Section 11.2: Limits and Continuity.

Notation: \( \lim_{{(x,y) \to (a,b)}} f(x,y) = L \)

The limit of the function \( f(x,y) \) is \( L \) as \( (x,y) \) approaches the point \( (a,b) \).

Existence/Nonexistence of the Limit. We say that

\[ \lim_{{(x,y) \to (a,b)}} f(x,y) \text{ EXISTS} \]

if

\[ f(x,y) \to \text{ a single, finite value} \]

when we allow \( (x,y) \) to approach \( (a,b) \) from all possible directions in two-dimensional space.
Problem in Determining Existence of a Limit of a Function of Two Variables.

You need "Advanced Calculus" or "Real Analysis" techniques to check existence.

At our level, we can only say when a limit

DOES NOT EXIST

most of the time. Some of the time, we can use the

SQUEEZE THEOREM

to say that a limit does in fact exist (see p. 116 and p. 118, Exercises 23-28 in Section 2, 3, text).
With each approach of \((x, y)\) towards \((a, b)\), \(f(x, y)\) must always approach the same finite value \(-L\).

If
\[
\lim_{(x, y) \to (a, b)} f(x, y) = \begin{cases} 
\pm \infty & \text{more than one value} \\
\text{no particular value} & \text{no particular value}
\end{cases}
\]

then we say
\[
\lim_{(x, y) \to (a, b)} f(x, y) \text{ DOES NOT EXIST (or DNE)}
\]
Exercises 11.2, p. 765

5.18. Find the limit, if it exists, or show that the limit does not exist.

\[ \lim_{(x,y) \to (5, -2)} (x^5 + 4x^3y - 5xy^2) \]

**Note:** If \( f(x, y) = x^5 + 4x^3y - 5xy^2 \) is a polynomial function, then

\[ \lim_{(x,y) \to (a,b)} f(x,y) = f(a, b) \]

So, here, we have

\[ \lim_{(x,y) \to (5, -2)} (x^5 + 4x^3y - 5xy^2) \]

\[ = (5)^5 + 4(5)^3(-2) - 5(5)(-2)^2 \]

\[ = -2025 \]
\[ \lim_{(x,y) \to (0,0)} \frac{x^2}{x^2 + y^2} \]

First try plugging \((x,y) = (0,0)\) into \(\frac{x^2}{x^2 + y^2}\): Get \(\frac{0^2}{0^2 + 0^2} = \frac{0}{0}\).

If you get an indeterminate form like \(\frac{0}{0}\), \(\frac{\infty}{\infty}\), \(\infty - \infty\), etc., try another method or show that the limit DNE.

We will try to show that the limit DNE by showing that the limit equals two different values when we allow \((x,y)\) to approach \((0,0)\) from two different directions.
\[ (x, y) \rightarrow (0, 0) \text{ along the x-axis} \]

\[ \lim_{(x, y) \rightarrow (0, 0)} \frac{x^2}{x^2 + y^2} = \lim_{(x, y) \rightarrow (0, 0)} \frac{x^2}{x^2 + 0^2} \]

\[ \lim_{(x, y) \rightarrow (0, 0)} \frac{x^2}{x^2} = \lim_{(x, y) \rightarrow (0, 0)} 1 \]

Limit of a constant is that constant.
\( (x, y) \to (0, 0) \) along the \( y \)-axis

\[
\lim_{(x, y) \to (0, 0) \text{ and } x = 0} \frac{x^2}{x^2 + y^2} = \lim_{(x, y) \to (0, 0) \text{ and } y \neq 0} \frac{0^2}{0^2 + y^2} = \lim_{(x, y) \to (0, 0)} \frac{0}{y^2} = \lim_{(x, y) \to (0, 0)} 0 = 0
\]

Limit of a constant = that constant
Then
\[ \lim_{{(x,y) \to (0,0)}} \frac{x^2}{x^2 + y^2} = \text{at least two different values depending on how} \]
\[ (x,y) \text{ approaches } (0,0) \]

\[ \therefore \text{This limit DNE} \]
\[ \lim_{{(x,y) \to (0,0)}} \frac{8x^2y^2}{x^4 + y^4} \]

First try plugging \((x,y) = (0,0)\) into \(\frac{8x^2y^2}{x^4 + y^4}\): Get \(\frac{8 \cdot 0^2 \cdot 0^2}{0^4 + 0^4} = \frac{0}{0} \), DNE.
\[(x, y) \to (0, 0) \text{ along the x-axis}\]

\[
\lim_{(x,y) \to (0,0)} \frac{8x^2 y^2}{x^4 + y^4} = \lim_{(x,y) \to (0,0)} \frac{8x^2 \cdot 0^2}{x^4 + 0^4}
\]

\[
= \lim_{(x,y) \to (0,0)} \frac{0}{x^4}
\]

\[
= \lim_{(x,y) \to (0,0)} \frac{0}{0}
\]

\[
= 0
\]
\[\lim_{(x,y) \to (0,0)} \frac{8x^2y^2}{x^4+y^4} = \lim_{(x,y) \to (0,0)} \frac{8 \cdot 0^2 y^2}{0^4+y^4} = \lim_{(x,y) \to (0,0)} \frac{0}{y^4} = \lim_{(x,y) \to (0,0)} 0 = 0\]

It does not necessarily mean that the limit exists!

Try another direction of approach:

Same as when \(x,y \to (0,0)\) along \(x = 0\).
\((x,y) \rightarrow (0,0)\) along the diagonal \(y = x\)

\[
\lim_{{(x,y) \to (0,0)}} \frac{8x^2 y^2}{x^4 + y^4} = \lim_{{(x,y) \to (0,0)}} \frac{8y^2 y^2}{y^4 + y^4}
\]

\(\text{and } x = y\)  
\(\text{and } x \neq 0, y \neq 0\)

\[
= \lim_{{(x,y) \to (0,0)}} \frac{8y^4}{2y^4} = \lim_{{(x,y) \to (0,0)}} 4 = 4
\]
Finally, then, we can say

\[ \lim_{(x,y) \to (0,0)} \frac{8x^2 y^2}{x^4 + y^4} = \text{at least two different values depending on how} \]
\[ (x,y) \text{ approaches } (0,0) \]

\[ \therefore \text{ This limit } \text{DNE} \]
\[
\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}}
\]

First try plugging \((x,y) = (0,0)\) into \(\frac{xy}{\sqrt{x^2 + y^2}}\): Get \(\frac{0\cdot0}{\sqrt{0^2 + 0^2}} = 0\).

This time, try to show limit exists using the SQUEEZE THEOREM, where:

\[
0 < \left| \frac{xy}{\sqrt{x^2 + y^2}} \right| < \frac{|xy|}{\sqrt{x^2 + y^2}}
\]

as long as \(x \neq 0\), \(y \neq 0\), and denominator is smaller

\[
|xy| = |x| \cdot |y|
\]

\[
\frac{|xy|}{\sqrt{x^2 + y^2}} < \frac{|x||y|}{\sqrt{x^2 + y^2}} = |y|
\]

as long as \(x \neq 0\), \(y \neq 0\),

\[
\frac{|xy|}{\sqrt{x^2 + y^2}} \leq \frac{|x||y|}{\sqrt{x^2 + y^2}} = |y|
\]

\[
\frac{|xy|}{\sqrt{x^2 + y^2}} \leq \frac{|x||y|}{\sqrt{x^2 + y^2}} = |y|
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\]

\[
\frac{|xy|}{\sqrt{x^2 + y^2}} \leq \frac{|x||y|}{\sqrt{x^2 + y^2}} = |y|
\]
So, we have, for \(-x \neq 0, y \neq 0,\)

\[
0 < \left| \frac{-xy}{\sqrt{x^2 + y^2}} \right| = \frac{|x||y|}{\sqrt{x^2 + y^2}} < \frac{|x||y|}{\sqrt{x^2}} \leq \frac{\sqrt{x^2}}{\sqrt{x^2}} = 1
\]

So this fraction is smaller than \(\frac{\sqrt{x^2}}{\sqrt{x^2}} = |x|\)

\[
\therefore 0 < \left| \frac{-xy}{\sqrt{x^2 + y^2}} \right| < |y| \Rightarrow
\]
\[ \lim_{(x,y) \to (0,0)} \frac{xy}{\sqrt{x^2+y^2}} \leq \lim_{(x,y) \to (0,0)} 1 = 1 \]

\[ 0 \leq \lim_{(x,y) \to (0,0)} \frac{xy}{\sqrt{x^2+y^2}} \leq 1 \]

\[ \therefore 0 \leq \lim_{(x,y) \to (0,0)} \frac{xy}{\sqrt{x^2+y^2}} \leq 0 \Rightarrow \]

We have to have\( \left| \frac{xy}{\sqrt{x^2+y^2}} \right| = 0 \Rightarrow \)

\[ \lim_{(x,y) \to (0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0 \]

\[ \text{If } |f(x,y)| \to 0, \text{ then } f(x,y) \to 0. \]
We need to determine the limits of functions so we can determine whether or not they are continuous.

**Definition:** A function $f(x, y)$ is continuous at the point $(a, b)$ if all three of the following hold:

1. $f(a, b)$ is defined (i.e., $f(a, b) = a$ single, finite value and $f(a, b) \neq \frac{0}{0}, \frac{\infty}{\infty}$, etc.).
2. $\lim_{(x, y) \to (a, b)} f(x, y)$ exists and is equal to some finite value $L$.
3. $\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b)$ (i.e., $L = f(a, b)$).

**The Three Criteria of Continuity**
Definition: A function \( f(x,y) \) is continuous on a subset of \( \mathbb{R}^2 \) if it is continuous at every point \((a,b)\) in \( S \).

Definition: A function \( f(x,y) \) is continuous if it is continuous at every point \((a,b)\) in \( \mathbb{R}^2 \) or is continuous everywhere.
1. Polynomial functions of two variables,
\[ f(x,y) = \text{sum of terms of the form } c x^m y^n, \]
c = a constant, 
m, n = nonnegative integers
are continuous. That is, they are continuous at every point \((a, b)\) in \(\mathbb{R}^2\) or continuous everywhere.

2. Rational functions of two variables,
\[ f(x,y) = \frac{\text{poly in } x \text{ and } y}{\text{poly in } x \text{ and } y} \]
are continuous on their domains.

3. Other functions can be continuous on their domains; e.g., \( f(x,y) = \sqrt{x+y} \).
E.g., \( f(x,y) = x^10 \cdot y - x^6 y^9 + 3 x - 2 y + 1 \) is continuous everywhere.

E.g., \( f(x,y) = \frac{x+1}{x^2 + y^2} \)

\[
\text{Domain of } f = \{ (x,y) \in \mathbb{R}^2 \mid \text{denom.} \neq 0 \} = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \neq 0 \} = \{ (x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0) \}.
\]

\( f(x,y) \) is continuous on the set \( \{ (x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0) \} \).
25-32. Determine the largest set on which the function is continuous.

\[ F(x, y) = \frac{1}{x^2 - y} \]

\[ F(x, y) \] is continuous on its domain

\[
\text{DOMAIN} = \{(x, y) \in \mathbb{R}^2 | \text{denom.} \neq 0 \} \\
= \{(x, y) \in \mathbb{R}^2 | x^2 - y \neq 0 \} \\
= \{(x, y) \in \mathbb{R}^2 | y \neq x^2 \}
\]

That is, \( F(x, y) \) is continuous everywhere except along the parabola \( y = x^2 \) in \( \mathbb{R}^2 \).
\[ f(x, y) = \begin{cases} \frac{x^2 y^3}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases} \]

The rational function \( \frac{x^2 y^3}{2x^2 + y^2} \) is continuous on its domain of:

\[ \text{Domain} = \{ (x, y) \in \mathbb{R}^2 \mid \text{denom.} \neq 0 \} \]
\[ = \{ (x, y) \in \mathbb{R}^2 \mid 2x^2 + y^2 \neq 0 \} \]
\[ = \{ (x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0) \} \]

Therefore, \( f(x, y) \) is at least continuous on the set \( \{ (x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0) \} \).

We will see whether or not \( f(x, y) \) is continuous at \( (0, 0) \) by checking the three criteria of continuity.
(1) \( f(0,0) = 1 \Rightarrow f(0,0) \) is defined

(2) \( \lim_{(x,y) \to (0,0)} \frac{x^3 y}{2x^2 + y^2} \)

Try applying the SQUEEZE THEOREM.

\[ 0 < \left| \frac{x^3 y}{2x^2 + y^2} \right| < \left| \frac{x^3 y}{y^2} \right| = \left| x^3 y \right| = x^2 |y| \Rightarrow \]

\[ 0 < \left| \frac{x^3 y}{2x^2 + y^2} \right| < x^2 |y| \Rightarrow \]

\[ \lim_{(x,y) \to (0,0)} 0 \leq \lim_{(x,y) \to (0,0)} \left| \frac{x^3 y}{2x^2 + y^2} \right| \leq \lim_{(x,y) \to (0,0)} x^2 |y| \Rightarrow \]

\[ 0 \leq \lim_{(x,y) \to (0,0)} \left| \frac{x^3 y}{2x^2 + y^2} \right| \leq 0 \Rightarrow \]

\[ 0 = \lim_{(x,y) \to (0,0)} \left| \frac{x^3 y}{2x^2 + y^2} \right| \]
\[
\lim_{{(x,y) \to (0,0)}} \frac{x^2y^3}{2x^2+y^2} = 0 = \circ \\
\lim_{{(x,y) \to (0,0)}} \frac{x^2y^3}{2x^2+y^2} = \circ \\
\lim_{{(x,y) \to (0,0)}} \frac{x^2y}{ax^2+by^2} \text{ exists}. \\
(3) \lim_{{(x,y) \to (0,0)}} \frac{x^2y}{2x^2+y^2} = f(0,0) \\
\frac{0}{0} \quad \text{No!!} \\
\text{f(x,y) does not satisfy all three criteria at (0,0), and so is not continuous at (0,0).} \\
\text{f(x,y) is only continuous on} \\
\{ (x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0) \} \}.
That is, $f(x,y)$ is continuous everywhere except at the origin.