Lecture

Section 11.7: Maximum and Minimum Values

Local Extrema

Functions of One Variable

RECALL (from Calculus I):
Let $f(x)$ be a real-valued function of a single variable. Then we have the following:

1. The point $x = p$ is a CRITICAL POINT if $f''(p) = 0$ or $f''(p) = \text{undefined (} \pm \infty \text{ or } 0)$

2. If $x = p$ is a CRITICAL POINT, then $f$ has a LOCAL MAXIMUM at $x = p$ if

**FIRST DERIVATIVE TEST**

- $f'(x) > 0$ for $x < p$ and $x \text{ close to } p$
- $f'(x) < 0$ for $x > p$ and $x \text{ close to } p$

E.g.,

\[
\begin{array}{c}
\text{graph of } f(x) \\
\text{cusp at } x = 0 \\
f'(0) = 0 \\
f''(0) \text{ undefined} \\
f(x) = -x^2, \quad p = 0
\end{array}
\]
**SECOND DERIVATIVE TEST**

- \( f''(p) < 0 \) (i.e., \( f \) is concave down at \( x = p \))

**Example:**

\[ f(x) = -x^2 \]
\[ f''(0) < 0 \]
\[ p = 0 \]

3. **If** \( x = p \) **is a CRITICAL POINT,** then **\( f \) has a LOCAL MINIMUM at** \( x = p \) **if**

**FIRST DERIVATIVE TEST**

- \( f'(x) < 0 \) for \( x < p \) and \( x \) "close to" \( p \)
- \( f'(x) > 0 \) for \( x > p \) and \( x \) "close to" \( p \)

**Example:**

\[ f(x) = x^2 \]
\[ p = 0 \]
SECOND DERIVATIVE TEST

- $f''(x) > 0$ (i.e., $f$ is concave up at $x = p$)

  e.g., $f'(a) > 0$

  $f''(a) = 0$

  $f(x) = x^2, \quad p = 0$

4. If $x = p$ is a CRITICAL POINT, then $f$ has NEITHER A LOCAL MAXIMUM NOR A LOCAL MINIMUM at $x = p$.

FIRST DERIVATIVE TEST

Either a. $f'(x) > 0$ for $x < p$ and $x$ "close to" $p$

b. $f'(x) > 0$ for $x > p$ and $x$ "close to" $p$

  e.g., $f'(a) = 0$

  $f'(x) > 0$

  $f(x) = x^3, \quad p = 0$
or b. \( f'(x) < 0 \) for \( x < p \) and \( x \) "close to" \( p \)
  - \( f'(x) < 0 \) for \( x > p \) and \( x \) "close to" \( p \)

\[ f(x) = -x^3, \quad p = 0 \]

**NOTES:**

1. When \( x = p \) is a critical point but \( f \) has neither a local maximum nor a local minimum, one usually does not but one may call \( x = p \) a **SADDLE POINT**.

2. We say that "THE SECOND DERIVATIVE FAILS" if

   \[ f''(p) = 0, \]

   Then we have 3 possibilities:

   - (a) \( f \) has neither a local maximum nor a local minimum at \( x = p \)
\[ f(x) = x^2; \]
\[ f'(0) = 0 \]
\[ f''(0) = 0 \]

(b) \( f \) has a local maximum at \( x = p \)

\[ f(x) = x^4; \]
\[ f'(0) = 0 \]
\[ f''(0) = 0 \]

(c) \( f \) has a local minimum at \( x = p \)

\[ f(x) = -x^4; \]
\[ f'(0) = 0 \]
\[ f''(0) = 0 \]
Functions of Two Variables

We begin with some definitions and examples of critical points, local maxima, local minima, and saddle points in three dimensions.

Definition. \( f(x,y) \) has a local maximum at the point \((a,b)\) if

\[ f(x,y) < f(a,b) \]

for all \((x,y)\) "close to" \((a,b)\).

NOTE: "All \((x,y)\) close to \((a,b)\)" means that there exists some \(r > 0\) such that all points \((x,y)\) in the disk with center \((a,b)\) and radius \(r\) are such that \(f(x,y) < f(a,b)\).
Definition. \( f(x, y) \) has a **Local Minimum** at the point \((a, b)\) if:

\[
f(x, y) > f(a, b)
\]

for all \((x, y)\) "close to" \((a, b)\).

**Note:** "All \((x, y)\) close to \((a, b)\)" means that there exists some \(r > 0\) such that all points \((x, y)\) in the disk with center \((a, b)\) and radius \(r\) are such that \(f(x, y) > f(a, b)\).
Definition: The point \((a, b)\) is a critical point or stationary point of \(f(x, y)\) if

\[ f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0 \]

or

\[ f_x(a, b) = \text{undefined} \quad \text{or} \quad f_y(a, b) = \text{undefined} \]

\(\left(\pm \infty \text{ or } \frac{a}{b}\right)\) (\(\pm \infty \text{ or } \frac{a}{b}\))
Examples.

1. \( f(x, y) = x^2 + y^2 \) has a LOCAL MAXIMUM at the point \((0, 0)\):

\[
\begin{align*}
  z &= x^2 + y^2 \quad \text{(surface)} \\
  x & \quad y \\
  z & \quad \text{O}
\end{align*}
\]

2. \( f(x, y) = -x^2 - y^2 \) has a LOCAL MINIMUM at the point \((0, 0)\):

\[
\begin{align*}
  z &= -x^2 - y^2 \quad \text{(surface)} \\
  x & \quad y \\
  z & \quad \text{O}
\end{align*}
\]
3. \( f(x, y) = y^2 - x^2 \) has **NEITHER A LOCAL MAXIMUM NOR A LOCAL MINIMUM** at the point \((0, 0)\). This point is called a **SADDLE POINT** (for obvious reasons).

\[
Z = y^2 - x^2 \quad \text{(Surface that looks like a saddle)}
\]

**Note:** In the \(yz\)-plane, the point \((0, 0)\) is a **LOCAL MINIMUM**; but in the \(xz\)-plane, the point \((0, 0)\) is a **LOCAL MAXIMUM**. So in three dimensions (or in \(\mathbb{R}^3\) or in "space"), the point \((0, 0)\) is **NEITHER A LOCAL MINIMUM NOR A LOCAL MAXIMUM**, since there does not exist a disk (with center \((0, 0)\) and radius something) such that all points \((x, y)\) in the disk are such that \(f(x, y) > f(0, 0)\) or such that all points \((x, y)\) in the disk are such that \(f(x, y) < f(0, 0)\).
For functions of two variables, \( f(x, y) \), we have what is called the "Second Derivatives Test," which can recognize when \( f(x, y) \) has a LOCAL MAXIMUM or LOCAL MINIMUM, and, unlike the Second Derivative Test for functions of one variable, \( f(x) \), can recognize when \( f(x, y) \) has NEITHER A LOCAL MAXIMUM NOR A LOCAL MINIMUM.

**Definition:** Let \( f(x, y) \) be a function of two variables whose second partial derivatives \( f_{xx}, f_{yy}, f_{xy}, \) and \( f_{yx} \) are all continuous. Then

\[
f_{xy} = f_{yx} \quad \text{(by Clairaut\'s Theorem, p. 773, text)}.
\]

The Hessian of \( f \) at \((a, b)\) is denoted by

\[
\Delta = \Delta(a, b)
\]

and is given by

\[
\Delta = f_{xx}f_{yy} - (f_{xy})^2
\]
or

\[ D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2 \]

or

\[ D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \]

or

\[ D(a,b) = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} \]
Second Derivatives Test,

1. \( D(a, b) > 0 \) \( \Rightarrow \) \( f(a, b) \) is a **local minimum** [value]
   
   \[ f_{xx}(a, b) > 0 \]

2. \( D(a, b) < 0 \) \( \Rightarrow \) \( f(a, b) \) is a **local maximum** [value]
   
   \[ f_{xx}(a, b) < 0 \]

3. \( D(a, b) < 0 \) \( \Rightarrow \) \( f(a, b) \) is **neither a local minimum nor a local maximum** and \( (a, b) \) is called a **saddle point**

4. \( D(a, b) = 0 \) \( \Rightarrow \) **test fails** and \( f(a, b) \) can be a **local minimum**, **local maximum**, or **neither**
Absolute Extrema
Functions of One Variable

RECALL (from Calculus I):

Let $f(x)$ be a real-valued function of a single variable. Then we have the following:

If $f$ is continuous on a closed interval $[a, b]$, then $f$ attains an **absolute maximum value** and an **absolute minimum value** either at the endpoint $x = a$, $x = b$, or at any critical point $x_0$ inside the interval $[a, b]$.

E.g., $f(x) = (x - 2)^2 + 1$

\[ y \]

\[ \begin{array}{c|c|c}
\text{Absolute Minimum Value} & \text{Absolute Maximum Value} \\
\hline
f(3) = 1 & f(1) = 5 \\
\end{array} \]

\[ x \]

0 1 2 3 4

Diagram showing the graph of $f(x) = (x - 2)^2 + 1$ with critical points and absolute extrema values.
STEP 1. Find all critical points of \( f \):

\[
f'(x) = 2(x-2)
\]

\[
f'(x) = 0 \Rightarrow 2(x-2) = 0 \Rightarrow x = 2
\]

STEP 2. Evaluate \( f(x) \) at all critical points and at two endpoints:

\[
f(2) = (2-2)^2 + 1 = 1
\]
\[
f(1) = (1-2)^2 + 1 = 2
\]
\[
f(4) = (4-2)^2 + 1 = 5
\]

STEP 3. Choose largest value in STEP 2 as \text{Absolute Maximum Value} of \( f \) and choose smallest value in STEP 2 as \text{Absolute Minimum Value} of \( f \):

\text{Absolute Maximum Value} = f(4) = 5

\text{Absolute Minimum Value} = f(2) = 1
Functions of Two Variables

The situation is similar for functions of two variables $f(x, y)$. However, we first need to define the following:

Definition. A closed and bounded set $D$ in $\mathbb{R}^2$ (or in the $xy$-plane) will be for us any region in $\mathbb{R}^2$ which has a boundary and does not extend indefinitely in any direction.

E.g., $D = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \}$

$D$ is closed and bounded.
\[ D = \text{region} \]
\[ = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0\} \]

D has a boundary BUT D extends indefinitely to the right.

\[ \therefore D \text{ is NOT closed and bounded.} \]
E.g.,

\[ D = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \} \]

\( D \) has an interior that does not extend indefinitely but is missing its boundary.

\therefore \ D \ is \ NOT \ closed \ and \ bounded.
So, we are now ready to make the following statement for functions of two variables, \( f(x, y) \):

If \( f \) is continuous on a closed and bounded set \( D \) in \( \mathbb{R}^2 \) then \( f \) attains an absolute maximum value and an absolute minimum value either along the boundary of \( D \) or at any critical points \((a, b)\), inside the set \( D \).

**Strategy to Find the Absolute Maximum and Minimum Values of \( f(x, y) \).**

**STEP 1.** Find all critical points of \( f \).

Set \( f_x = 0 \) and \( f_y = 0 \) and solve two equations in \( x \) and \( y \).

**STEP 2.** Evaluate \( f(x, y) \) at all critical points.

**STEP 3.** Evaluate \( f(x, y) \) along the boundary of \( D \) and find largest and smallest values of \( f(x, y) \) along \( D \).
STEP 4. Let

**ABSOLUTE MAXIMUM VALUE** = largest value of $f(x,y)$ from STEPS 2 and 3

**ABSOLUTE MINIMUM VALUE** = smallest value of $f(x,y)$ from STEPS 2 and 3.

5–14 Find the local maximum and minimum values and saddle point(s) of the function.

5. \( f(x, y) = 9 - 2x + 4y - x^2 - 4y^2 \)

First compute the partials of \( f \):

\[
\begin{align*}
    f_x &= \frac{\partial}{\partial x} (9 - 2x + 4y - x^2 - 4y^2) \\
    &= -2 - 2x \\

    f_y &= \frac{\partial}{\partial y} (9 - 2x + 4y - x^2 - 4y^2) \\
    &= 4 - 8y \\

    f_{xx} &= \frac{\partial^2}{\partial x^2} (9 - 2x + 4y - x^2 - 4y^2) \\
    &= \frac{\partial}{\partial x} (-2 - 2x) \\
    &= -2
\end{align*}
\]
\[
\begin{align*}
f_{yy} &= \frac{\partial^2}{\partial y^2} \left( 9 - 2x + 4y - x^2 - 4y^2 \right) \\
&= -\frac{\partial}{\partial y} (4 - 8y) \\
&= -8
\end{align*}
\]

\[
\begin{align*}
f_{xy} &= f_{yx} = \frac{\partial^2}{\partial x \partial y} \left( 9 - 2x + 4y - x^2 - 4y^2 \right) \\
&= \frac{\partial}{\partial x} (4 - 8y) \\
&= 0
\end{align*}
\]

Since \( f \) is a polynomial function and so continuous.

Second find all critical points:

\[
\begin{align*}
f_x &= 0 \\
\Rightarrow & \quad -2 + 2x = 0 \\
\Rightarrow & \quad x = 1 \\
\end{align*}
\]

\[
\begin{align*}
f_y &= 0 \\
\Rightarrow & \quad 4 - 8y = 0 \\
\Rightarrow & \quad y = \frac{1}{2} \\
\end{align*}
\]

\((-1, \frac{1}{2}) \) is one critical point.

Third compute the Hessian of \( f \) at \((-1, \frac{1}{2})\):

\[
\begin{align*}
D(-1, \frac{1}{2}) &= f_{xx}(-1, \frac{1}{2}) f_{yy}(-1, \frac{1}{2}) - \left[ f_{xy}(-1, \frac{1}{2}) \right]^2 \\
&= (-2)(-8) - (0)^2 \\
&= 16 > 0
\end{align*}
\]
Finally apply SECOND DERIVATIVES TEST:

\[ D = 16 > 0 \]
\[ f_{xx} = -2 < 0 \]

\[ \Rightarrow f(-1, \frac{1}{2}) \text{ is a LOCAL MAXIMUM VALUE} \]

Note that

\[ f(-1, \frac{1}{2}) = 9 - 2(-1) + 4\left(\frac{1}{2}\right) - (-1)^2 - 4\left(\frac{1}{2}\right)^2 \]

\[ = 9 + 2 + 2 - 1 \]

\[ = 11 \]
\[ f(x, y) = x^2 + y^2 + x^2y + 4 \]

First compute the partials of \( f \):

\[ f_x = \frac{\partial}{\partial x} (x^2 + y^2 + x^2y + 4) \]
\[ = 2x + 2xy \]

\[ f_y = \frac{\partial}{\partial y} (x^2 + y^2 + x^2y + 4) \]
\[ = 2y + x^2 \]

\[ f_{xx} = \frac{\partial^2}{\partial x^2} (x^2 + y^2 + x^2y + 4) \]
\[ = \frac{\partial}{\partial x} (2x + 2xy) \]
\[ = 2 + 2y \]

\[ f_{yy} = \frac{\partial^2}{\partial y^2} (x^2 + y^2 + x^2y + 4) \]
\[ = \frac{\partial}{\partial y} (2 + 2y) \]
\[ = 2 \]
\[ f_{xy} = f_{yx} = \frac{2}{2\times 2y} \left( x^2 + y^2 + x^2 y + 4 \right) \]

Since \( f \) is a polynomial function and so continuous,
\[ \frac{2}{2x} \left( 2y + x^2 \right) = \frac{2}{2x} \]

Second and all critical points:

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 2x + 2xy = 0 \\
\frac{\partial f}{\partial y} &= 2y + x^2 = 0
\end{align*}
\]

\[ 2x(1+y) = 0 \]

\[ x = 0 \text{ or } y = -1 \]

\[ 2y + x^2 = 0 \]

\[ y = -1 : 2y + x^2 = 2(-1) + x^2 = 0 \Rightarrow \]

\[ x^2 = 2 \Rightarrow x = \pm \sqrt{2} \]

\[ x = 0 \text{ or } y = -1 \]

\[ y = 0 : 2y + x^2 = 2y + 0 = 0 \Rightarrow \]

\[ x = 0 \text{ or } y = -1 \]

\[ y = 0 : 2y + x^2 = 2(0) + x^2 = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0 \]

\[ x = 0 \text{ or } y = -1 \]

\[ y = 0 : 2y + x^2 = 2(0) + x^2 = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0 \]

\[ x = 0 \text{ or } y = -1 \]

\[ y = 0 : 2y + x^2 = 2y + 0 = 0 \Rightarrow \]

\[ x = 0 \text{ or } y = -1 \]

\[ y = 0 : 2y + x^2 = 2y + x^2 = 0 \Rightarrow \]

\[ x^2 = 2 \Rightarrow x = \pm \sqrt{2} \]

\[ x = 0 \text{ or } y = -1 \]

\[ y = 0 : 2y + x^2 = 2y + 0 = 0 \Rightarrow \]

\[ x = 0 \text{ or } y = -1 \]

\[ y = 0 : 2y + x^2 = 2y + x^2 = 0 \Rightarrow \]

\[ x^2 = 2 \Rightarrow x = \pm \sqrt{2} \]

\[ x = 0 \text{ or } y = -1 \]

\[ y = 0 : 2y + x^2 = 2y + x^2 = 0 \Rightarrow \]

\[ x^2 = 2 \Rightarrow x = \pm \sqrt{2} \]

\[ x = 0 \text{ or } y = -1 \]

\[ y = 0 : 2y + x^2 = 2y + x^2 = 0 \Rightarrow \]

\[ x^2 = 2 \Rightarrow x = \pm \sqrt{2} \]

\[ x = 0 \text{ or } y = -1 \]

\[ y = 0 : 2y + x^2 = 2y + x^2 = 0 \Rightarrow \]

\[ x^2 = 2 \Rightarrow x = \pm \sqrt{2} \]

\[ x = 0 \text{ or } y = -1 \]

\[ y = 0 : 2y + x^2 = 2y + x^2 = 0 \Rightarrow \]

\[ x^2 = 2 \Rightarrow x = \pm \sqrt{2} \]

\[ x = 0 \text{ or } y = -1 \]

\[ y = 0 : 2y + x^2 = 2y + x^2 = 0 \Rightarrow \]

\[ x^2 = 2 \Rightarrow x = \pm \sqrt{2} \]

\[ x = 0 \text{ or } y = -1 \]

\[ y = 0 : 2y + x^2 = 2y + x^2 = 0 \Rightarrow \]

\[ x^2 = 2 \Rightarrow x = \pm \sqrt{2} \]

\[ x = 0 \text{ or } y = -1 \]

\[ y = 0 : 2y + x^2 = 2y + x^2 = 0 \Rightarrow \]

\[ x^2 = 2 \Rightarrow x = \pm \sqrt{2} \]

\[ x = 0 \text{ or } y = -1 \]

\[ y = 0 : 2y + x^2 = 2y + x^2 = 0 \Rightarrow \]

\[ x^2 = 2 \Rightarrow x = \pm \sqrt{2} \]

\[ x = 0 \text{ or } y = -1 \]

\[ y = 0 : 2y + x^2 = 2y + x^2 = 0 \Rightarrow \]

\[ x^2 = 2 \Rightarrow x = \pm \sqrt{2} \]

\[ x = 0 \text{ or } y = -1 \]

\[ y = 0 : 2y + x^2 = 2y + x^2 = 0 \Rightarrow \]

\[ x^2 = 2 \Rightarrow x = \pm \sqrt{2} \]

\[ x = 0 \text{ or } y = -1 \]

\[ y = 0 : 2y + x^2 = 2y + x^2 = 0 \Rightarrow \]

\[ x^2 = 2 \Rightarrow x = \pm \sqrt{2} \]

\[ x = 0 \text{ or } y = -1 \]

\[ y = 0 : 2y + x^2 = 2y + x^2 = 0 \Rightarrow \]

\[ x^2 = 2 \Rightarrow x = \pm \sqrt{2} \]
Third compute the Hessians of \( f \) at \((0, 0), \(\pm \sqrt{2}, -1\), and \((-\sqrt{2}, -1)\):

\[
D(0, 0) = f_{xx}(0, 0) f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = [2 + 2(0)](2) - [2(0)]^2 = 4 > 0
\]

\[D(0, 0) = 4 > 0 \Rightarrow f(0, 0) \text{ is a local minimum value}\]

\[f_{xx}(0, 0) = 2 > 0\]

Note that

\[ f(0, 0) = 0^2 + 0^2 + 0^2 + 0 + 4 = 4 \]

\[
D(\sqrt{2}, -1) = f_{xx}(\sqrt{2}, -1) f_{yy}(\sqrt{2}, -1) - [f_{xy}(\sqrt{2}, -1)]^2
\]

\[= [2 + 2(-1)](2) - [2(\sqrt{2})]^2 = -8 < 0 \]

\[D(\sqrt{2}, -1) = -8 < 0 \Rightarrow (\sqrt{2}, -1) \text{ is a saddle point}\]
\( D(-\sqrt{2}, -1) = f_{xx}(-\sqrt{2}, -1) f_{yy}(-\sqrt{2}, -1) - \left[ f_{xy}(-\sqrt{2}, -1) \right]^2 \)

\[
= \left[ 2 + 2(-1) \right] \left[ 2 \right] - \left[ 2(-\sqrt{2}) \right]^2
\]

\[= -8 < 0 \]

\[ D(-\sqrt{2}, -1) = -8 < 0 \implies (-\sqrt{2}, -1) \text{ is a SADDLE POINT} \]
9. \( f(x, y) = xy - 2x - y \)

First compute the partials of \( f \):

\[
\begin{align*}
f_x &= \frac{\partial}{\partial x} (xy - 2x - y) \\
     &= y - 2
\end{align*}
\]

\[
\begin{align*}
f_y &= \frac{\partial}{\partial y} (xy - 2x - y) \\
     &= x - 1
\end{align*}
\]

\[
\begin{align*}
f_{xx} &= \frac{\partial^2}{\partial x^2} (xy - 2x - y) \\
       &= \frac{\partial}{\partial x} (y - 2) \\
       &= 0
\end{align*}
\]

\[
\begin{align*}
f_{yy} &= \frac{\partial^2}{\partial y^2} (xy - 2x - y) \\
       &= \frac{\partial}{\partial y} (x - 1) \\
       &= 0
\end{align*}
\]

\[
\begin{align*}
f_{xy} &= f_{yx} = \frac{\partial^2}{\partial x \partial y} (xy - 2x - y) \\
       &= \frac{\partial}{\partial x} (x - 1) \\
       &= 1
\end{align*}
\]

Since \( f \) is a polynomial function and so continuous.
Second find all CRITICAL POINTS:
\[ f_x = 0 \Rightarrow y - 2 = 0 \Rightarrow y = 2 \]
\[ f_y = 0 \Rightarrow x - 1 = 0 \Rightarrow x = 1 \]

(1,2) ONE CRITICAL POINT

Third compute the Hessian of \( f \) at (1,2):

\[
D(1,2) = f_{xx}(1,2) f_{yy}(1,2) - [f_{xy}(1,2)]^2
\]

\[
= (0)(0) - (1)^2
\]

\[
= -1 < 0
\]

\[ D(1,2) = -1 < 0 \Rightarrow (1,2) \text{ is a SADDLE POINT} \]
LIKE EXERCISES 31 AND 32:

Find the shortest distance from the point \((2, 1, -1)\) to the plane \(x + y - z = 1\).

And give the point on the plane \(x + y - z = 1\) that is closest to the point \((2, 1, -1)\).

This is an "ABSOLUTE MINIMUM VALUE" of \(f(x, y)\) problem, without however, \(f(x, y)\) being restricted to a closed and bounded set \(D\).

We need to create an \(f(x, y)\) from the information given in this problem and then find its minimum value.

For our \(f(x, y)\), we will define it as a "distance squared function" which gives the formula for the "distance squared" between the point \((2, 1, -1)\) and all points \((x, y, z)\) on the plane \(x + y - z = 1\).
Then we will find the minimum value of \( f(x, y) \) or of the "distance squared." This will, in turn, give us the minimum distance (i.e., shortest distance) between the point \((2, 1, -1)\) and the plane \(x + y - z = 1\), where we will simply take the square root of the minimum "distance squared."

So, we first rewrite the equation of the plane as

\[
z = x + y - 1,
\]

where we are "solving for \( z \) in terms of \( x \) and \( y \)."

Then we write down the "distance-squared" formula for the "distances squared" between the point \((2, 1, -1)\) and any point \((x, y, z)\) on the plane \(z = x + y - 1:\n
\[
d^2 = (x - 2)^2 + (y - 1)^2 + (z - (-1))^2.
\]

We then substitute (*) into (**) to get
\[ d^2 = (x-2)^2 + (y-1)^2 + (x+y-1)^2 \]

or

\[ d^2 = (x-2)^2 + (y-1)^2 + (x+y)^2. \]

We next set \( f(x,y) \) equal to the right-hand side of \((***)\):

\[ f(x,y) = (x-2)^2 + (y-1)^2 + (x+y)^2 \]

We finally need to find the ABSOLUTE MINIMUM VALUE of \( f(x,y) \), though without \( f(x,y) \) being restricted to a closed and bounded set \( D \) in \( \mathbb{R}^2 \). We note that the ABSOLUTE MINIMUM will occur at a CRITICAL POINT (or LOCAL MINIMUM) of \( f(x,y) \), since the shortest distance between the point \((2,1,-1)\) and the plane \( z = x+y-1 \) is, of course, surrounded by longer distances, and there is only one shortest distance.

So, we will only try to find the sole CRITICAL POINT of \( f(x,y) \) and then plug that into \( f(x,y) \) to obtain \( d^2 \).
\[ f(x, y) = (x - 2)^2 + (y - 1)^2 + (x + y)^2 \]

\[ f_x = \frac{\partial}{\partial x} \left[ (x - 2)^2 + (y - 1)^2 + (x + y)^2 \right] \]
\[ = 2(x - 2) + 2(x + y) \cdot 1 \]
\[ = -4x + 2y - 4 \]
\[ f_y = \frac{2}{3y} \left[ (x-2)^2 + (y-1)^2 + (x+y)^2 \right] \]
\[ = 0 + -2(y-1)' \cdot \frac{2}{3y} (y-1) + 2(x+y)' \cdot \frac{2}{3y} (x+y) \]
\[ = 2(y-1)' + 2(x+y)' \cdot \frac{2}{3y} (x+y) \]
\[ = 2y - 2 + 2x + 2y \]
\[ = 2x + 4y - 2 \]

Set \( f_x = 0 \) and \( f_y = 0 \) and solve simultaneously to obtain CRITICAL POINT:

\[ f_x = 0 \quad \Rightarrow \quad 4x + 2y - 4 = 0 \quad \Rightarrow \quad 2x + y = 2 \]
\[ f_y = 0 \quad \Rightarrow \quad \frac{2x + y = 2}{x + 2y = 1} \quad \Rightarrow \quad \frac{y = 2 - 2x}{x + 2y = 1} \quad \Rightarrow \quad \text{Substitute in} \]
\[ y = 2 - 2x \quad \Rightarrow \quad y = 2 - 2x \quad \Rightarrow \quad y = 2 - 2x \quad \Rightarrow \quad x + 4 - 4x = 1 \]
\[ y = 2 - 2x \quad \Rightarrow \quad y = 2 - 2x \quad \Rightarrow \quad x = 1 \]
\[ -3x = -3 \quad \Rightarrow \quad x = 1 \]
\[ y = 2 - 2(1) \quad \Rightarrow \quad y = 0 \quad \Rightarrow \quad x = 1 \]

\((1, 0)\) ONE CRITICAL POINT

From the above discussion, the shortest distance between \((2, 1, -1)\) and the plane \(z = x + y - 1\) is

\[ d = \sqrt{d^2} \]

\[ = \sqrt{f(1, 0)} \]

\[ = \sqrt{(1-2)^2 + (0-1)^2 + (1+0)^2} \]

\[ = \sqrt{1 + 1 + 1} \]

\[ = \sqrt{3} \]
Moreover, the point on the plane that is closest to the point \((2,1,-1)\) is

\[
(x, y, z) = (x, y, \frac{x+y-2}{\sqrt{\text{CRITICAL EQUATION OF PLANE}}})
\]

\[
= (1, 0, 1+0-1)
\]

\[
= (1, 0, -0)
\]