Lecture

$\boxed{\text{CHAPTER 4}}$ DEs of Higher Order

We will look at specific methods of solution of

(1) SCALAR nth-order DEs

primarily 2nd-order DEs for simplicity

(2) SYSTEMS of n 1st-order DEs

We will SKIP Sections 4.5 (an alternative method to the one given in Section 4.4) and 4.7 (on a specific type of DE

Lecture

Section 4.1. Preliminary Theory: Linear Equations.

I will primarily state RESULTS WITHOUT PROOF.

This section made up of 3 subsections:

I. Existence and uniqueness of solutions of

A. IVPs (initial value problems)
B. BVPs (boundary value problems)

II. Can have 0, 1, or >1 soln.

0: Soln. does not exist
1: Soln. exists and is unique
>1: Soln. is not unique

II. Homogeneous (linear) DEs ($\neq$ DE in 2.4 where have $Mdx + Ndy = 0$ and can make substitution $y = ux$): $a_1 y^{(n)} + \cdots + a_0 y = \underline{\underline{0}}$

A. Superposition principle:
Sum of solns. of a linear DE is also a soln.

B. Linear dependence/independence:
nth-order DE has n DISTINCT (i.e., linearly independent) solns.

C. General soln. of a homogeneous linear nth-order DE:

"Linear combination" of n
- linearly independent solns.
of DE or

$$\underbrace{y(x) = c_1 y_1(x) + \cdots + c_2 y_2(x)}_{\text{GEN. SOLN.}}$$

III. Nonhomogeneous (linear) DEs :

$$a_1 y^{(n)} + \cdots + a_0 y = \underbrace{g(x)}_{\text{not } \underbrace{\text{identically equal to } 0}_{\text{means equal to } 0 \text{ for all } x}}$$

A. General soln. :
soln. of homogeneous version +
particular soln.

$$a_1 y^{(n)} + \cdots + a_0 y = 0 \leftarrow y_c$$

$$a_1 y^{(n)} + \cdots + a_0 y = g(x) \leftarrow y_p$$

$$y(x) = y_c(x) + y_p(x)$$

B. Superposition principle :
Sum of particular solns. is also
a particular soln.

Lecture

Subsection 4.1.1. Initial Value and Boundary Value Problems (IVPs and BVPs)

Linear nth-order IVP:

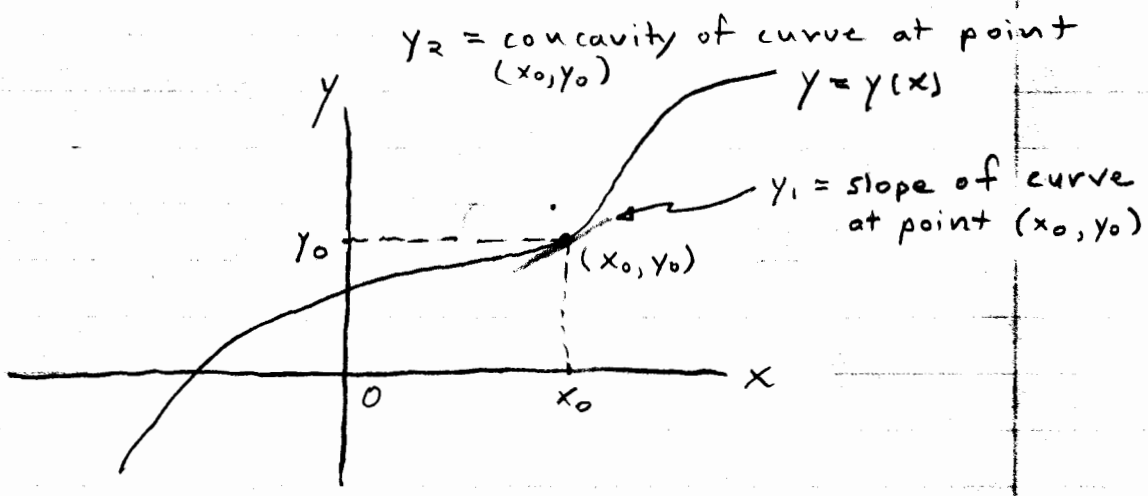
$$\begin{cases} a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x) \\ \left.\begin{array}{l} y(x_0) = y_0 \\ y'(x_0) = y_1 \\ \vdots \\ y^{(n-1)}(x_0) = y_{n-1} \end{array}\right\} \text{ n INITIAL CONDITIONS} \end{cases}$$

General Soln.: Will end up with n arbitrary constants $C_1, C_2, \ldots, C_n$. In a sense, these constants are "constants of integration" which come from the "n integrations" of $y^{(n)} = d^n y/dx^n$ to obtain y:

$$y^{(n)} = -\frac{a_{n-1}(x)}{a_n(x)}y^{(n-1)} - \cdots - \frac{a_1(x)}{a_n(x)}y' - \frac{a_0(x)}{a_n(x)} + \frac{g(x)}{a_n(x)}$$

Particular Soln.: Need n initial conditions to solve for $C_1, C_2, \ldots, C_n$.

Solution Curve of IVP:



$y_3 - y_{n-1}$ further describe curve and its curvature at point (x_0, y_0)

<u>Note</u>. When we say

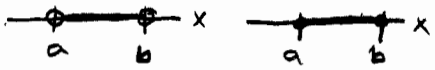
CONSIDER THE IVP ON THE INTERVAL I

we will mean

CONSIDER THE PART. SOLN. $y = y(x)$ AS HOLDING FOR ALL x IN I (AND I ONLY)

I will be of the form

(a, b) $\quad$ $[a, b]$



(a, ∞) $\quad$ $[a, \infty)$

$(-\infty, a)$ $\quad$ $(-\infty, a]$

open intervals $\quad$ closed intervals

Existence and Uniqueness of a Linear nth-Order IVP

PICARD'S EXISTENCE AND UNIQUENESS THEOREM (for Linear nth-Order IVPs).

Consider the following IVP on the interval I:

$$(*)\begin{cases} a_n(x)y^{(n)} + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y' + a_0(x)y = g(x) \\ y(x_0) = y_0 \\ y'(x_0) = y_1 \\ \vdots \\ y^{(n-1)}(x_0) = y_{n-1} \end{cases}$$

IF (1) $a_0(x), a_1(x), \ldots, a_n(x), g(x)$ are all continuous on I

(2) $a_n(x) \neq 0$ on I

THEN $(*)$ has exactly one particular solution (i.e., there exists a part. soln. and it is unique or the only solution that works).

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<u>Examples.</u>

1. (HW Exercise 5, p. 107.)

Given that

$$y = c_1 x + c_2 x \ln x$$

is a 2-parameter (c_1, c_2) family of solns. of

$$x^2 y'' - x y' + y = 0 \quad \text{on the interval } (-\infty, \infty),$$

Find a member of the family satisfying the initial conditions

$$y(1) = 3, \quad y'(1) = -1.$$

Let $\boxed{x_0 = 1}$.

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Replace $(-\infty, \infty)$ by an interval I such that (1) x_0 is contained in I, and (2) $y = c_1 x + c_2 x \ln x$ or $\ln x$ is defined on I:

Choose $\boxed{I = (0, \infty)}$

Then have IVP

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$$\begin{cases} x^2 y'' - x y' + y = 0 & \text{on } I = (0, \infty) \\ \quad a_2(x) = x^2 \quad a_1(x) = -x \quad a_0(x) = 1 \quad g(x) = 0 \\ y(1) = 3 \\ y'(1) = -1 \end{cases}$$

1. $a_0(x)$, $a_1(x)$, $a_2(x)$, $g(x)$ all cont. on $I = (0, \infty)$

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2. $a_2(x) \neq 0$ on $I = (0, \infty)$

$\therefore$ This IVP has exactly one soln. We can find it by solving for c_1 and c_2:

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$$y(x) = c_1 x + c_2 x \ln x$$

$$y'(x) = c_1 + c_2\left(x \cdot \frac{1}{x} + 1 \cdot \ln x\right) = c_1 + c_2(1 + \ln x)$$

$$y(1) = 3 \Rightarrow 3 = y(1) = c_1(1) + c_2(1)\overbrace{\ln(1)}^{=0}$$
$$\Rightarrow 3 = c_1 + 0$$
$$\Rightarrow c_1 = 3$$

$$y'(1) = -1 \Rightarrow -1 = y'(1) = \overbrace{c_1}^{=3} + c_2(1 + \overbrace{\ln(1)}^{=0})$$
$$\Rightarrow -1 = 3 + c_2$$
$$\Rightarrow c_2 = -4$$

$\therefore$ Part. soln. is $\boxed{y(x) = 3 - 4x\ln x}$

2. (HW Exercise 6, p. 107.)

Given that

$$y = c_1 + c_2 x^2 \quad .$$

is a 2-parameter (c_1, c_2) family of solns. of

$$xy'' - y' = 0 \quad \text{on the interval } (-\infty, \infty),$$

show that no member of the family satisfies the initial conditions

$$y(0) = 0, \quad y'(0) = 1.$$

Explain why this does not violate PICARD'S THEOREM.

Let $\boxed{x_0 = 0}$

Choose interval I such that (1) x_0 is contained in I, and (2) $y = c_1 + c_2 x^2$ is defined on I:

$$\text{Choose } \boxed{I = (-\infty, \infty)}$$

Then have IVP

$$\begin{cases} xy'' - y = 0 & \text{on } I = (-\infty, \infty) \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

$$\underbrace{x}_{a_2(x)=x}y'' + \underbrace{0}_{a_1(x)=0}\cdot y' \underbrace{-}_{a_0(x)=-1} y = \underbrace{0}_{g(x)=0}$$

1. $a_0(x), a_1(x), a_2(x), g(x)$ all cont. on $I = (-\infty, \infty)$
2. It is <u>not</u> true that $a_2(x) \neq 0$ on $I = (-\infty, \infty)$ since $a_2(x) = 0$ when $x = 0$, which is contained in I.

$\therefore$ Cannot say whether or not this IVP has a solution and, if it does, whether or not the solution is the only one.

Try to solve for c_1 and c_2 anyway:

$$y(x) = c_1 + c_2 x^2$$

$$y'(x) = 2c_2 x$$

$$y(0) = 0 \Rightarrow 0 = y(0) = c_1 + c_2(0)^2$$
$$\Rightarrow c_1 = 0$$

$$y'(0) = 1 \Rightarrow 1 = y'(0) = 2c_2(0)$$
$$\Rightarrow 1 = 0 \quad \text{IMPOSSIBLE!}$$

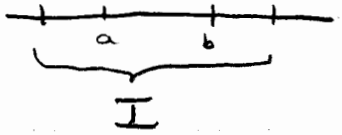
$\therefore$ There can be no particular solution. ✓

Existence and Uniqueness of a Linear 2nd-Order Boundary Value Problem (BVP)

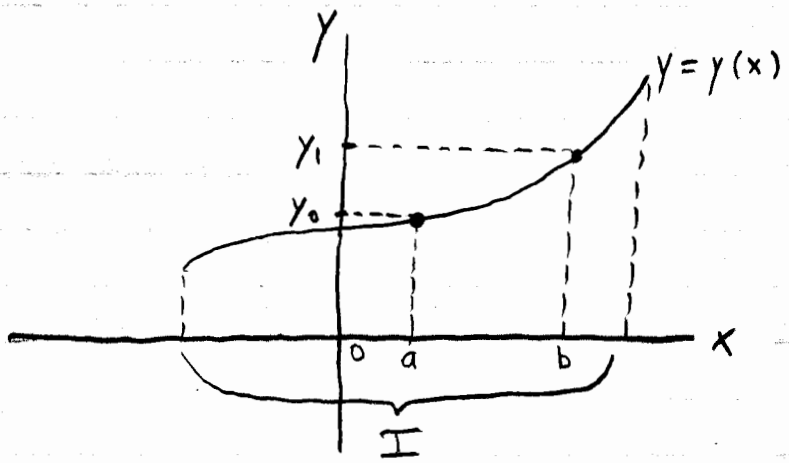
Linear 2nd-order BVP :

$$\begin{cases} a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \quad \text{on} \quad I \\ \left.\begin{array}{l} y(a) = y_0 \\ y(b) = y_1 \end{array}\right\} \text{2 BOUNDARY CONDITIONS} \end{cases}$$

$x = a$ and $x = b$ are contained in the interval I



Solution curve :



<u>Note</u>: The solution $y(x)$ of the DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

is composed of 2 parts.

$$y(x) = \underbrace{c_1 y_1(x) + c_2 y_2(x)}_{} + \underbrace{y_p(x)}_{}$$

y_1 and y_2 are each solns. of the DE

$a_2 y'' + a_1 y' + a_0 y = 0$

y_p is a specific soln. of the DE

$a_2 y'' + a_1 y' + a_0 y = g(x)$

<u>Theorem</u>.

Consider the following BVP on the interval I:

$$(*) \begin{cases} a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \\ y(a) = y_0 \\ y(b) = y_1 \end{cases}$$

Suppose the soln. of the DE is

$$y = c_1 y_1 + c_2 y_2 + y_p$$

Consider the system of algebraic equations

$$(**)\quad \begin{cases} c_1 y_1(a) + c_2 y(a) = y_0 - y_p(a) \\ c_1 y_1(b) + c_2 y(b) = y_1 - y_p(b) \end{cases}$$

to be solved for c_1 and c_2.

THEN the BVP $(*)$ has as many solutions as the system $(**)$.

So, one of the following is true:

(1) The BVP $(*)$ has NO soln.
(2) The BVP $(*)$ has EXACTLY ONE soln.
(3) The BVP $(*)$ has MORE THAN ONE soln. (usually infinitely many solns.)

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<u>Example.</u>

DE: $y'' + y = 0$

General solution: $y(x) = c_1 \cos x + c_2 \sin x$

$\boxed{\text{Case 1}}$: $y(0) = 2, \quad y(\pi) = 1$

Then $c_1 = 2$, $c_1 = -1$ $\Rightarrow$ IMPOSSIBLE $\Rightarrow$ BVP has <u>no soln.</u>

$\boxed{\text{Case 2}}$: $y(0) = 2$, $y(\frac{\pi}{2}) = 3$

Then $c_1 = 2$, $c_2 = 3$ $\Rightarrow$ BVP has <u>exactly one soln.</u>

3/21/01 W lecture $\boxed{\text{Case 3}}$: $y(0) = 2$, $y(\pi) = -2$

Then $c_1 = 2$, $c_1 = 2$ $\Rightarrow$ $c_1 = 2$ and c_2 is arbitrary $\Rightarrow$ BVP has <u>infinitely many solns.</u>

$\boxed{\text{Case 2}}$: $y(0) = 2$, $y(\frac{\pi}{2}) = 3$

$$2 = y(0) = c_1 \overbrace{\cos 0}^{=1} + c_2 \overbrace{\sin 0}^{=0} \Longrightarrow$$

$$2 = c_1$$

$$3 = y(\tfrac{\pi}{2}) = c_1 \overbrace{\cos \tfrac{\pi}{2}}^{=0} + c_2 \overbrace{\sin \tfrac{\pi}{2}}^{=1} \Longrightarrow$$

$$3 = c_2$$