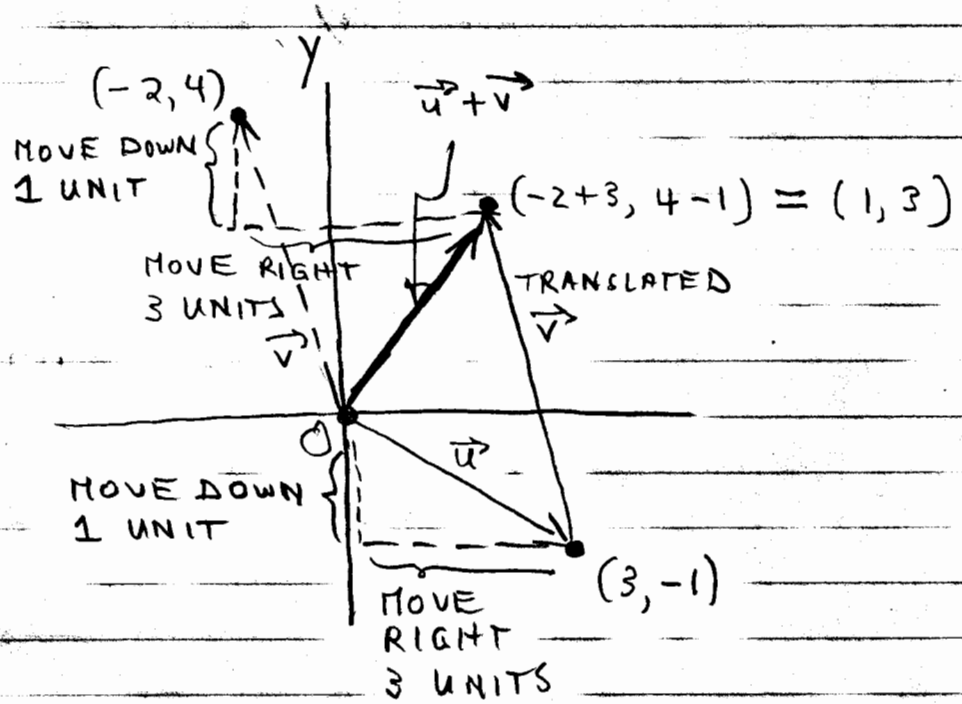
$$\vec{u} + \vec{v} = \langle 3, -1 \rangle + \langle -2, 4 \rangle$$

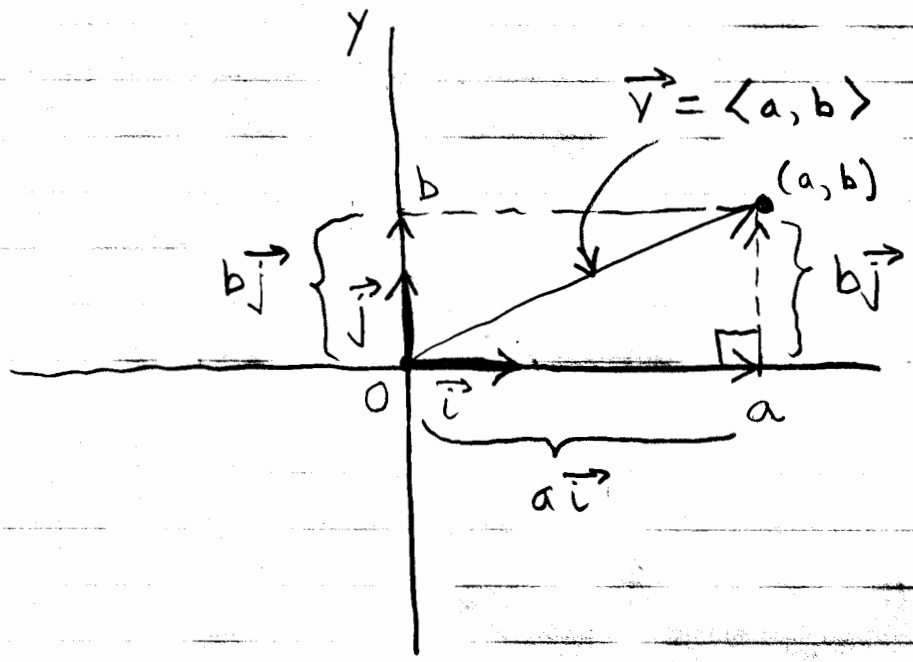
$$= \langle 3 + (-2), (-1) + 4 \rangle$$

$$= \boxed{\langle 1, 3 \rangle}$$

GEOMETRICALLY:


y
(-2,4)
4
$\vec{v}$
3
x
-2
0
$\vec{u}$
-1
(3,-1)

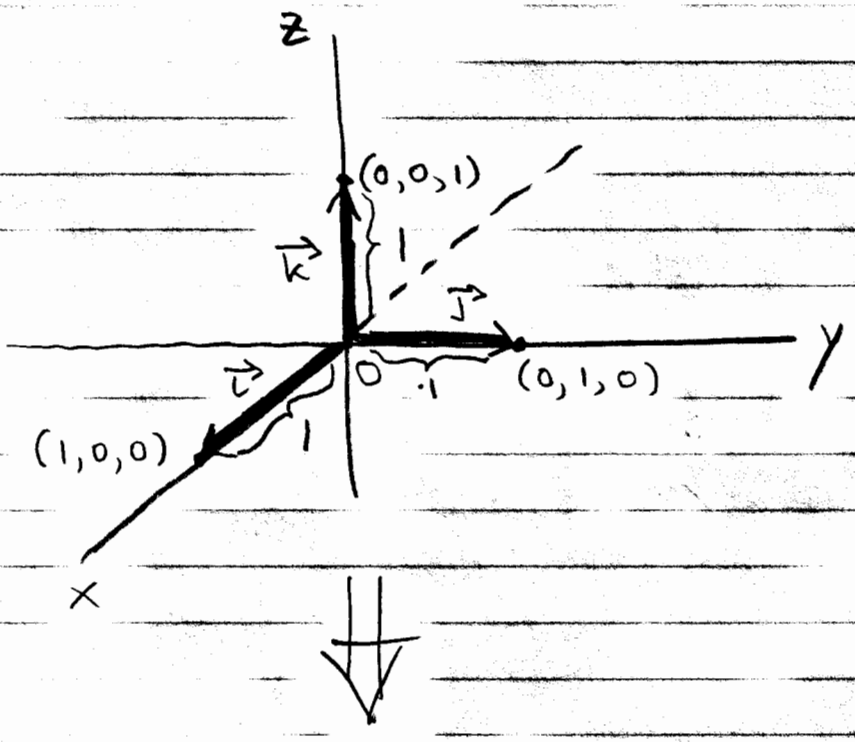

y
(-2,4)
$\vec{u}+\vec{v}$
MOVE DOWN
1 UNIT
(-2+3, 4-1) = (1, 3)
MOVE RIGHT
3 UNITS
$\vec{v}$
TRANSLATED
$\vec{v}$
0
$\vec{u}$
MOVE DOWN
1 UNIT
(3,-1)
MOVE
RIGHT
3 UNITS

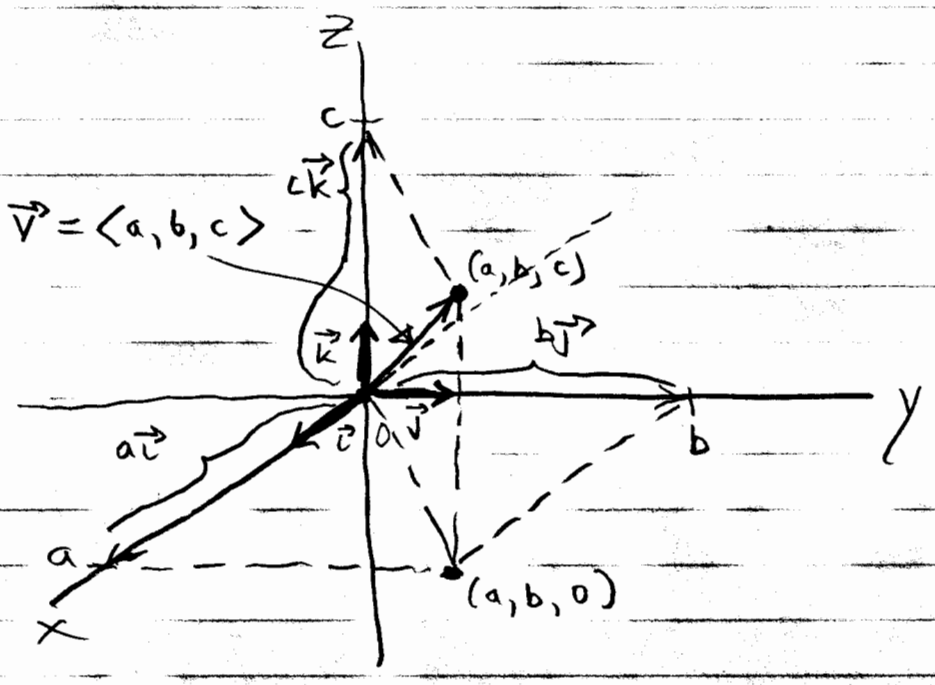


$$\vec{v} = \langle a, b \rangle = a\vec{i} + b\vec{j}$$

Similarly, every vector $\vec{v} = \langle a, b, c \rangle$ in three dimensions can be represented as a "LINEAR COMBINATION" of the three BASIS VECTORS

$$\vec{i} = \langle 1, 0, 0 \rangle, \quad \vec{j} = \langle 0, 1, 0 \rangle, \quad \vec{k} = \langle 0, 0, 1 \rangle$$





One can show that

$$\vec{v} = \langle a, b, c \rangle = a\vec{i} + b\vec{j} + c\vec{k}$$

So, we have the following:

① $\vec{a} = \vec{\imath} - 2\vec{\jmath} + \vec{k} \Longrightarrow$

$\vec{a} = (1)\vec{\imath} + (-2)\vec{\jmath} + (1)\vec{k} \Longrightarrow$

$\vec{a} = \langle 1, -2, 1 \rangle \Longrightarrow$

$|\vec{a}| = \|\vec{a}\| = \sqrt{(1)^2 + (-2)^2 + (1)^2}$

↑ Text's notation ↑ My notation

$= \sqrt{1+4+1}$

$= \boxed{\sqrt{6}}$

② $\vec{a} + \vec{b} = (\vec{\imath} - 2\vec{\jmath} + \vec{k}) + (\vec{\jmath} + 2\vec{k})$

↓ Remove parentheses and collect "like terms"

$= \vec{\imath} + (-2\vec{\jmath} + \vec{\jmath}) + (\vec{k} + 2\vec{k})$

↓ Factor out $\vec{\imath}, \vec{\jmath}, \vec{k}$

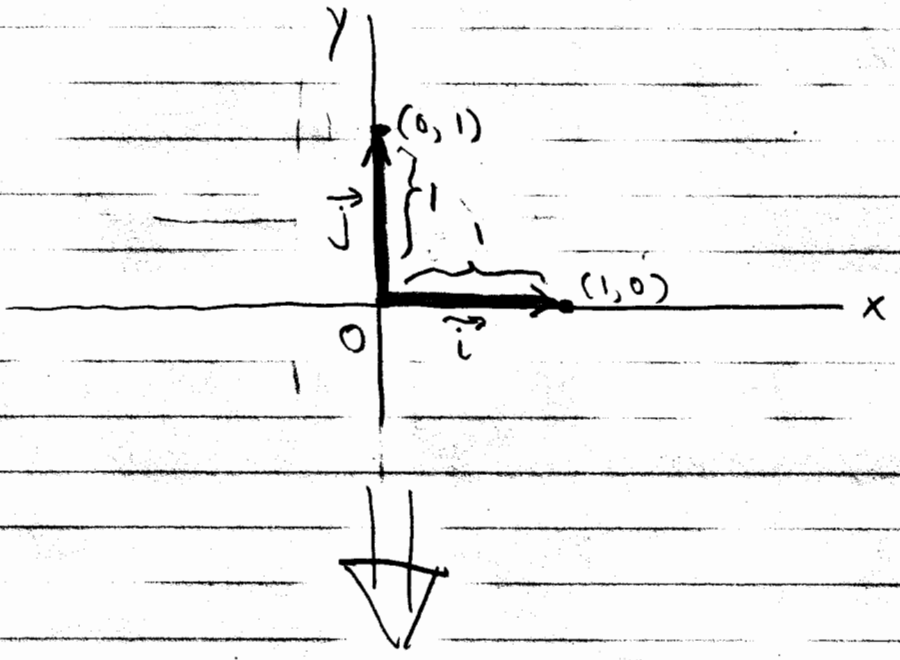
$= \vec{\imath} + (-2+1)\vec{\jmath} + (1+2)\vec{k}$

17. Find $|\vec{a}|$, $\vec{a}+\vec{b}$, $\vec{a}-\vec{b}$, $2\vec{a}$, and $3\vec{a}+4\vec{b}$.

$$\vec{a} = \vec{i} - 2\vec{j} + \vec{k}, \quad \vec{b} = \vec{j} + 2\vec{k} \quad (= 0\vec{i} + \vec{j} + 2\vec{k})$$

Every vector $\vec{v} = \langle a, b \rangle$ in two dimensions can be represented as a "LINEAR COMBINATION" of the two BASIS VECTORS

$$\vec{i} = \langle 1, 0 \rangle \text{ and } \vec{j} = \langle 0, 1 \rangle$$



$$= \vec{i} + (-1)\vec{j} + 3\vec{k}$$

$$= \boxed{\vec{i} - \vec{j} + 3\vec{k}}$$

③ $\vec{a} - \vec{b} = (\vec{i} - 2\vec{j} + \vec{k}) - (\vec{j} + 2\vec{k})$

$$= \vec{i} - 2\vec{j} + \vec{k} - \vec{j} - 2\vec{k}$$

$$= \vec{i} + (-2\vec{j} - \vec{j}) + (\vec{k} - 2\vec{k})$$

$$= \vec{i} + (-2 - 1)\vec{j} + (1 - 2)\vec{k}$$

$$= \vec{i} + (-3)\vec{j} + (-1)\vec{k}$$

$$= \boxed{\vec{i} - 3\vec{j} - \vec{k}}$$

④ $2\vec{a} = 2(\vec{i} - 2\vec{j} + \vec{k})$

↓ Distribute the 2

$$= 2\vec{i} - 2(2\vec{j}) + 2\vec{k}$$

$$= \boxed{2\vec{i} - 4\vec{j} + 2\vec{k}}$$

(5) $3\vec{a} + 4\vec{b} = 3(\vec{i} - 2\vec{j} + \vec{k}) + 4(\vec{j} + 2\vec{k})$

$= 3\vec{i} - 6\vec{j} + 3\vec{k} + 4\vec{j} + 8\vec{k}$

$= \boxed{3\vec{i} - 2\vec{j} + 11\vec{k}}$

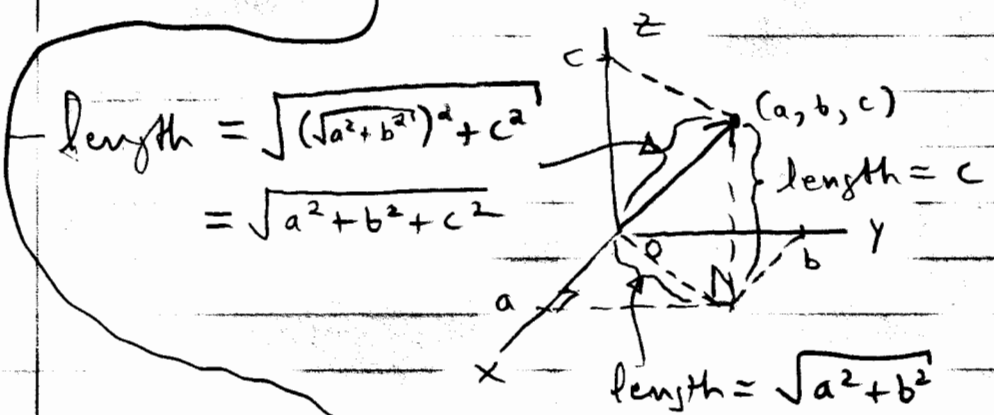
19. Find a unit (= one) vector with the same direction as

$$8\vec{i} - \vec{j} + 4\vec{k}$$

Let $\vec{v} = 8\vec{i} - \vec{j} + 4\vec{k} = \langle 8, -1, 4 \rangle$. Then

$\|\vec{v}\|$ = length of $\vec{v}$

$$= \sqrt{(8)^2 + (-1)^2 + (4)^2}$$



length = $\sqrt{(\sqrt{a^2+b^2})^2 + c^2}$

$= \sqrt{a^2+b^2+c^2}$

$$= \sqrt{64+1+16}$$

$$= \sqrt{81}$$

$$= 9$$

We want to "shrink" $\vec{v}$ so that it it has length 1. We do this by multiplying $\vec{v}$ by the appropriate scalar α:

$$\alpha\vec{v} = 1 \;,\; \alpha = ?$$

How about

$$\alpha = \frac{1}{\|\vec{v}\|} = \frac{1}{9} ?$$

This will work:

$$\frac{1}{\|\vec{v}\|}\cdot\vec{v} = \frac{1}{9}\left(8\vec{i} - \vec{j} + 4\vec{k}\right)$$

$$= \boxed{\frac{8}{9}\vec{i} - \frac{1}{9}\vec{j} + \frac{4}{9}\vec{k}}$$

UNIT VECTOR WITH SAME DIRECTION AS $\vec{v}$

Check:

$$\left\| \frac{8}{9}\vec{i} - \frac{1}{9}\vec{j} + \frac{4}{9}\vec{k} \right\| = \left\| \left\langle \frac{8}{9}, -\frac{1}{9}, \frac{4}{9} \right\rangle \right\|$$

$$= \sqrt{\left(\frac{8}{9}\right)^2 + \left(-\frac{1}{9}\right)^2 + \left(\frac{4}{9}\right)^2}$$

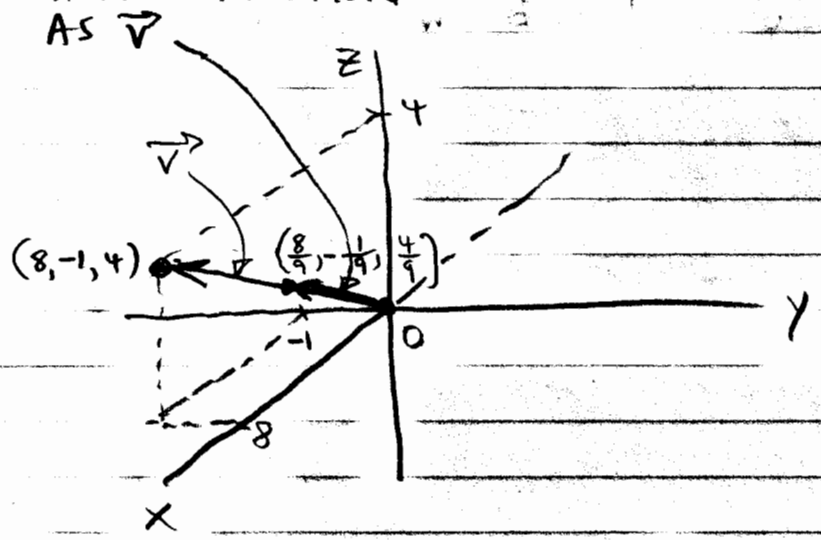
$$= \sqrt{\frac{64}{81} + \frac{1}{81} + \frac{16}{81}}$$

$$= \sqrt{\frac{81}{81}}$$

$$= \sqrt{1}$$

$$= 1 \checkmark$$

UNIT VECTOR WITH SAME DIRECTION AS $\vec{v}$



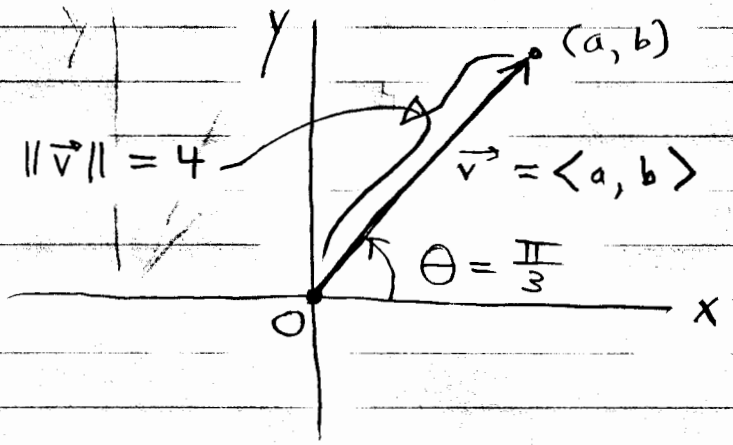
21. If $\vec{v}$ lies in the first quadrant and makes an angle $\pi/3$ with the positive x-axis and

$$|\vec{v}| = 4,$$

find $\vec{v}$ in component form.

Given

① $\|\vec{v}\|$ = length of $\vec{v}$ = 4
② θ = "direction" of $\vec{v}$ = $\frac{\pi}{3}$

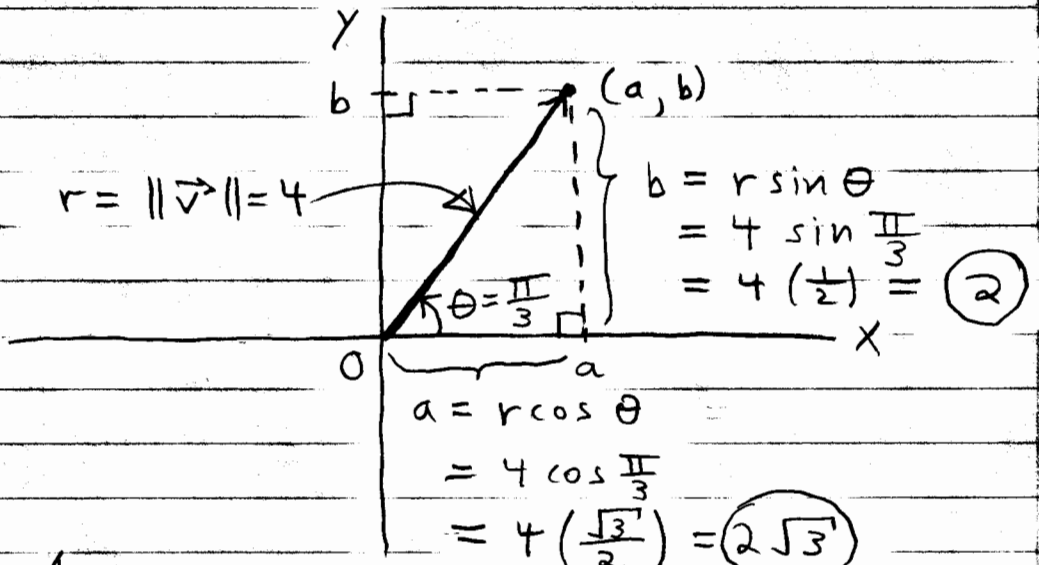


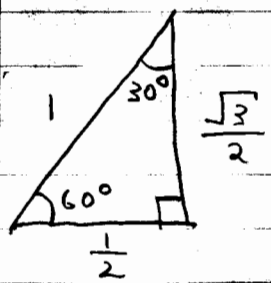
we can compute a and b. We can then write $\vec{v}$ in component form:

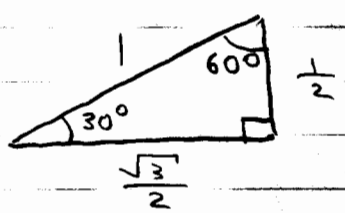
$$\vec{v} = a\vec{i} + b\vec{j}$$

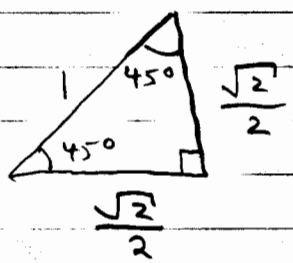
This is called "RESOLVING A VECTOR IN TWO DIMENSIONS INTO ITS COMPONENTS."

Recall: $\frac{\pi}{3} = \frac{180^\circ}{3} = 60^\circ$ and









$\therefore \quad \vec{v} = a\vec{i} + b\vec{j} \Longrightarrow$

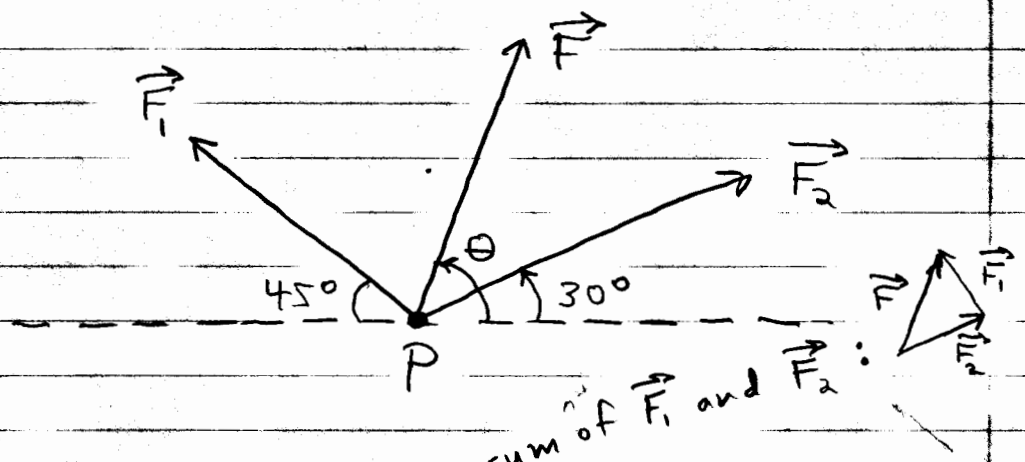
$$\boxed{\vec{v} = 2\sqrt{3}\,\vec{i} + 2\vec{j}}$$

or

$$\boxed{\vec{v} = \langle 2\sqrt{3}, 2 \rangle}$$

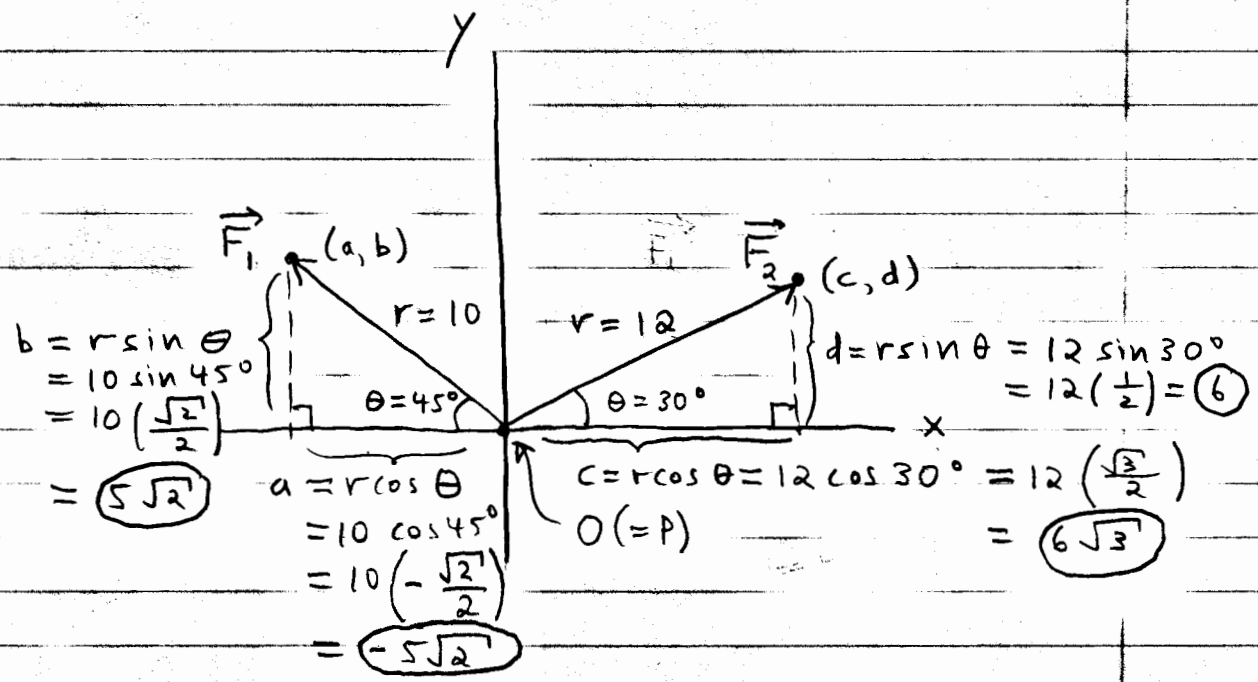
SOLUTION IN TEXT

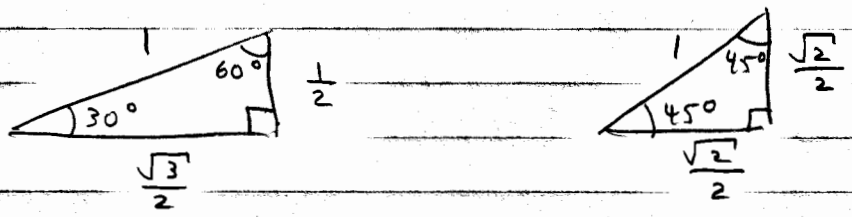
23. Two forces $\vec{F}_1$ and $\vec{F}_2$ with magnitudes 10 lb and 12 lb, act on an object at a point P as shown in the figure:



$= \text{sum of } \vec{F}_1 \text{ and } \vec{F}_2$:

Find the resultant force $\vec{F}$ acting at P as well as its magnitude and its direction, i.e., $\|\vec{F}\|$ and θ.





$$\therefore \left.\begin{aligned} \vec{F_1} &= a\vec{i} + b\vec{j} \\ \vec{F_2} &= c\vec{i} + d\vec{j} \end{aligned}\right\} \Longrightarrow$$

$$\left.\begin{aligned} \vec{F_1} &= 5\sqrt{2}\,\vec{i} + 5\sqrt{2}\,\vec{j} \\ \vec{F_2} &= 6\sqrt{3}\,\vec{i} + 6\vec{j} \end{aligned}\right\} \Longrightarrow$$

$$\vec{F} = \vec{F_1} + \vec{F_2}$$

$$= (-5\sqrt{2}\,\vec{i} + 5\sqrt{2}\,\vec{j}) + (6\sqrt{3}\,\vec{i} + 6\vec{j})$$

$$= \boxed{(-5\sqrt{2} + 6\sqrt{3})\,\vec{i} + (5\sqrt{2} + 6)\,\vec{j}}$$

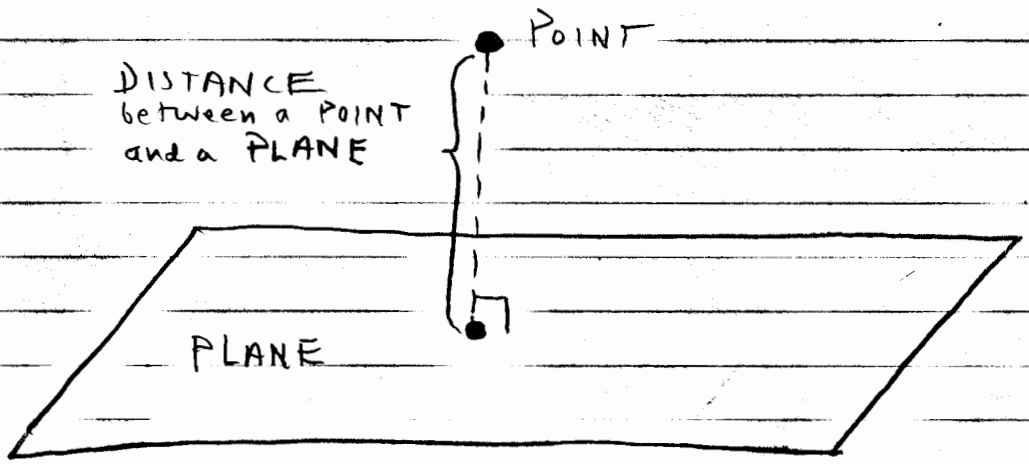
Exercises 9.1, pp. 651-652

3. Which of the points

$P(6, 2, 3)$, $Q(-5, -1, 4)$, and $R(0, 3, 8)$

is closest to the xz-plane? Which point lies in the yz-plane?

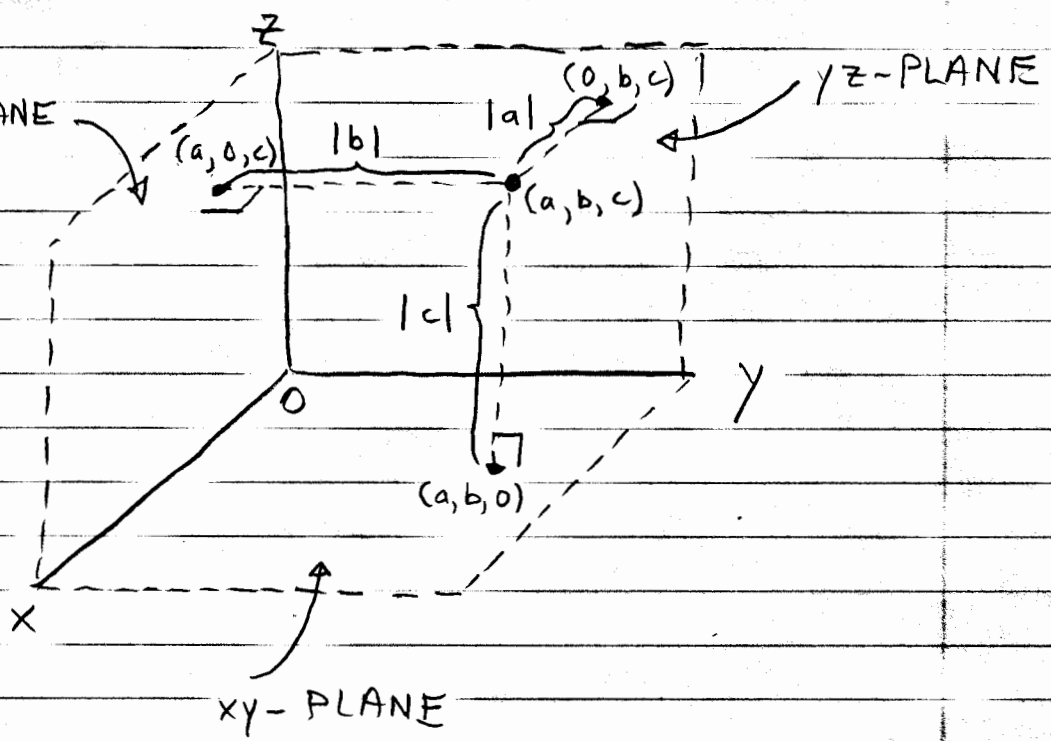
The DISTANCE between a POINT and a PLANE (where the point is not in the plane) is equal to the LENGTH OF THE PERPENDICULAR dropped from the point to the plane:



The DISTANCE between a POINT (a, b, c) and the xz-PLANE is $|b|$.

The DISTANCE between a POINT (a, b, c) and the yz-PLANE is $|a|$.

The DISTANCE between a POINT (a, b, c) and the xy-PLANE is $|c|$.



So, we have the following;

① DISTANCE between P and xz-plane $= |2| =$ ②

② DISTANCE between Q and xz-plane $= |-1| =$ ①

③ DISTANCE between R and xy-plane $= |3| =$ ③

④ DISTANCE between P and yz-plane $= |6| =$ ⑥

⑤ DISTANCE between Q and yz-plane $= |-5| =$ ⑤

⑥ DISTANCE between R and yz-plane $= |0| =$ ⓪

$\therefore$ **Q is closest to the xz-plane.**

R is closest to the yz-plane.
(In fact, R is a point <u>in</u> the yz-plane!)

5. Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation

$$x + y = 2.$$

Of course, in $\mathbb{R}^2$ (i.e., in two dimensions or in the xy-plane) the graph of the equation

$$x + y = 2$$

or

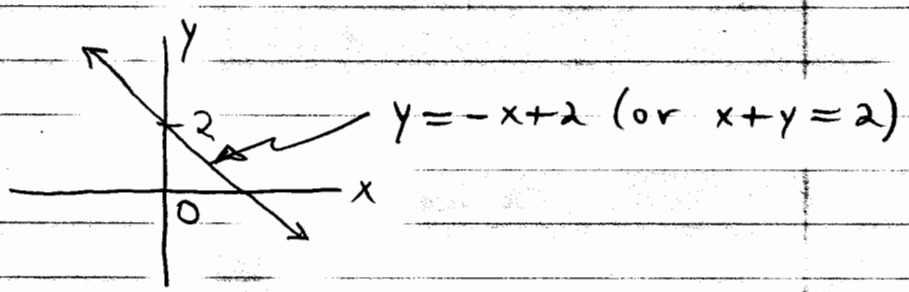
$$y = -x + 2$$

is a straight line with slope

$$m = -1$$

and y-intercept

$$b = 2:$$



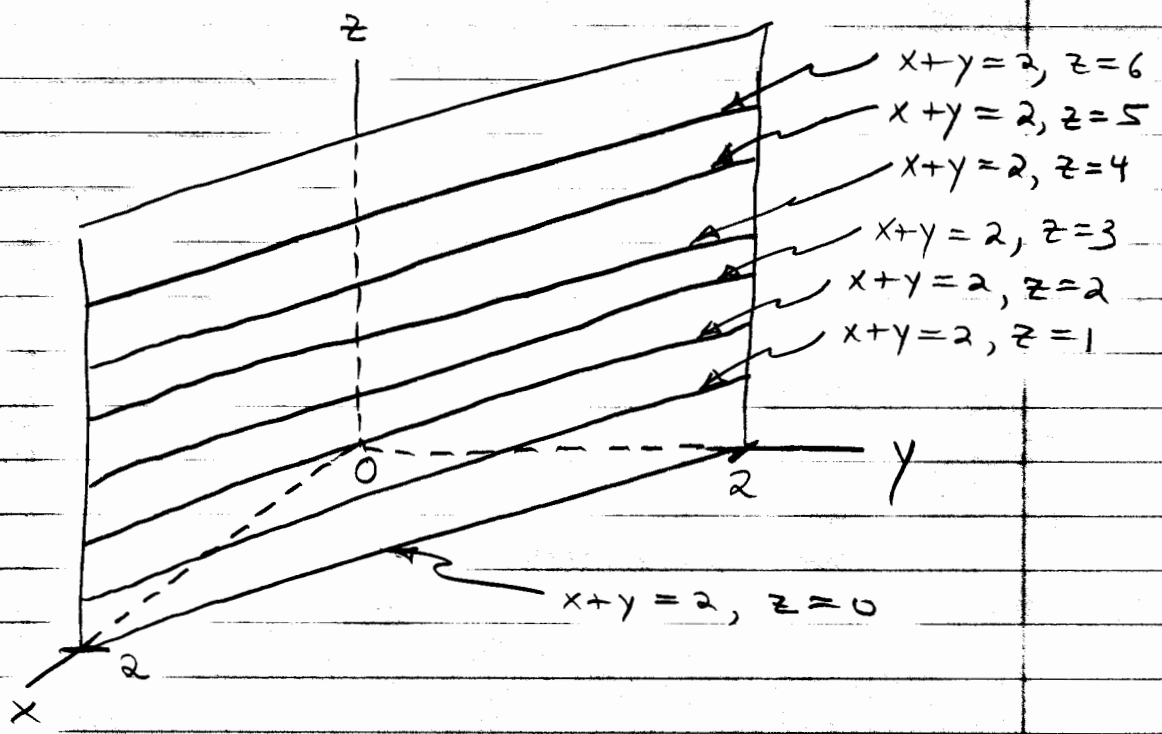
In $\mathbb{R}^3$ (i.e., in three dimensions or in the xyz-space), the graph of the equation

$$x + y = 2$$

is now a surface, where the graph now is made up the collection of lines

$$x + y = 2$$

for each value of z:



∴ The graph of

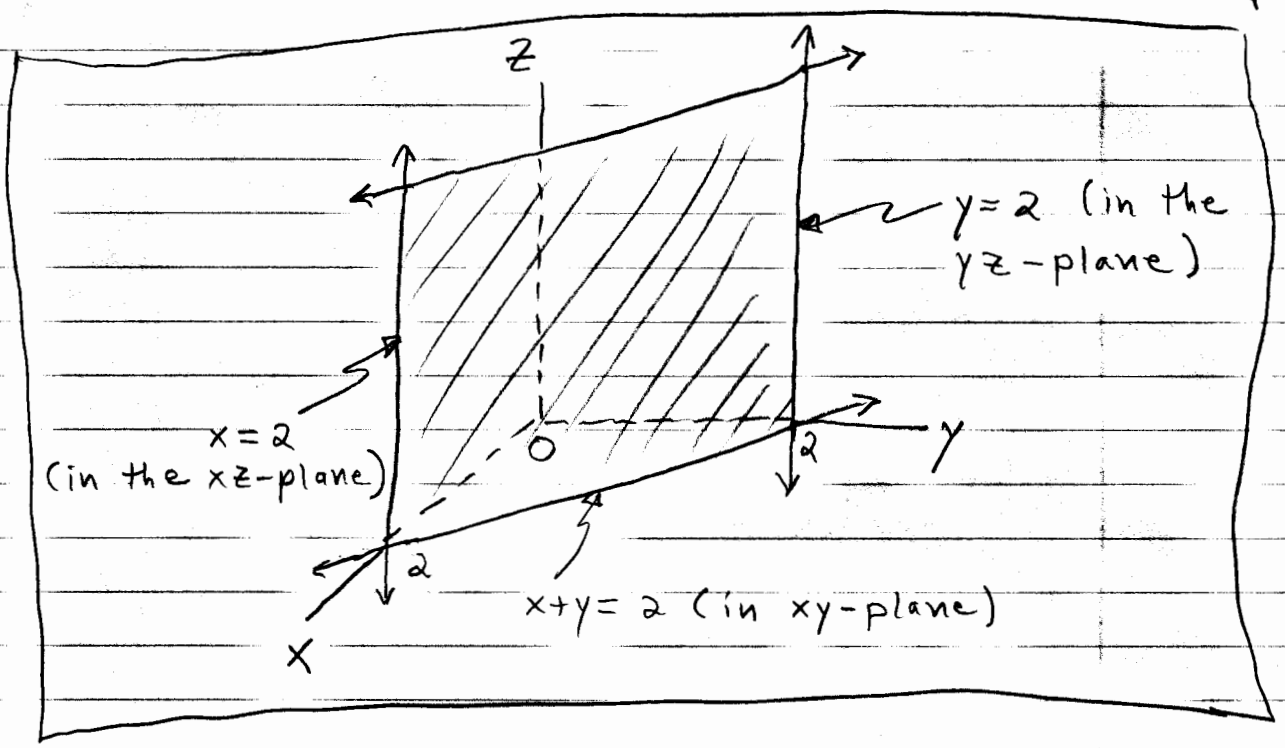
$$x + y = 2$$

is the VERTICAL PLANE which

① intersects the xy-plane in the line $x+y=2$ (where $x+y=2$ and $z=0$)

② intersects the xz-plane in the line $x=2$ (where $x+y=2$ and $y=0 \Rightarrow x+0=2 \Rightarrow x=2$)

③ intersects the yz-plane in the line $y=2$ (where $x+y=2$ and $x=0 \Rightarrow 0+y=2 \Rightarrow y=2$)

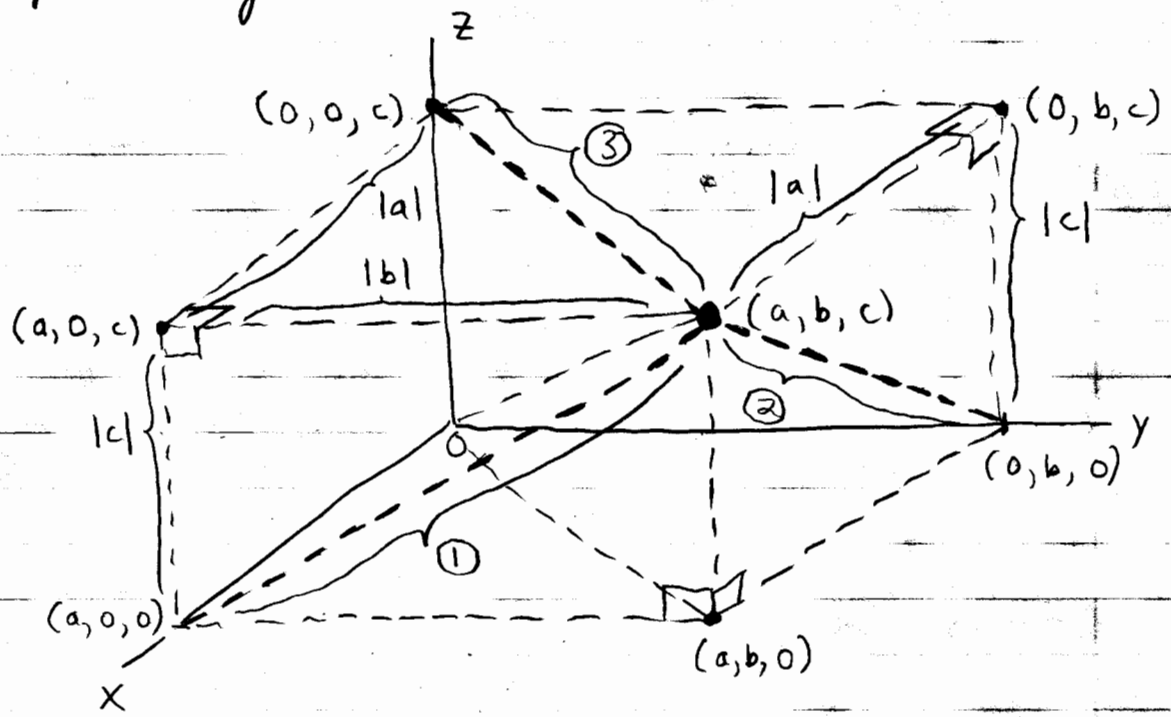


LIKE EXERCISE 8: Find the distance from $(-1, 3, 4)$ to each of the following.

(a) The xy-plane
(b) The yz-plane
(c) The xz-plane
(d) The x-axis
(e) The y-axis
(f) The z-axis

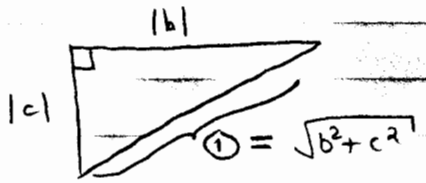
In general, we have the following for some arbitrary point (a, b, c):

(SEE SOLUTION to Exercise 3 for the DISTANCES between a point and the xy-PLANE, yz-PLANE, and xz-PLANE, respectively.)

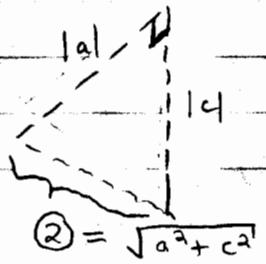


[HINT: This drawing was obtained by drawing a rectangular box "around" the point (a, b, c), as is done in FIGURE 5 on p. 648 of the text.]

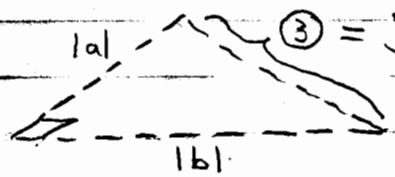
① = DISTANCE between (a, b, c) and x-axis
$= \sqrt{b^2 + c^2}$



② = DISTANCE between (a, b, c) and y-axis
$= \sqrt{a^2 + c^2}$



③ = DISTANCE between (a, b, c) and z-axis
$= \sqrt{a^2 + b^2}$



(a) DISTANCE between $(-1, 3, 4)$ and xy-plane
$= |4| = \boxed{4}$ (the $|z$-coordinate$|$)

(b) DISTANCE between $(-1, 3, 4)$ and yz-plane
$= |-1| = \boxed{1}$ (the $|x$-coordinate$|$)

(c) DISTANCE between $(-1, 3, 4)$ and xz-plane
$= |3| = \boxed{3}$ (the $|y$-coordinate$|$)

(d) DISTANCE between $(-1, 3, 4)$ and x-axis
$= \sqrt{(3)^2 + (4)^2} = \sqrt{9+16} = \sqrt{25} = \boxed{5}$

(e) DISTANCE between $(-1, 3, 4)$ and y-axis
$= \sqrt{(-1)^2 + (4)^2} = \sqrt{1+16} = \boxed{\sqrt{17}}$

(f) DISTANCE between $(-1, 3, 4)$ and z-axis
$= \sqrt{(-1)^2 + (3)^2} = \sqrt{1+9} = \boxed{\sqrt{10}}$

EXERCISE 11 BUT WITH ADDITIONAL PART TO IT: Find an equation of the sphere that passes through the point $(4, 3, -1)$ and has center $(3, 8, 1)$.

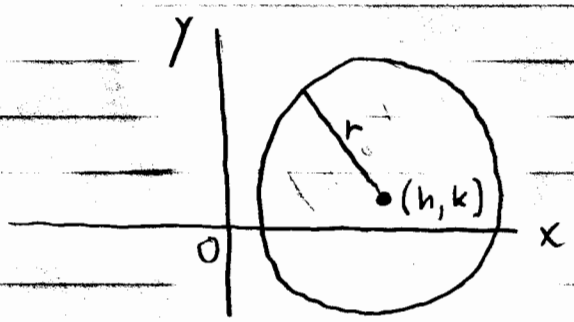
Then describe its intersection with each of the coordinate planes.

The equation of a circle (in two dimensions or in $\mathbb{R}^2$) is

$$(x-h)^2 + (y-k)^2 = r^2$$

Center $= (h, k)$, radius $= r$

and its graph is the curve

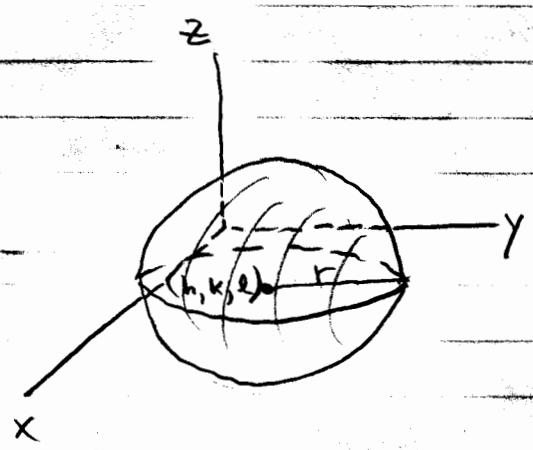


The equation of a sphere (in three dimensions or in $\mathbb{R}^3$) is

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

center $= (h, k, l)$, radius $= r$

and its graph is the surface



So, in this problem, we need to determine the following:

① Center $= (h, k, l)$

② Radius $= r$

We are <u>given</u> the center, and it is

Center $= (h, k, l) =$ $(3, 8, 1)$.

By definition, we have that

radius = DISTANCE between any POINT ON THE SPHERE and the CENTER OF THE SPHERE

= DISTANCE between $(4, 3, -1)$ and $(3, 8, 1)$

$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

$$= \sqrt{(4-3)^2 + (3-8)^2 + (-1-1)^2}$$

$$= \sqrt{1 + 25 + 4}$$

$$= \sqrt{30}$$

$\therefore$ The equation of our sphere is

$$\boxed{(x-3)^2 + (y-8)^2 + (z-1)^2 = 30}$$

$(\sqrt{30})^2$

Next we look at the intersection of the sphere,

$$(x-3)^2 + (y-8)^2 + (z-1)^2 = 30,$$

with each of the coordinate planes,

the xy-plane, the yz-plane, and the xz-plane.

Note the following:

① INTERSECTION = set of all points (x, y, z) which satisfy both the equation of the SPHERE and the equation of the COORDINATE PLANE

② Equation of xy-plane is

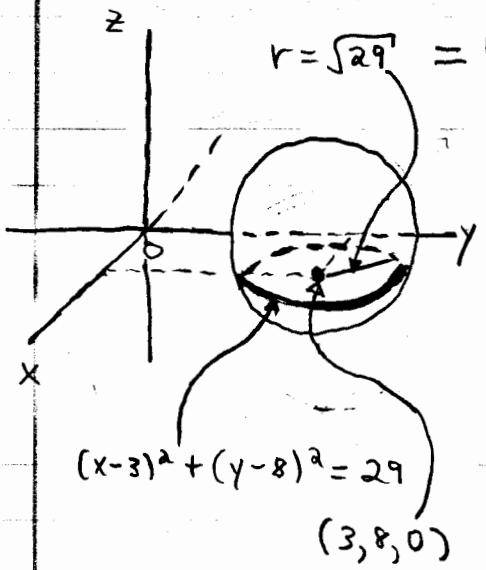
$$z = 0.$$

Thus,

INTERSECTION of SPHERE and xy-PLANE = all points (x, y, z) satisfying the equation $(x-3)^2 + (y-8)^2 + (z-1)^2 = 30$ and the equation $z = 0$

= all points $(x, y, 0)$ satisfying the equation $(x-3)^2 + (y-8)^2 + (0-1)^2 = 30$

= all points $(x, y, 0)$ satisfying the equation $(x-3)^2 + (y-8)^2 = 29$



= the CIRCLE in the xy-plane with equation

$$(x-3)^2 + (y-8)^2 = 29$$

with

CENTER = $(3, 8, 0)$
RADIUS = $\sqrt{29}$

③ Equation of yz-plane is

$$x = 0.$$

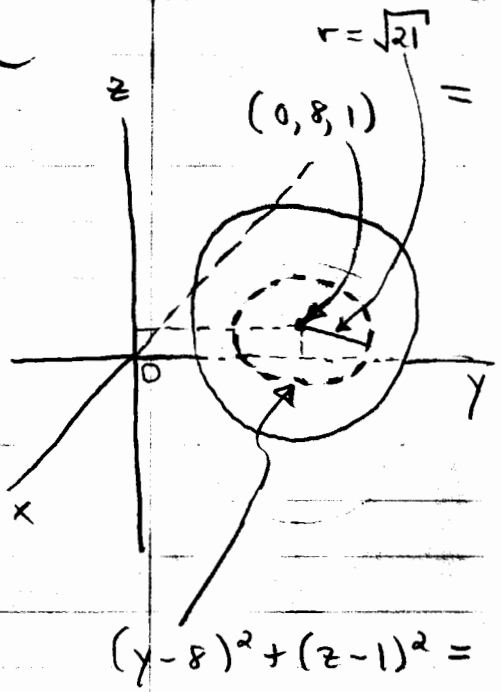
Thus,

INTERSECTION of SPHERE and yz-PLANE

= all points (x, y, z) satisfying the equation
$(x-3)^2 + (y-8)^2 + (z-1)^2 = 30$
and the equation $x = 0$

= all points $(0, y, z)$ satisfying the equation
$(0-3)^2 + (y-8)^2 + (z-1)^2 = 30$

= all points $(0, y, z)$ satisfying the equation
$(y-8)^2 + (z-1)^2 = 21$



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the CIRCLE in the yz-plane with equation

$$(y-8)^2 + (z-1)^2 = 21$$

with

CENTER = $(0, 8, 1)$
RADIUS = $\sqrt{21}$

④ Equation of the xz-plane is

$y = 0$.

Thus,

INTERSECTION of SPHERE and xz-PLANE

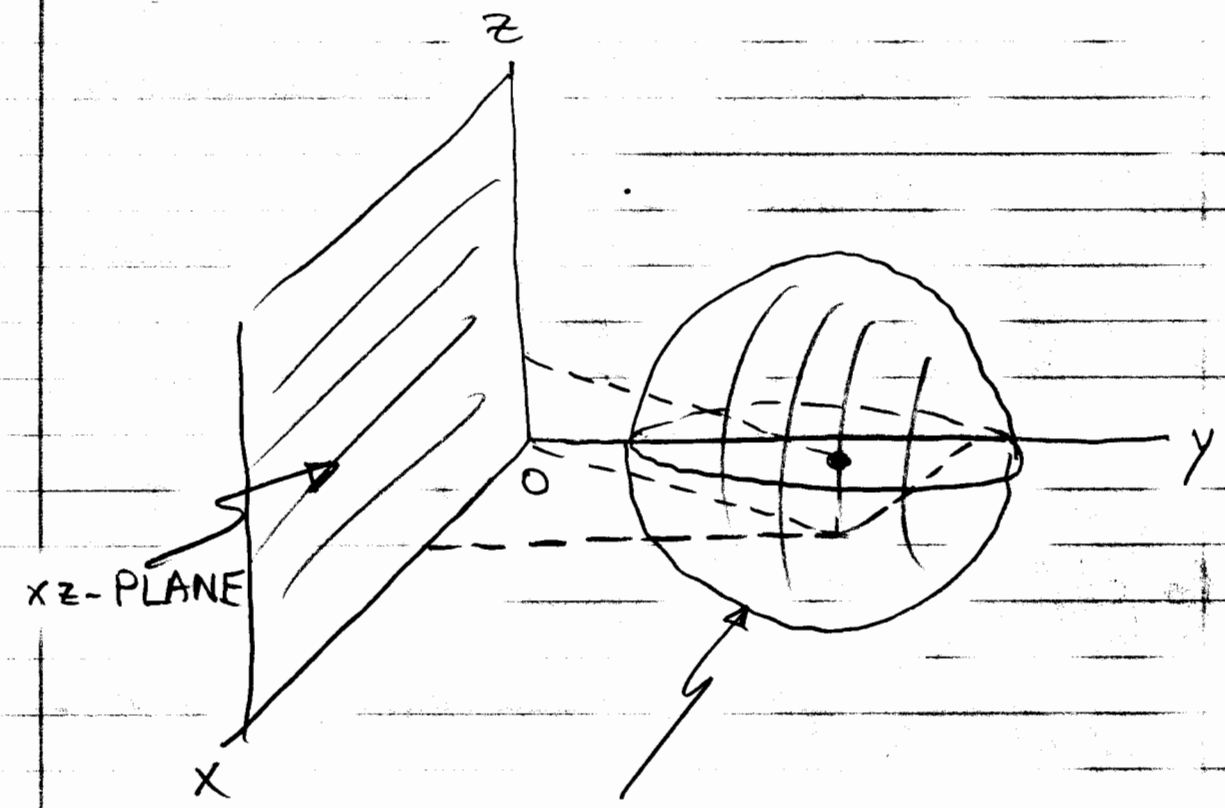
= all points (x, y, z) satisfying the equation $(x-3)^2 + (y-8)^2 + (z-1)^2 = 30$ and the equation $y = 0$

= all points $(x, 0, z)$ satisfying the equation $(x-3)^2 + (0-8)^2 + (z-1)^2 = 30$

= all points $(x, 0, z)$ satisfying the equation $(x-3)^2 + (z-1)^2 = -34$

= no points!!! since there are no values for x and z such that the sum of the two squares, $(x-3)^2$ and $(z-1)^2$, equals a negative number (the square of any <u>real</u> number is <u>nonnegative</u>)

Therefore the SPHERE does not intersect the xz-PLANE!



13. Show that the equation

$$x^2 + y^2 + z^2 = x + y + z$$

represents a sphere, and find its center and radius.

We must get our equation into the form

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

where h, k, and l can be positive or negative and where we really want

$$(x^2 - 2hx + h^2) + (y^2 - 2ky + k^2) + (z^2 - 2lz + l^2) = r^2$$

So, we need to do some rearranging and "COMPLETING OF SQUARES"!

$$x^2 + y^2 + z^2 = x + y + z \Longrightarrow$$

$$(x^2 - x) + (y^2 - y) + (z^2 - z) = 0$$

$$\left[x^2 - \overbrace{2\left(\tfrac{1}{2}\right)}^{=1}x + \left(\tfrac{1}{2}\right)^2\right] + \left[y^2 - \overbrace{2\left(\tfrac{1}{2}\right)}^{=1}y + \left(\tfrac{1}{2}\right)^2\right]$$

$$+ \left[z^2 - \overbrace{2\left(\tfrac{1}{2}\right)}^{=1}z + \left(\tfrac{1}{2}\right)^2\right] = 0 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2$$

$\Longrightarrow$

$$\left(x-\tfrac{1}{2}\right)^2+\left(y-\tfrac{1}{2}\right)^2+\left(z-\tfrac{1}{2}\right)^2=\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4}\Rightarrow$$

$$\boxed{\left(x-\tfrac{1}{2}\right)^2+\left(y-\tfrac{1}{2}\right)^2+\left(z-\tfrac{1}{2}\right)^2=\tfrac{3}{4}}$$

$$\text{Center} = (h,k,l) = \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right)$$

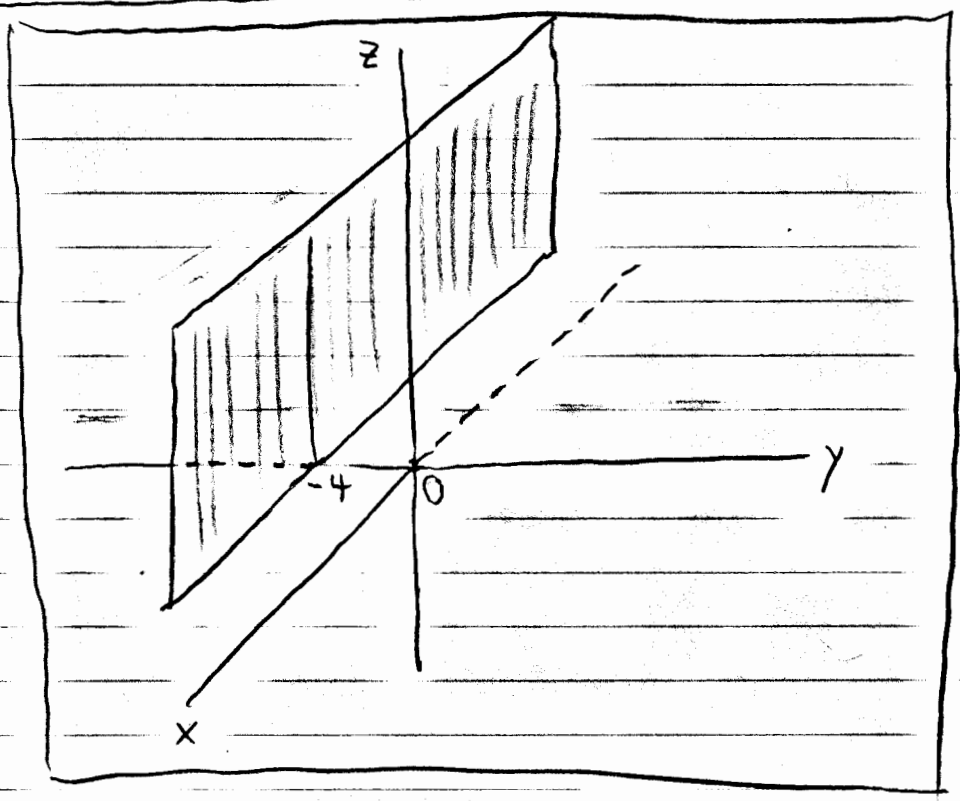
$$\text{Radius} = r = \sqrt{\tfrac{3}{4}} = \tfrac{\sqrt{3}}{2}$$

Describe in words the region of $\mathbb{R}^3$ represented by the equation or inequality in the following exercises:

19. $y = -4$

The region of $\mathbb{R}^3$ represented by $y = -4$ is the set of all points $(x, y, -4)$ $\Rightarrow$

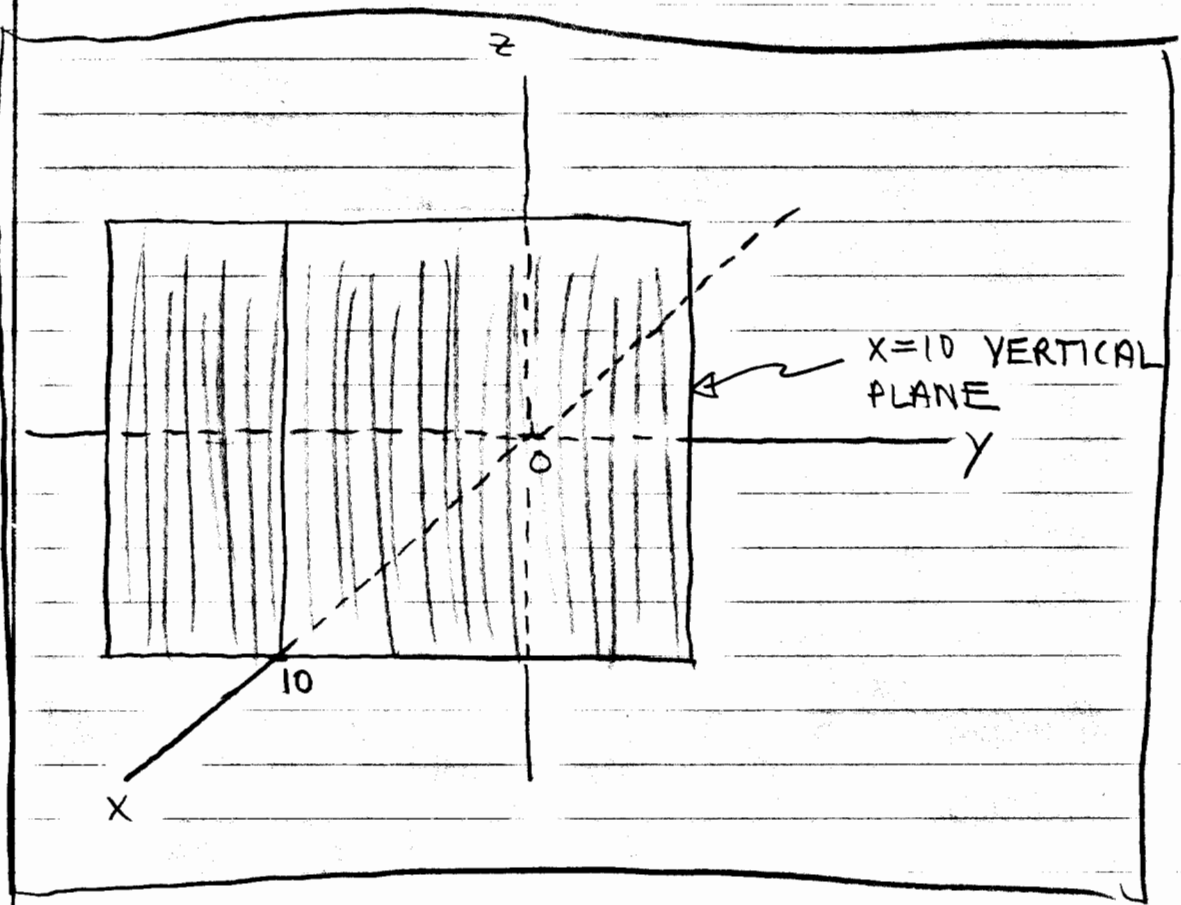
is the vertical plane parallel to the xz-plane (= set of all points $(x, y, 0)$) and 4 units to the left of it



20. $x = 10$

The region of $\mathbb{R}^3$ represented by $x=10$ is the set of all points $(10, y, z)$ $\Longrightarrow$

is the VERTICAL PLANE parallel to the yz-plane (= set of all points $(0, y, z)$) and 10 units in front of it



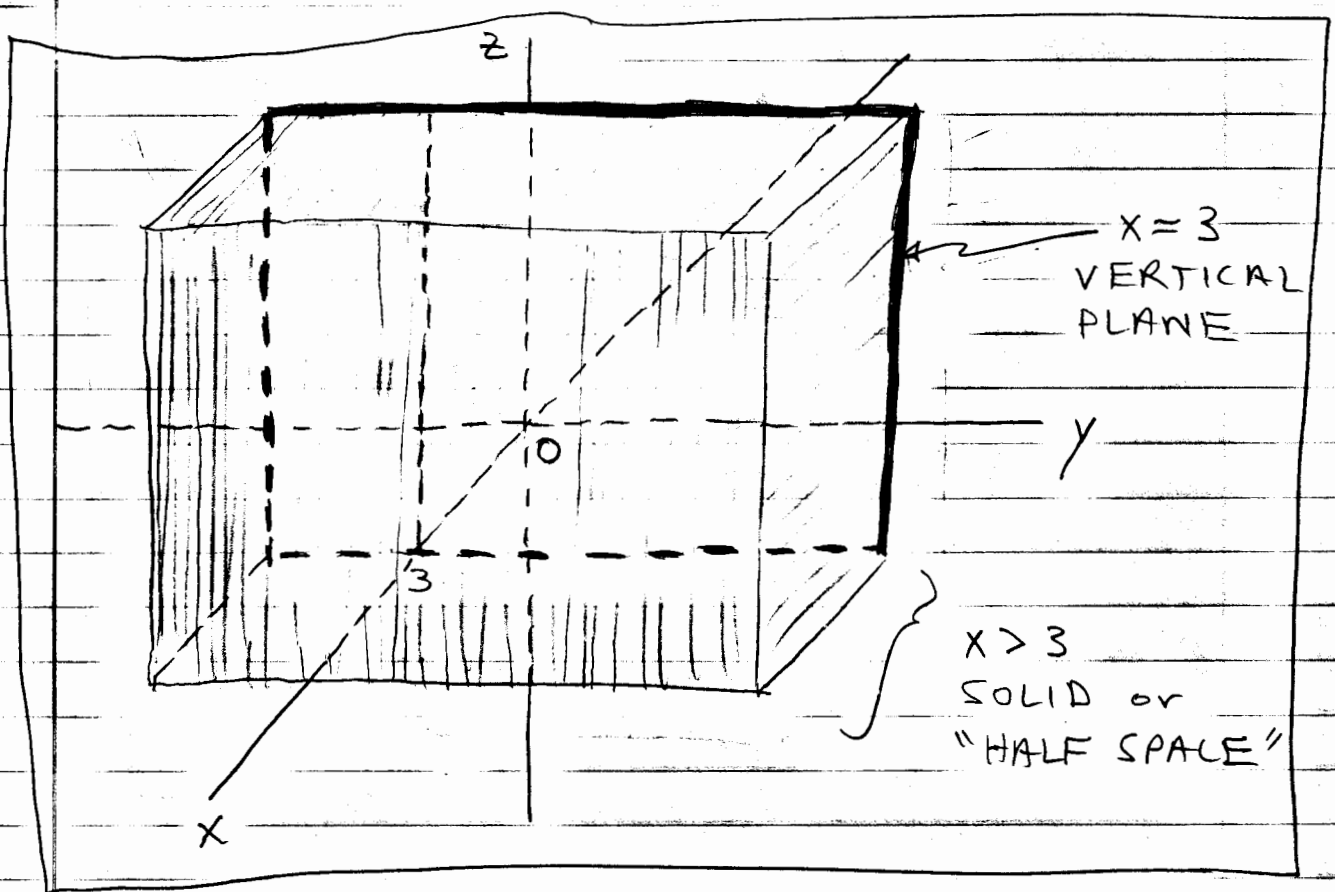
21. $x > 3$

The region of $\mathbb{R}^3$ represented by $x > 3$ is is the set of all points (a, y, z), $a > 3$, in front of the VERTICAL PLANE that is

(1) parallel to the yz-plane (= set of all points $(0, y, z)$), and

(2) 3 units in front of the yz-plane.

Note that the set of all points (x, y, z) satisfying $x > 3$ is a SOLID, not a SURFACE. This SOLID is bounded in back by the $x = 3$ plane and unbounded in front (and on the top, bottom, and sides).



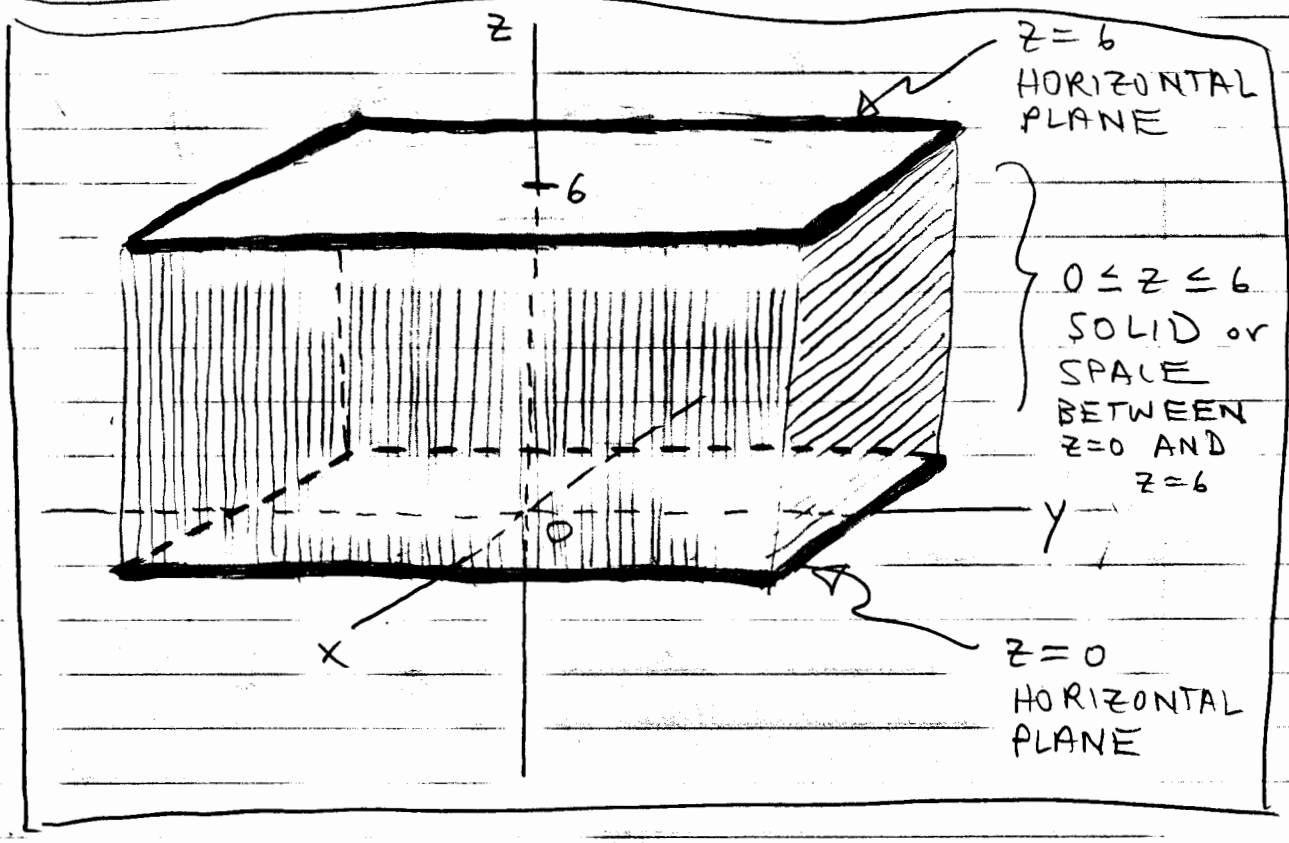
23. $0 \leq z \leq 6$

The region of $\mathbb{R}^3$ represented by $0 \leq z \leq 6$ is the set of all points (x, y, c), $0 \leq c \leq 6$, between the two HORIZONTAL PLANES

(1) $z = 0$: the xy-plane (= set of all points $(x, y, 0)$);

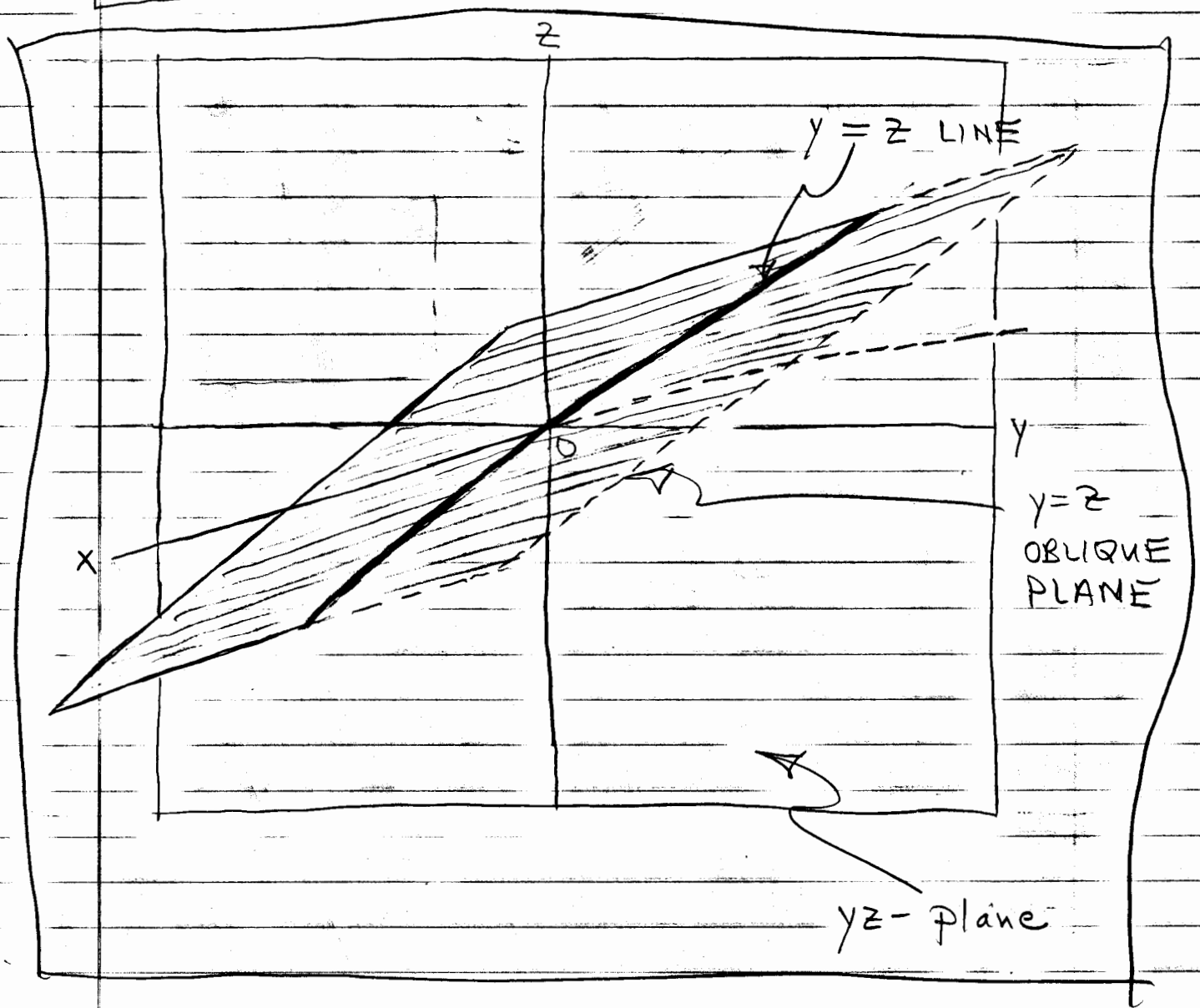
(2) $z = 6$: parallel to the xy-plane and 6 units above it.

Note that the set of all points (x, y, z) satisfying $0 \leq z \leq 6$ is a SOLID not a SURFACE. This SOLID is bounded below by the $z = 0$, bounded above by the $z = 6$ plane, and unbounded in front and on the sides.



24. $y = z$

The region of $\mathbb{R}^3$ represented by $y = z$ is the set of all points $(x, y, y) \Rightarrow$ is the OBLIQUE PLANE intersecting the yz-plane along the line $y = z$ in the yz-plane.



LIKE EXERCISE 26 : $9 \le x^2+y^2+z^2 \le 16$

The region of $\mathbb{R}^3$ represented by

$$9 \le x^2+y^2+z^2 \le 16$$

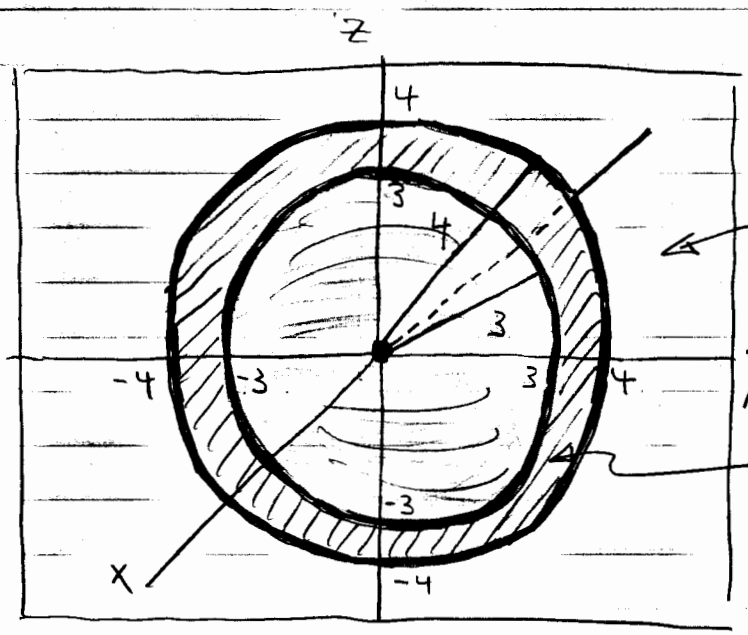
is the set of all points (x,y,z) between the two SPHERES

(1) $x^2+y^2+z^2 = 9$: center $= (0,0,0)$, radius $= 3$;

(2) $x^2+y^2+z^2 = 16$: center $= (0,0,0)$, radius $= 4$.

Note that this set of points (x,y,z) satisfying $9 \le x^2+y^2+z^2 \le 16$ forms the outer shell of the sphere $x^2+y^2+z^2 = 16$ with thickness 1 $(= \sqrt{16} - \sqrt{9} = 4-3)$ units.

CROSS-SECTION Along yz-Plane of the Region of $\mathbb{R}^3$ Represented by $9 \le x^2+y^2+z^2 \le 16$



LIKE EXERCISE 28 : $xy = 0$

The region of $\mathbb{R}^3$ represented by

$xy = 0$ ($\Rightarrow$ $x=0$ or $y=0$ or both)

is the set of all points (x, y, z) of the form

(1) $(0, b, c)$: b = anything (including 0)
c = anything (including 0), or

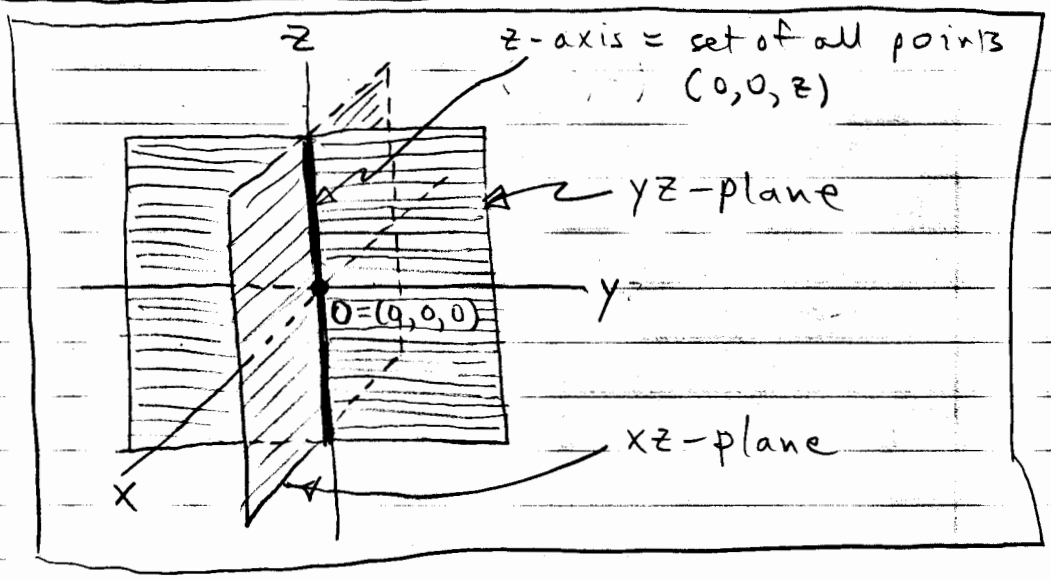
(2) $(a, 0, c)$: a = anything (including 0)
c = anything (including 0).

So, this set of points (x, y, z) satisfying $xy = 0$ forms the union of two planes that intersect each other:

(1) the yz-plane (= set of all points $(0, y, z)$),

(2) the xz-plane (= set of all points $(x, 0, z)$),

(3) intersection of the yz-plane and the xz-plane is the z-axis.



34. [NOTE: This is an assigned HW#1 problem, which I began on the board at the end of the Wednesday September 10th lecture. However, the solution given below, which you may use for your HW #1 writeup, is a revised version of what was done in lecture and is based on the suggestions of a few of your classmates.]

Consider the points P such that the distance from P to A(-1, 5, 3) is twice the distance from P to B(6, 2, -2).
Show that the set of all such points is a sphere, and find its center and radius.

Let

$$P = P(x, y, z).$$

Then we wish to find all points (x, y, z) that satisfy the following equation:

$$|\overrightarrow{PA}| = 2 \cdot |\overrightarrow{PB}|$$

or

$$\sqrt{(x+1)^2 + (y-5)^2 + (z-3)^2}$$

$$= 2 \cdot \sqrt{(x-6)^2 + (y-2)^2 + (z+2)^2}.$$

We next "do some algebra" on the above expression until we get it in the form of the equation of a sphere:

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

center $= (h, k, l)$. $r =$ radius; r^2 positive!, h, k, l positive or negative.

So,

$$\sqrt{(x+1)^2+(y-5)^2+(z-3)^2} = 2\cdot\sqrt{(x-6)^2+(y-2)^2+(z+2)^2} \Rightarrow$$

$$\left[\sqrt{(x+1)^2+(y-5)^2+(z-3)^2}\right]^2 = \left[2\cdot\sqrt{(x-6)^2+(y-2)^2+(z+2)^2}\right]^2 \Rightarrow$$

$$(x+1)^2+(y-5)^2+(z-3)^2 = 4\left[(x-6)^2+(y-2)^2+(z+2)^2\right] \Rightarrow$$

$$(x^2+2x+1)+(y^2-10y+25)+(z^2-6z+9)$$

$$= 4(x^2-12x+36)+4(y^2-4y+4)+4(z^2+4z+4) \Rightarrow$$

$$(x^2+2x+1)+(y^2-10y+25)+(z^2-6z+9)$$

$$= (4x^2-48x+144)+(4y^2-16y+16)+(4z^2+16z+16) \Rightarrow$$

COLLECT ALL LIKE TERMS (terms with x^2, x, y^2, y, z^2, and z and constant terms) on the left-hand side

$$(-3x^2+50x)+(-3y^2+6y)+(-3z^2-22z)$$

$$+\underbrace{(1+25+9-144-16-16)}_{} = 0 \Rightarrow$$

$$= -141$$

"MOVE" CONSTANT TERM to right-hand side"

$$\downarrow$$

$$(-3x^2+50x)+(-3y^2+6y)+(-3z^2-22z)=141 \Rightarrow$$

$$\downarrow$$

MULTIPLY BOTH SIDES by $-\frac{1}{3}$

$$\downarrow$$

$$\underbrace{\left(x^2-\frac{50}{3}x\right)}_{=x^2-2\left(\frac{25}{3}\right)x}+\underbrace{\left(y^2-2y\right)}_{=y^2-2(1)y}+\underbrace{\left(z^2+\frac{22}{3}z\right)}_{=z^2+2\left(\frac{11}{3}\right)z}=-47 \Rightarrow$$

$$\downarrow$$

"COMPLETE SQUARES":

$$x^2+2ax \longrightarrow x^2+2ax+a^2-a^2$$
$$\longrightarrow (x^2+2ax+a^2)-a^2 \longrightarrow$$
$$(x+a)^2-a^2$$

$$\downarrow$$

$$\left\{\left[x^2-2\left(\frac{25}{3}\right)x+\left(\frac{25}{3}\right)^2\right]-\left(\frac{25}{3}\right)^2\right\}$$
$$+\left\{\left[y^2-2(1)y+(1)^2\right]-(1)^2\right\}$$
$$+\left\{\left[z^2+2\left(\frac{11}{3}\right)z+\left(\frac{11}{3}\right)^2\right]-\left(\frac{11}{3}\right)^2\right\}=-47 \Rightarrow$$

$$\left(x-\frac{25}{3}\right)^2+\left(y-1\right)^2+\left(z+\frac{11}{3}\right)^2$$

$$-\left(\frac{625}{9}+1+\frac{121}{9}\right)=-47 \Longrightarrow$$

$$=\frac{625}{9}+\frac{9}{9}+\frac{121}{9}=\frac{755}{9}$$

$$\left(x-\frac{25}{3}\right)^2+\left(y-1\right)^2+\left(z+\frac{11}{3}\right)^2=\frac{755}{9}-47 \Longrightarrow$$

$$=\frac{755}{9}-\frac{423}{9}$$

$$=\frac{332}{9}$$

$$\boxed{\left(x-\frac{25}{3}\right)^2+\left(y-1\right)^2+\left(z+\frac{11}{3}\right)^2=\frac{332}{9}}$$

This equation is of a SPHERE with

CENTER $=\left(\frac{25}{3}, 1, -\frac{11}{3}\right)$

RADIUS $=\frac{\sqrt{332}}{3}$

35. Find an equation of the set of all points equidistant from the [SAME] points [AS IN EXERCISE 34]

$A(-1,5,3)$ and $B(6,2,-2)$.

Describe the set.

Let

$$P = P(x,y,z).$$

Then we wish to find all points (x,y,z) that satisfy the following:

$$\overrightarrow{PA} = \overrightarrow{PB} \Longrightarrow$$

$$\sqrt{(x+1)^2+(y-5)^2+(z-3)^2} = \sqrt{(x-6)^2+(y-2)^2+(z+2)^2} \Longrightarrow$$

$$\left(\sqrt{(x+1)^2+(y-5)^2+(z-3)^2}\right)^2 = \left(\sqrt{(x-6)^2+(y-2)^2+(z+2)^2}\right)^2 \Longrightarrow$$

$$(x+1)^2+(y-5)^2+(z-3)^2 = (x-6)^2+(y-2)^2+(z+2)^2 \Longrightarrow$$

$(x^2+2x+1)+(y^2-10y+25)+(z^2-6z+9)$

$=(x^2-12x+36)+(y^2-4y+4)+(z^2+4z+4) \Rightarrow$

$(2x+1)+(-10y+25)+(-6z+9)$

$=(-12x+36)+(-4y+4)+(4z+4) \Rightarrow$

$(14x-6y-10z)+\underbrace{(1+25+9-36-4-4)}_{=-9}=0 \Rightarrow$

$$\boxed{14x-6y-10z-9=0}$$

This is the equation of a PLANE (SEE Equation (7) in Section 9.5 of the text).