Section 4.6. Variation of Parameters.

Suppose you have a non-homogeneous linear DE

\[ a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1y' + a_0y = g(x) \]

\[ = g_1(x) + \cdots + g_k(x) \]

But either

1. \( a_0(x), a_1(x), \ldots, a_n(x) \) are NOT ALL CONSTANT COEFFICIENTS

or

2. \( g_1(x), g_2(x), \ldots, g_k(x) \) DO NOT ALL GENERATE A FINITE FAMILY OF DERIVATIVES, i.e., they are not \( e^{bx} \), \( bx^n + \cdots + b_1x + b_0 \), \( \sin bx \), \( \cos bx \), or product of these.

E.g., \( \tan x \rightarrow \{ \sec^2 x, 2\sec^2 x + \tan x, 4\sec^2 x + \tan x + 2\sec^4 x, \ldots \} \)

\( f''(x) \) all distinct (linearly independent)
Then you CANNOT use the Method of Undetermined Coefficients to find $y_p$ in $y = y_c + y_p$.

Instead you should use the

**METHOD OF VARIATION OF PARAMETERS.**

This method may remind you of the Method of Reduction of Order, but it is based on different ideas.

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**Motivation for the Method of Variation of Parameters for a Second-Order DE**

[SEE 4.6. The text actually uses this motivation to compute $y_p$. I will use a formula that summarizes everything up.]

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \text{ on } I$$

Please in **STANDARD FORM**:

$$y'' + P(x)y' + Q(x)y = f(x) \text{ (defined on an } I \text{ where } a_2(x) \text{ is nonzero)}$$
For simplicity, suppose we know $y_0$:

$$y'' + P(x)y' + Q(x)y = 0$$

**General Solution:** $y_0 = c_1y_1 + c_2y_2$

Generalize these ARBITRARY CONSTANTS $c_1$, $c_2$ to ARBITRARY FUNCTIONS $u_1(x)$, $u_2(x)$ and write

$$y_p = u_1(x)y_1 + u_2(x)y_2$$

[Analogous to the motivation for the Method of Solving First-Order Linear Equations in 2.3 — We skipped this.]

**Question:** Will we be able to find $u_1(x)$, $u_2(x)$ such that $y_p$ is a particular solution of (*)?

**Answer:** The answer turns out to be YES AS LONG AS WE ASSUME
\[
\begin{align*}
\begin{cases}
 y_1 u_1' + y_2 u_2' &= 0, \\
 y_1' u_1 + y_2' u_2' &= f(x).
\end{cases}
\end{align*}
\]

(SEE p. 142 of the text for why we need to assume this.)

Cramer's rule then says we can solve this "system of 2 algebraic equations in 2 unknowns" \(u_1'\) and \(u_2'\) where

\[
 u_1' = \frac{0 \ y_2' - f(x) y_2}{y_1 y_2' - y_1' y_2} = \frac{-f(x) y_2}{y_1 y_2' - y_1' y_2}
\]

Wronskian of \(y_1\) and \(y_2\)

\[W(y_1, y_2) \neq 0 \text{ for all } x \text{ in } I\]

\[
 u_2' = \frac{y_1 f(x) - y_1 y_2 y_2'}{y_1 y_2' - y_1' y_2} = \frac{y_1 f(x)}{y_1 y_2' - y_1' y_2}
\]

Wronskian of \(y_1\) and \(y_2\)

\[W(y_1, y_2) \neq 0 \text{ for all } x \text{ in } I\]

To find \(u_1\) and \(u_2\), all we need to do is integrate \(u_1'\) and \(u_2'\) with respect to \(x\):
\[ u_1 = \left( u_1' \right)' dx = \int \begin{vmatrix} 0 & y_1 \\ f(x) & y_2' \\ \end{vmatrix} \frac{dx}{y_1' y_2'} \]

\[ u_2 = \left( u_2' \right)' dx = \int \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \\ \end{vmatrix} \frac{dx}{y_1' y_2'} \]

(\( u_1' \) and \( u_2' \) are integrable because we will assume they are both continuous.)

**Summary in a Formula:**

\[ y_p = y_1 u_1 + y_2 u_2 \]

\[ = y_1 \int \begin{vmatrix} 0 & y_2' \\ f(x) & y_2 \\ \end{vmatrix} \frac{dx}{y_1' y_2'} + y_2 \int \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \\ \end{vmatrix} \frac{dx}{y_1' y_2'} \]

\[ = y_1 \int \frac{-f(x)y_2}{y_1 y_2' - y_1' y_2} dx - y_2 \int \frac{y_1 f(x)}{y_1 y_2' - y_1' y_2} dx \]
Example. (HW Exercise 1, p. 146.)

Solve $y'' + y = \sec x$ using Variation of Parameters.
State interval over which general solution is defined.

Note: $\sec x = \frac{1}{\cos x}$

General solution: $y = y_c + y_p$

**STEP 1. Find $y_c$ using "CHARACTERISTIC EQUATION METHOD":**

$y'' + y = 0$

$y = e^{mx}: y'' + y = 0 \Rightarrow m^2 e^{mx} + e^{mx} = 0 \Rightarrow$

$m^2 + 1 = 0 \Rightarrow m^2 = -1 \Rightarrow m = \pm \sqrt{-1} = \pm i \Rightarrow m_1 = -i, m_2 = i \Rightarrow$

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lecture
\[ y_1 = \mathrm{e}^{-ix}, \quad y_2 = \mathrm{e}^{ix} \]

Since
\[ e^{-ix} = \cos x - i\sin x, \quad e^{ix} = \cos x + i\sin x \]

Let
\[ y_1 = \cos x, \quad y_2 = \sin x \]

Then
\[ y_c = c_1 y_1 + c_2 y_2 \implies \]
\[ y_c = c_1 \cos x + c_2 \sin x \]

**STEP 2: Find \( y_p \) using the METHOD OF VARIATION OF PARAMETERS:**

Either go through the steps on p.142 of the text OR use the FORMULA. We will use
The formula here:

$$\gamma_p = \gamma_1 \left( \frac{\frac{0}{f'(x)} \gamma_2}{\gamma_1 \gamma_2} \right) + \gamma_2 \left( \frac{\frac{0}{f'(x)} \gamma_1}{\gamma_1 \gamma_2} \right)$$

$$= \cos x \left( \frac{\frac{0}{\sec x} \sin x}{\cos x \sin x} \right) dx + \sin x \left( \frac{\frac{0}{\sec x} \cos x}{\cos x \cos x} \right) dx$$

$$= \cos x \left( \frac{-\sec x (\sin x)}{(\cos x) (\cos x) - (-\sin x) (\sin x)} \right) dx$$

$$+ \sin x \left( \frac{\cos x \sec x}{(\cos x) (\cos x) - (-\sin x) (\sin x)} \right) dx$$

$$= \cos x \left( \frac{-\tan x}{\cos x + \sin x} \right) dx + \sin x \left( \frac{\cos x}{\cos x + \sin x} \right) dx$$

$$= \cos x \left( \frac{-\tan x}{1} \right) dx + \sin x \left( \frac{1}{1} \right) dx$$

See back of text, Table of Integrals, #12:

$$\int \tan u \, du = -\ln |\cos u| + C$$
\[ -\cos x \left( -\ln \cos x + \frac{1}{\sqrt{C_2}} \right) + \sin x \left( x + \frac{1}{\sqrt{C_2}} \right) \]

- \(C_1\cos x\) and \(C_2\sin x\) will be "absorbed" by the terms \(c_1\cos x\) and \(c_2\sin x\) in the general solution.

\[ y = y_c + y_p = c_1 y_1 + c_2 y_2 + y_p = c_1 \cos x + c_2 \sin x + y_p \]

**STEP 3. General Solution:**

\[ y = c_1 y_1 + c_2 y_2 + y_1 \int u_1' \, dx + y_2 \int u_2' \, dx \Rightarrow \]

\[ y = c_1 \cos x + c_2 \sin x + \cos x \ln \cos x + x \sin x \]

**STEP 4. Interval over which general solution is defined:**
The general solution
\[ y = c_1 \cos x + c_2 \sin x + \cos x \ln |\cos x| + x \sin x \]
is defined whenever \( \cos x \neq 0 \) \( \Rightarrow \)
whenver \( x \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots \) \( \Rightarrow \)

\[ ... -\frac{3\pi}{2} -\frac{\pi}{2} 0 \frac{\pi}{2} \frac{3\pi}{2} \frac{5\pi}{2} \ldots \]

So, for \( I \),

**Choose any open interval between these zeros of \( \cos x \) \( \Rightarrow \)

Choose \(( -\frac{\pi}{2}, \frac{\pi}{2} )\)

\[ I = ( -\frac{\pi}{2}, \frac{\pi}{2} ) \]