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Section 4.3. Homogeneous Linear Equations with Constant Coefficients.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

CONSTANT SOLUTIONS

E.g., $7y'' + 2y' - 5y = 0$ ✓

$x^2 y'' + \frac{1}{x} y' + y = \cos x$ ✗

Example. (HW Exerise 17, p. 119.)

Find the general soln. of

$$3y'' + 2y' + y = 0.$$

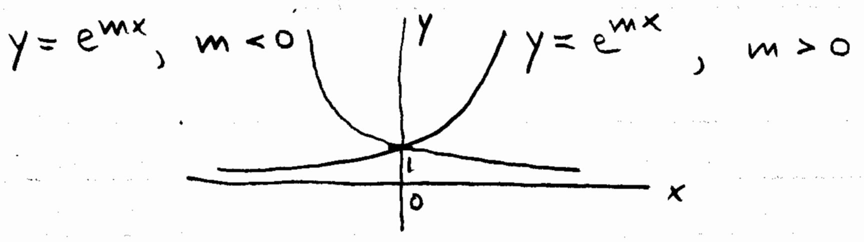
Assume: The DE has a soln. of the form

$$y = e^{mx}.$$

Substitute $y = e^{mx}$ into the DE and see what happens:

$$3m^2\cancel{e^{mx}} + 2m\cancel{e^{mx}} + \cancel{e^{mx}} = 0 \implies$$

Can divide both sides of this equation by e^{mx} since $e^{mx} \neq 0$ for all x and any m



$$\boxed{3m^2 + 2m + 1 = 0}$$ CHARACTERISTIC EQUATION

We just converted a DE problem into an ALGEBRA problem. We next solve for m.

DIGRESSION:

Recall: • Solns. or roots of the quadratic eq.

$$Ax^2 + Bx + C = 0$$

are always given by the quadratic formula

$$x_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Have one of the following:

(1) 2 distinct distinct real roots
$x_1 \neq x_2 \quad (B^2 - 4AC > 0)$

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(2) 1 repeated real root
$x_1 = x_2 \quad (B^2 - 4AC = 0)$

(3) Complex conjugate pair of roots,
$x_1 = \alpha + i\beta, \quad x_2 = \alpha - i\beta$
$(B^2 - 4AC < 0)$

• $i \stackrel{\text{def}}{=} \sqrt{-1} \Rightarrow i^2 = -1$

• $z = a + ib, \quad a, b$ real
$a =$ real part of z
$b =$ imaginary part of z

Solving for m cont'd:

$$3m^2 + 2m + 1 = 0$$

$$m_{1,2} = \frac{-2 \pm \sqrt{4-12}}{6} = \frac{-2 \pm \sqrt{-8}}{6} = \frac{-2 \pm 2\sqrt{2}\, i}{6}$$

$$= -\frac{1}{3} \pm \frac{\sqrt{2}}{3} i$$

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∴ We get 2 (distinct) solns.:

$$y_1(x) = e^{m_1 x} = e^{\left(-\frac{1}{3} - \frac{\sqrt{2}}{3} i\right) x}$$

$$y_2(x) = e^{m_2 x} = e^{\left(-\frac{1}{3} + \frac{\sqrt{2}}{3} i\right) x}$$

($\{y_1, y_2\}$ is a lin. indep. set since

$$\frac{y_1}{y_2} = \frac{e^{\left(-\frac{1}{3} - \frac{\sqrt{2}}{3} i\right) x}}{e^{\left(-\frac{1}{3} + \frac{\sqrt{2}}{3} i\right) x}} = \frac{e^{-x/3}\, e^{i\sqrt{2}x/3}}{e^{-x/3}\, e^{i\sqrt{2}x/3}} = e^{-i\frac{\sqrt{2}}{3}x - i\frac{\sqrt{2}}{3}x}$$

$$= e^{-i\frac{2\sqrt{2}}{3}x} \neq \text{constant}.)$$

General Soln.:

$$y = c_1 e^{\left(-\frac{1}{3} - \frac{\sqrt{2}}{3} i\right) x} + c_2 e^{\left(-\frac{1}{3} + \frac{\sqrt{2}}{3} i\right) x}$$

However, BY CONVENTION, we want $y(x)$ to appear as the linear combination of of REAL, NOT UNREAL, FUNCTIONS (with the arbitrary constants now being real or complex).

To do this we consider

EULER'S FORMULA
(in complex analysis - accept on faith)

$$e^{i\theta} \stackrel{\text{def}}{=} \cos\theta + i\sin\theta \quad , \quad \theta \text{ a real number}$$

$$\left(e^{-i\theta} = e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta) \right.$$
$$= \cos\theta + i(-\sin\theta)$$
$$\left. = \cos\theta - i\sin\theta \right)$$

$$\therefore \quad e^{\left(-\frac{1}{3} - \frac{\sqrt{2}}{3}i\right)x} = e^{-\frac{1}{3}x} \cdot e^{-\frac{\sqrt{2}}{3}ix}$$
$$= e^{-\frac{1}{3}x} e^{-i\frac{\sqrt{2}}{3}x}$$
$$= e^{-\frac{1}{3}x}\left(\cos\frac{\sqrt{2}}{3}x - i\sin\frac{\sqrt{2}}{3}x\right)$$

$$e^{\left(-\frac{1}{3} + \frac{\sqrt{2}}{3}i\right)x} = e^{-\frac{1}{3}x}\left(\cos\frac{\sqrt{2}}{3}x + i\sin\frac{\sqrt{2}}{3}x\right)$$

$$\therefore \quad y(x) = c_1 e^{-\frac{1}{3}x}\left(\cos\frac{\sqrt{2}}{3}x - i\sin\frac{\sqrt{2}}{3}x\right) + c_2 e^{-\frac{1}{3}x}\left(\cos\frac{\sqrt{2}}{3}x + i\sin\frac{\sqrt{2}}{3}x\right)$$

$$= \underbrace{(c_1 + c_2)}_{\text{REPLACE BY NEW } c_1} e^{-\frac{1}{3}x}\cos\frac{\sqrt{2}}{3}x + \underbrace{i(c_2 - c_1)}_{\text{REPLACE BY NEW } c_2} e^{-\frac{1}{3}x}\sin\frac{\sqrt{2}}{3}x \Rightarrow$$

$$\boxed{y = c_1 e^{-\frac{1}{3}x}\cos\frac{\sqrt{2}}{3}x + c_2 e^{-\frac{1}{3}x}\sin\frac{\sqrt{2}}{3}x}$$

In general :

If $y_1 = e^{(\alpha - i\beta)x}$ and $y_2 = e^{(\alpha + i\beta)x}$

then the gen. soln. is written as

$$y = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x$$

Two Distinct Real Roots

Example. (HW Exercise 8, p. 119.)

Find the general solution of

$$y'' - 3y' + 2y = 0$$

$y = e^{mx}$; $y'' - 3y' + 2y = 0 \implies$

$$m^2 \cancel{e^{mx}} - 3m\cancel{e^{mx}} + 2\cancel{e^{mx}} = 0 \implies$$

$$m^2 - 3m + 2 = 0 \implies$$

$$(m-1)(m-2) = 0 \implies$$

$$m_1 = 1, \quad m_2 = 2 \implies$$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \implies$$

$$\boxed{y = c_1 e^{x} + c_2 e^{2x}}$$ DONE!

Repeated Roots

Example. (HW Exercise 9, p. 119.)

Find the general solution of

$$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0.$$

$y = e^{mx}$: $\quad y'' + 8y' + 16y = 0 \Rightarrow$

$$m^2\cancel{e^{mx}} + 8m\cancel{e^{mx}} + 16\cancel{e^{mx}} = 0 \Rightarrow$$

$$m^2 + 8m + 16 = 0 \Rightarrow$$

$$(m+4)(m+4) = 0 \Rightarrow$$

$$m_1 = -4, \quad m_2 = -4$$

-4 IS A REPEATED ROOT

$\therefore$ $y_1 = e^{-4x}$ and $y_2 = e^{-4x}$ so that the general solution is

$$y = c_1 e^{-4x} + c_2 e^{-4x} \; ?$$

NO!

With $y_1 = e^{-4x}$ and $y_2 = e^{-4x}$, $\{y_1, y_2\}$ is linearly dependent, so no general solution can be formed from y_1 and y_2.

We have one solution,

$$y_1 = e^{-4x}.$$

We need a second solution, y_2.

It turns out that a second solution can still be found from $y_1 = e^{-4x}$ using the

REDUCTION OF ORDER FORMULA (4.2)

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

$$(y'' + P(x)y' + Q(x)y = 0)$$

So, we have

$$y_2 = e^{-4x} \int \frac{e^{-\int 8\,dx}}{(e^{-4x})^2}\,dx$$

$$= e^{-4x \cdot 2} = e^{-8x}$$

$$= e^{-4x} \int \frac{e^{-8x}}{e^{-8x}}\, dx$$

$$= e^{-4x} \int 1\, dx$$

$$= e^{-4x} \left(x \cancel{+ C} \right) \Rightarrow$$

$$y_2 = x e^{-4x}$$

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$\therefore$ General soln.:

$$\boxed{y = c_1 e^{-4x} + c_2 x e^{-4x}}$$

NOTE: For second-order DEs, always "tag on" an x to make the second soln. distinct.
If you have something like

$$y^{(4)} + 16y''' + 96y'' + 256y' + 256y = 0 \Rightarrow$$

$$m^4 + 16m^3 + 96m^2 + 256m + 256 = 0 \Rightarrow$$

$$(m+4)^4 = 0 \Rightarrow$$

$$m_1 = m_2 = m_3 = m_4 = -4,$$

then "tag on" increasing powers of x:

$y_1 = e^{-4x}$, $y_2 = xe^{-4x}$, $y_3 = x^2e^{-4x}$,

$y_4 = x^3e^{-4x}$.