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Section 4.6. Variation of Parameters.

Suppose you have a

NONHOMOGENEOUS LINEAR DE

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1 y' + a_0 y = \underbrace{g(x)}$$

$$= g_1(x) + \cdots + g_k(x)$$

BUT either

(1) $a_0(x), a_1(x), \ldots, a_n(x)$ are NOT ALL CONSTANT COEFFICIENTS

or

(2) $g_1(x), g_2(x), \ldots, g_k(x)$ DO NOT ALL GENERATE A FINITE FAMILY OF DERIVATIVES, i.e., they are not e^{bx}, $b_n x^n + \cdots + b_1 x + b_0$, $\sin bx$, $\cos bx$, or products of these

E.g., $\tan x \longrightarrow \underbrace{\{\overset{f(x)}{\sec^2 x}, \overset{f'(x)}{2\sec^2 x \tan x}, \overset{f''(x)}{4\sec^2 x \tan x + 2\sec^4 x}, \ldots\}}_{\text{all distinct (linearly independent)}}$

Then you CANNOT use the Method of Undetermined Coefficients to find y_p in $y = y_c + y_p$.

Instead you should use the

METHOD OF VARIATION OF PARAMETERS.

This method may remind you of the Method of Reduction of Order, but it is based on different ideas.

Motivation for the Method of Variation of Parameters for a Second-Order DE

[SEE 4.6. The text actually uses this motivation to compute y_p. I will use a formula that summarizes everything up.]

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \text{ on } I$$

Place in STANDARD FORM:

(*) $$y'' + P(x)y' + Q(x)y = f(x)$$ (defined on an I where $a_2(x)$ is never 0)

For simplicity, suppose we know y_c:

$$y'' + P(x)y' + Q(x)y = 0$$

General Soln.: $y_c = c_1 y_1 + c_2 y_2$

Generalize these ARBITRARY CONSTANTS

$$c_1, c_2$$

to ARBITRARY FUNCTIONS

$$u_1(x), u_2(x)$$

and write

$$y_p = u_1(x) y_1 + u_2(x) y_2$$

[Analogous to the motivation for the Method of Solving First-Order Linear Equations in 2.3 — We skipped this.]

<u>Question</u>: Will we be able to find $u_1(x)$, $u_2(x)$ such that y_p is a particular soln. of (*)?

<u>Answer</u>: The answer turns out to be YES AS LONG AS WE ASSUME

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0, \\ y_1' u_1' + y_2' u_2' = f(x). \end{cases}$$

(SEE p. 142 of the text for why we need to assume this.)

Cramer's rule then says we can solve this "system of 2 algebraic equations in 2 unknowns" u_1' and u_2' where

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$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\underbrace{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}_{\substack{\text{Wronskian of } y_1 \text{ and } y_2 \\ = W(y_1, y_2) \\ \neq 0 \text{ for all } x \text{ in } I}}} = \frac{-f(x)\, y_2}{y_1 y_2' - y_1' y_2},$$

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$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\underbrace{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}_{\substack{\text{Wronskian of } y_1 \text{ and } y_2 \\ = W(y_1, y_2) \\ \neq 0 \text{ for all } x \text{ in } I}}} = \frac{y_1\, f(x)}{y_1 y_2' - y_1' y_2}.$$

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To find u_1 and u_2, all we need to do is integrate u_1' and u_2' with respect to x:

$$u_1 = \int u_1' \, dx = \int \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \, dx ,$$

$$u_2 = \int u_2' \, dx = \int \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \, dx .$$

(u_1' and u_2' are integrable because we will assume they are both continuous.)

SUMMARY IN A FORMULA:

$$\boxed{\begin{aligned} y_p &= y_1 u_1 + y_2 u_2 \\ &= y_1 \int \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \, dx + y_2 \int \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \, dx \\ &= y_1 \int \frac{-f(x)\, y_2}{y_1 y_2' - y_1' y_2} \, dx - y_2 \int \frac{y_1 f(x)}{y_1 y_2' - y_1' y_2} \, dx \end{aligned}}$$

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<u>Example</u>. (HW Exercise 1, p. 146.)

Solve $y'' + y = \sec x$ using Variation of Parameters.
State interval over which general solution is defined.

<u>Note</u>: $\sec x = \dfrac{1}{\cos x}$

<u>General solution</u>: $y = y_c + y_p$

<u>STEP 1</u>. Find y_c using "CHARACTERISTIC EQUATION METHOD":

$$y'' + y = 0$$

$y = e^{mx}$: $y'' + y = 0 \Rightarrow$

$$m^2 \cancel{e^{mx}} + \cancel{e^{mx}} = 0 \Rightarrow$$

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$$m^2 + 1 = 0 \Rightarrow$$

$$m^2 = -1 \Rightarrow$$

$$m = \pm\sqrt{-1} = \pm i \Rightarrow$$

$$m_1 = -i \ , \ m_2 = i \Rightarrow$$

$$y_1 = e^{-ix}, \quad y_2 = e^{ix}$$

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Since

$$e^{-ix} = \boxed{\cos x} - i\boxed{\sin x}$$
$$e^{ix} = \boxed{\cos x} + i\boxed{\sin x}$$

let

$$y_1 = \cos x, \quad y_2 = \sin x$$

We absolutely must re-represent y_1 and y_2 as REAL functions, since we are going to find y_p in terms of y_1 and y_2 using REAL integration

Then

$$y_c = c_1 y_1 + c_2 y_2 \Rightarrow$$

$$y_c = c_1 \cos x + c_2 \sin x.$$

<u>STEP 2</u>. Find y_p using the METHOD OF VARIATION OF PARAMETERS:

Either go through the steps on p. 142 of the text OR use the FORMULA. We will use

the FORMULA here,

$$y_p = y_1 \int \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} dx + y_2 \int \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} dx$$

$$= \cos x \int \frac{\begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} dx + \sin x \int \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} dx$$

$$= \cos x \int \frac{-(\sec x)(\sin x)}{(\cos x)(\cos x) - (-\sin x)(\sin x)} dx$$

$$+ \sin x \int \frac{(\cos x)(\sec x)}{(\cos x)(\cos x) - (-\sin x)(\sin x)} dx$$

$$= \cos x \int \frac{-\frac{1}{\cos x} \cdot \sin x}{\cos^2 x + \sin^2 x} dx + \sin x \int \frac{\cos x \cdot \frac{1}{\cos x}}{\cos^2 x + \sin^2 x} dx$$

$$= \cos x \underbrace{\int \frac{-\tan x}{1} dx}_{} + \sin x \int \frac{1}{1} dx$$

SEE back of text, TABLE OF INTEGRALS, #12 :

$$\int \tan u \, du = -\ln|\cos u| + C$$

$$= -\cos x\left(-\ln|\cos x|\,\cancel{A}/\cancel{C_1}\right) + \sin x\left(x\,\cancel{A}/\cancel{C_2}\right)$$

$-C_1\cos x$ and $C_2\sin x$ will be "absorbed" by the terms $c_1\cos x$ and $c_2\sin x$ in the general solution

$$\begin{aligned} y &= y_c + y_p \\ &= c_1y_1 + c_2y_2 + y_p \\ &= c_1\cos x + c_2\sin x + y_p \end{aligned}$$

$$= \boxed{\cos x\,\ln|\cos x| + x\sin x}$$

STEP 3. General solution:

$$\begin{aligned} y &= y_c + y_p \\ &= c_1y_1 + c_2y_2 + y_1\int u_1'\,dx + y_2\int u_2'\,dx \Rightarrow \end{aligned}$$

$$\boxed{y = c_1\cos x + c_2\sin x + \cos x\,\ln|\cos x| + x\sin x}$$

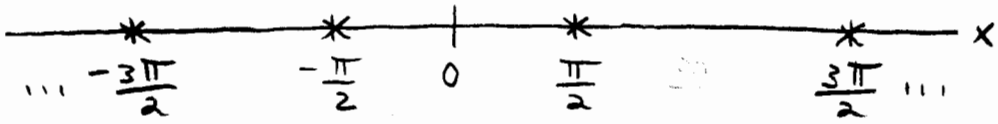
STEP 4. Interval over which general solution is defined:

The general soln,

$$y = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x$$

is defined

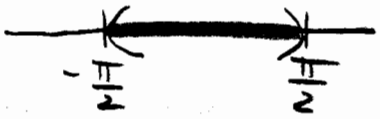
wherever $\cos x \neq 0 \Longrightarrow$
wherever $x \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots \Longrightarrow$



So, for I,

CHOOSE ANY OPEN INTERVAL BETWEEN THESE ZEROS OF $\cos x \Longrightarrow$

CHOOSE $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



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So, let

$$\boxed{I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$$