Lecture

Section 11.2: Limits and Continuity.

Notation: $\underbrace{\lim_{(x,y)\to(a,b)} f(x,y) = L}$

The limit of the function $f(x,y)$ is L as (x,y) approaches the point (a,b)

Existence / Nonexistence of the Limit.

We say that

$$\lim_{(x,y)\to(a,b)} f(x,y) \quad \text{EXISTS}$$

if

$f(x,y) \longrightarrow$ a <u>single</u>, <u>finite</u> value

when we allow (x,y) to approach (a,b) from <u>all</u> possible directions in two-dimensional space;

Problem in Determining Existence of a Limit of a Function of Two Variables.

You need "Advanced Calculus" or "Real Analysis" techniques to check existence.

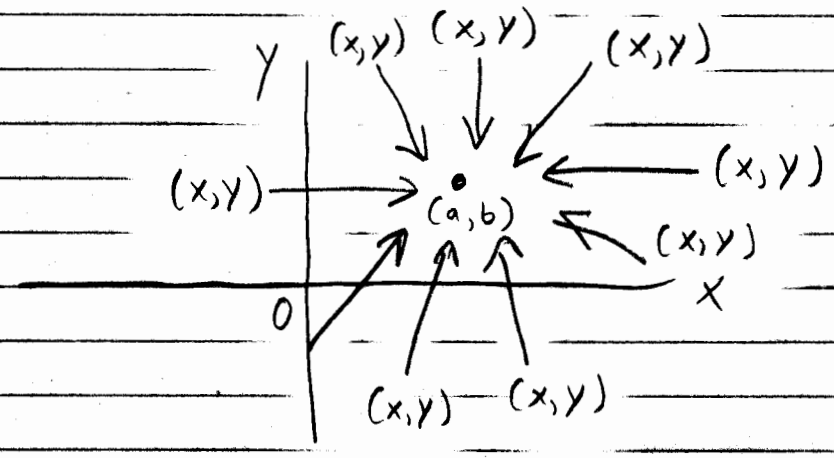
At our level, we can only say when a limit

DOES NOT EXIST

most of the time. Some of the time, we can use the

SQUEEZE THEOREM

to say that a limit does in fact exist (SEE p. 116 and p. 118, Exercises 23 - 28, in Section 2.3, text).



With each approach of (x,y) towards (a,b), $f(x,y)$ must always approach the same finite value - L!

If

$$\lim_{(x,y)\to(a,b)} f(x,y) = \begin{cases} \pm\infty \\ \text{more than one value} \\ \text{no particular value} \end{cases}$$

then we say

$$\lim_{(x,y)\to(a,b)} f(x,y) \quad \text{DOES NOT EXIST (or DNE)}$$

Exercises 11.2, p. 765

5–18 Find the limit, if it exists, or show that the limit does not exist.

5. $\lim_{(x,y)\to(5,-2)} (x^5 + 4x^3y - 5xy^2)$

NOTE: $f(x,y) = x^5 + 4x^3y - 5xy^2$ is a POLYNOMIAL FUNCTION and if $f(x,y)$ is a polynomial function, then

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)\,!$$

So, here, we have

$$\lim_{(x,y)\to(5,-2)} (x^5 + 4x^3y - 5xy^2)$$

$$= (5)^5 + 4(5)^3(-2) - 5(5)(-2)^2$$

$$= \boxed{2025}$$

7. $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

First try plugging $(x,y)=(0,0)$ into $\frac{x^2}{x^2+y^2}$: Get $\frac{0^2}{0^2+0^2} = \frac{0}{0}$.

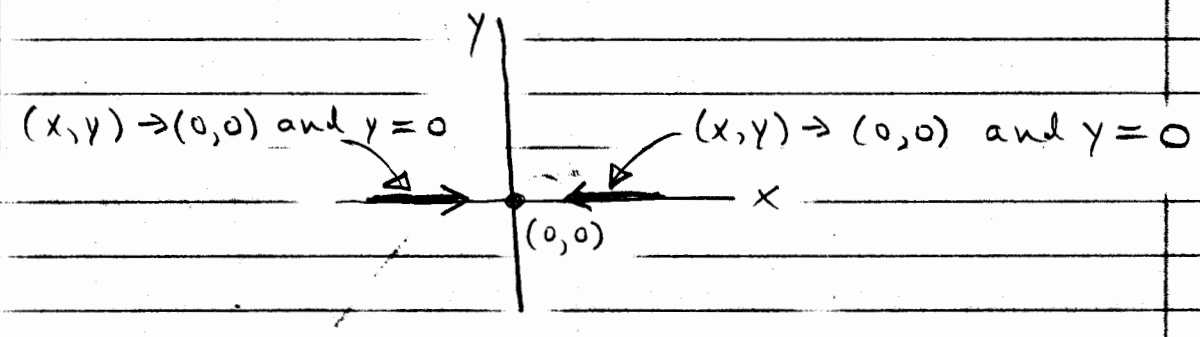
If you get an indeterminate form like $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty-\infty$, 0^∞, etc., try another method or show that the limit DNE.

We will try to show that the limit DNE by showing that the limit equals <u>two</u> different values when we allow (x,y) to approach $(0,0)$ from <u>two</u> different directions.

$C \leq$

y^2

$\boxed{(x,y) \to (0,0) \text{ along the } x\text{-axis}}$



$$\lim_{\substack{(x,y)\to(0,0)\\ \text{and } y=0\\ (\text{and } x\neq 0)}} \frac{x^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+0^2}$$

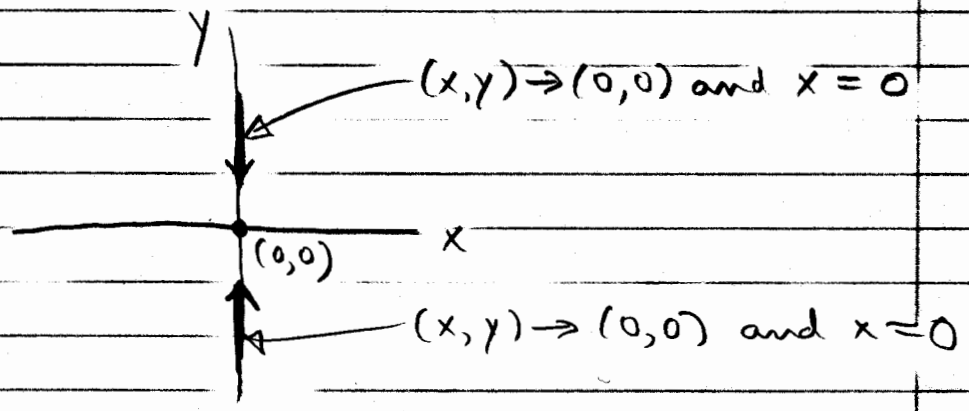
$$= \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2}$$

$$= \lim_{(x,y)\to(0,0)} 1$$

limit of a constant = that constant

$$= \boxed{1}$$

$\boxed{(x,y) \to (0,0) \text{ along the } y\text{-axis}}$



$$\lim_{\substack{(x,y)\to(0,0)\\ \text{and } x=0\\ (\text{and } y\neq 0)}} \frac{x^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{0^2}{0^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{0}{y^2}$$

$$= \lim_{(x,y)\to(0,0)} 0$$

limit of a constant = that constant

$$= \boxed{0}$$

Then

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \text{at least two different values depending on how } (x,y) \text{ approaches } (0,0)$$

$\therefore$ This limit $\boxed{\text{DNE}}$

9. $\lim_{(x,y)\to(0,0)} \frac{8x^2y^2}{x^4+y^4}$

$\lim_{(x,y)\to(0,0)} \frac{8x^2y^2}{x^4+y^4}$

First try plugging $(x,y)=(0,0)$ into $\frac{8x^2y^2}{x^4+y^4}$; Get $\frac{8\cdot 0^2\cdot 0^2}{0^4+0^4} = \frac{0}{0}$.

We will try to show the limit DNE.

$\boxed{(x,y) \to (0,0) \text{ along the } x\text{-axis}}$

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{and } y=0 \\ (\text{and } x\neq 0)}} \frac{8x^2y^2}{x^4+y^4} = \lim_{(x,y)\to(0,0)} \frac{8x^2\cdot 0^2}{x^4+0^4}$$

$$= \lim_{(x,y)\to(0,0)} \frac{0}{x^4}$$

$$= \lim_{(x,y)\to(0,0)} 0$$

$$= \boxed{0}$$

$\boxed{(x,y) \to (0,0) \text{ along the } y\text{-axis}}$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{and } x=0\\ (\text{and } y\neq 0)}} \frac{8x^2y^2}{x^4+y^4} = \lim_{(x,y)\to(0,0)} \frac{8\cdot 0^2 y^2}{0^4+y^4}$$

$$= \lim_{(x,y)\to(0,0)} \frac{0}{y^4}$$

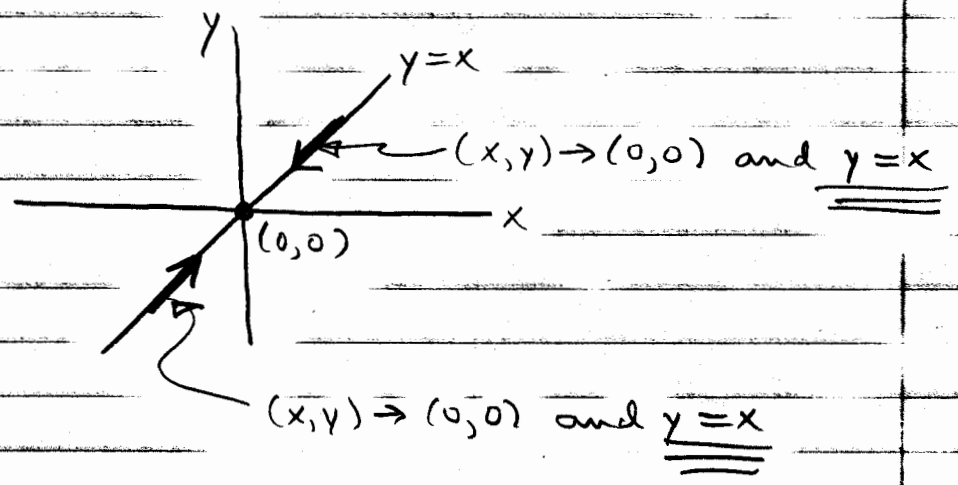
$$= \lim_{(x,y)\to(0,0)} 0$$

$$= \boxed{0}$$ SAME AS WHEN $(x,y)\to(0,0)$ along x-axis!

Still does not necessarily mean that the limit exists!

Try still another direction of approach of (x,y) towards $(0,0)$:

$\boxed{(x,y)\to(0,0) \text{ along the diagonal } y=x}$



$$\lim_{\substack{(x,y)\to(0,0)\\ \text{and } x=y\\ (\text{and } x\neq 0,\, y\neq 0)}} \frac{8x^2y^2}{x^4+y^4} = \lim_{(x,y)\to(0,0)} \frac{8y^2y^2}{y^4+y^4}$$

$$= \lim_{(x,y)\to(0,0)} \frac{8\cancel{y^4}}{2\cancel{y^4}}$$

$$= \lim_{(x,y)\to(0,0)} 4$$

$$= \boxed{4}$$

Finally, then, we can say

$$\lim_{(x,y)\to(0,0)} \frac{8x^2y^2}{x^4+y^4} =$$ at least two different values depending on how (x,y) approaches $(0,0)$.

$\therefore$ This limit $\boxed{\text{DNE}}$

11, $\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}}$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$$

First try plugging $(x,y)=(0,0)$ into $\dfrac{xy}{\sqrt{x^2+y^2}}$: Get $\dfrac{0\cdot 0}{\sqrt{0^2+0^2}} = \dfrac{0}{0}$.

This time, try to show limit EXISTS using the SQUEEZE THEOREM, where $\left|\dfrac{xy}{\sqrt{x^2+y^2}}\right|$ can be "trapped" between two functions that go to the same limit as $(x,y)\to(0,0)$:

$$0 < \left|\frac{xy}{\sqrt{x^2+y^2}}\right| < \left|\frac{xy}{\sqrt{x^2}}\right| = \frac{|x||y|}{|x|} = |y|$$

as long as $x\neq 0$, $y\neq 0$

as long as $x\neq 0$, $y\neq 0$

smaller denominator than $\dfrac{xy}{\sqrt{x^2+y^2}}$

$\sqrt{x^2} = |x|$

So, we have, for $x \neq 0$, $y \neq 0$,

$$0 < \left| \frac{xy}{\sqrt{x^2+y^2}} \right| = \frac{|x||y|}{\sqrt{x^2+y^2}}$$

$$< \frac{|x||y|}{\sqrt{x^2}}$$

smaller than $\sqrt{x^2+y^2}$ so this fraction larger

$$= \frac{\cancel{|x|}|y|}{\cancel{|x|}}$$

$\sqrt{x^2} = |x|$

$$= |y|$$

$$\therefore 0 < \left| \frac{xy}{\sqrt{x^2+y^2}} \right| < |y| \Longrightarrow$$

strict inequalities < or > change to ≤ or ≥ when take limits

$$\lim_{(x,y)\to(0,0)} 0 \le \lim_{(x,y)\to(0,0)} \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \le \lim_{(x,y)\to(0,0)} |y| =$$

$$\begin{array}{ccc} \| & & \| \\ 0 & & 0 \end{array}$$

$$\therefore \quad 0 \le \lim_{(x,y)\to(0,0)} \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \le 0 \Longrightarrow$$

We have to have (absolute value signs are [illegible])

$$\lim_{(x,y)\to(0,0)} \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = 0 \Longrightarrow$$

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0}$$

If $|f(x,y)| \to 0$, then $f(x,y) \to 0$.

Continuity

We need to determine the limits of functions so we can determine whether or not they are continuous.

Definition. A function $f(x, y)$ is continuous at the point (a, b) if all three of the following hold:

(1) $f(a,b)$ is defined (i.e., $f(a,b)$ = a single, finite value and $f(a,b) \neq \frac{0}{0}, \frac{1}{0}$, etc.).

(2) $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists (and is equal to some finite value L).

(3) $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

(i.e., $L = f(a,b)$).

THE THREE CRITERIA OF CONTINUITY

<u>Definition</u>. A function $f(x,y)$ is <u>continuous on a subset S of $\mathbb{R}^2$</u> if it is continuous at every point (a,b) in S.

<u>Definition</u>. A function $f(x,y)$ is <u>continuous</u> if it is continuous at every point (a,b) in $\mathbb{R}^2$, or continuous everywhere.

Rules of Thumb

1. Polynomial functions of two variables,

   $f(x,y)$ = sum of terms of the form $cx^m y^n$,

   c = a constant,
   m, n = nonnegative integers,

   are <u>continuous</u>. That is, they are continuous at every point (a,b) in $\mathbb{R}^2$, or continuous everywhere.

2. Rational functions of two variables,

   $$f(x,y) = \frac{\text{poly in } x \text{ and } y}{\text{poly in } x \text{ and } y}$$

   are <u>continuous on their domains</u>.

3. Other functions can be continuous on their domains; e.g., $f(x,y) = \sqrt{x+y}$.

E.g., $f(x,y) = x^{10}y - x^6y^9 + 3x - 2y + 1$

is $\boxed{\text{continuous everywhere}}$.

E.g., $f(x,y) = \dfrac{x+1}{x^2+y^2}$

$$\text{DOMAIN of } f = \{(x,y) \in \mathbb{R}^2 \mid \text{denom.} \neq 0\}$$
$$= \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \neq 0\}$$
$$= \{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\}.$$

$\therefore$ $f(x,y)$ is continuous on the set

$$\boxed{\{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\}}.$$

Exercises 11.2, p. 766.

25-32 Determine the largest set on which the function is continuous.

25. $F(x,y) = \dfrac{1}{x^2 - y}$.

$F(x,y)$ is continuous on its

$$\text{DOMAIN} = \{(x,y) \in \mathbb{R}^2 \mid \text{denom.} \neq 0\}$$
$$= \{(x,y) \in \mathbb{R}^2 \mid x^2 - y \neq 0\}$$
$$= \boxed{\{(x,y) \in \mathbb{R}^2 \mid y \neq x^2\}}$$

That is, $F(x,y)$ is continuous everywhere except along the parabola $y = x^2$ in $\mathbb{R}^2$.

31. $f(x,y) = \begin{cases} \dfrac{x^2y^3}{2x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$

The rational function $\dfrac{x^2y^3}{2x^2+y^2}$ is continuous on its

$$\text{DOMAIN} = \{(x,y) \in \mathbb{R}^2 \mid \text{denom.} \neq 0\}$$
$$= \{(x,y) \in \mathbb{R}^2 \mid 2x^2+y^2 \neq 0\}$$
$$= \{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\}.$$

Therefore,

$f(x,y)$ is <u>at least</u> continuous on the set $\{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\}$.

We will see whether or not $f(x,y)$ is continuous at $(0,0)$, by checking "the three criteria of continuity":

(1) $f(0,0) = 1 \Rightarrow f(0,0)$ is defined ✓

(2) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^3}{2x^2+y^2}$ exists?

Try applying the SQUEEZE THEOREM:

$$0 < \left|\frac{x^2y^3}{2x^2+y^2}\right| < \left|\frac{x^2y^3}{y^2}\right|$$

$$= |x^2y|$$

$$= x^2|y| \Rightarrow$$

$$0 < \left|\frac{x^2y^3}{2x^2+y^2}\right| < x^2|y| \Rightarrow$$

$$\underset{\substack{\| \\ 0}}{\lim_{(x,y)\to(0,0)} 0} \le \lim_{(x,y)\to(0,0)} \left|\frac{x^2y^3}{2x^2+y^2}\right| \le \underset{\substack{\| \\ 0}}{\lim_{(x,y)\to(0,0)} x^2|y|} \Rightarrow$$

$$0 \le \lim_{(x,y)\to(0,0)} \left|\frac{x^2y^3}{2x^2+y^2}\right| \le 0 \Rightarrow$$

$$\lim_{(x,y)\to(0,0)} \left| \frac{x^2y^3}{2x^2+y^2} \right| = 0 \Longrightarrow$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{2x^2+y^2} = 0 \Longrightarrow$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{2x^2+y^2} \text{ exists.}$$

$$(3) \quad \underbrace{\lim_{(x,y)\to(0,0)} \frac{x^2y}{2x^2+y^2}}_{=\,0} \overset{?}{=} \underbrace{f(0,0)}_{=\,1}$$

No!!

$\therefore$ $f(x,y)$ does not satisfy all three criteria at $(0,0)$, and so is <u>not</u> continuous at $(0,0)$

$\therefore$ $f(x,y)$ is only continuous on

$$\boxed{\{(x,y)\in\mathbb{R}^2 \mid (x,y)\neq(0,0)\}.}$$

That is, $f(x,y)$ is continuous everywhere except at the origin.