CHAPTER 4  DEs of Higher Order

We will look at specific methods of solution of

(1) SCALAR $n$th-order DEs

primarily 2nd-order DEs

for simplicity

(2) SYSTEMS of $n$ 1st-order DEs

We will SKIP Sections 4.5 (an alternative method to the one given in Section 4.4) and 4.7 (on a specific type of DE)
Section 4.1. Preliminary Theory: Linear Equations.

I will primarily state RESULTS WITHOUT PROOF.

This section made up of 3 subsections:

I. Existence and uniqueness of solutions of
   A. IVPs (initial value problems)
   B. BVPs (boundary value problems)

II. Can have 0, 1, or > 1 soln.

   Soln. does not exist
   Soln. exists and is unique
   Soln. is not unique

II. Homogeneous (linear) DEs (≠ DE in 2.4 when they have $a_0x + a_1y = 0$ and can make substitution $y = ux$):

   $a_0y^{(n)} + \cdots + a_ny = 0$

   A. Superposition principle:
      Sum of solns of a linear DE is also a soln.
   B. Linear dependence/independence:
      nth-order DE has $n$ DISTINCT (i.e., linearly independent) solns.
   C. General soln. of a homogeneous linear nth-order DE;
"Linear combination" of n linearly independent solns. of DE or
\[ y(x) = C_1 y_1(x) + \ldots + C_n y_n(x) \]

GEN. SOLN.

III. Nonhomogeneous (linear) DEs:
\[ a_1 y^{(n)} + \ldots + a_n y = g(x) \]
not identically equal to 0

means equal to 0 for all \( x \)

A. General soln.:
Soln. of homogeneous version + particular soln.
\[ a_1 y^{(n)} + \ldots + a_n y = 0 \quad \rightarrow \quad y_c \]
\[ a_1 y^{(n)} + \ldots + a_n y = g(x) \quad \rightarrow \quad y_p \]
\[ y(x) = y_c(x) + y_p(x) \]

B. Superposition principle:
Sum of particular solns. is also a particular soln.
Subsection 4.1.1. Initial Value and Boundary Value Problems (IVPs and BVPs)

Linear $n$th-order IVP:

\[
\begin{align*}
  & a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x) \\
  & y(x_0) = y_0 \\
  & y'(x_0) = y_1 \\
  & \vdots \\
  & y^{(n-1)}(x_0) = y_{n-1}
\end{align*}
\]

$\text{n INITIAL CONDITIONS}$

General Solution: Will end up with $n$ arbitrary constants $C_1, C_2, \ldots, C_n$. In a sense, these constants are "constants of integration" which come from the "$n$ integrations" of $y^{(n)} = \frac{d^n y}{dx^n}$ to obtain $y$:

\[
y^{(n)} = -\frac{a_{n-1}(x)}{a_n(x)} y^{(n-1)} - \cdots - \frac{a_1(x)}{a_n(x)} y' - \frac{a_0(x)}{a_n(x)} y + \frac{g(x)}{a_n(x)}
\]

Particular Solution: Need $n$ initial conditions to solve for $C_1, C_2, \ldots, C_n$. 

Solution Curve of IVP:

- $y_2 = \text{concavity of curve at point } (x_0, y_0)$
- $y = y(x)$
- $y_1 = \text{slope of curve at point } (x_0, y_0)$

$y_2 - y_{n-1}$ further describe curve and its curvature at point $(x_0, y_0)$. 
Note: When we say

\textbf{Consider the IVP on the interval} I

we will mean

\textbf{Consider the Part. Soln.} \ y = y(x) \ as \ holding \ for \ all \ x \ in \ I \ (and \ I \ only)

I will be of the form

\begin{align*}
(a, b) & \quad [a, b] \\
(a, \infty) & \quad [a, \infty) \\
(-\infty, a) & \quad (-\infty, a] \\
\uparrow & \quad \uparrow \\
\text{open intervals} & \quad \text{closed intervals}
\end{align*}
Existence and Uniqueness of a Linear nth-Order IVP

Picard's Existence and Uniqueness Theorem (for linear nth-Order IVPs).

Consider the following IVP on the interval $I$:

$$
\begin{cases}
  a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots + a_1(x) y' + a_0(x) y = g(x) \\
y(x_0) = y_0 \\
y'(x_0) = y_1 \\
\vdots \\
y^{(n-1)}(x_0) = y_{n-1}
\end{cases}
$$

If
1. $a_0(x), a_1(x), \ldots, a_n(x), g(x)$ are all continuous on $I$
2. $a_n(x) \neq 0$ on $I$

then $(*)$ has exactly one particular solution (i.e., there exists a particular solution and it is unique or the only solution that works).
Examples.

1. (HW Exercise 5, p. 107.)

Given that

\[ y = c_1 x + c_2 x \ln x \]

is a 2-parameter \((c_1, c_2)\) family of solutions of

\[ x^2 y'' - xy' + y = 0 \]

on the interval \((-\infty, \infty)\).

Find a member of the family satisfying the initial conditions

\[ y(1) = 3, \quad y'(1) = -1. \]

Let \(x_0 = 1\).

Replace \((-\infty, \infty)\) by an interval \(I\) such that

1. \(x_0\) is contained in \(I\), and
2. \(y = c_1 x + c_2 x \ln x\) or \(x \ln x\) is defined on \(I\).

Choose \(I = (0, \infty)\).

Then have IVP
\[ \begin{aligned} x^2 y'' - xy' + y &= 0 \quad \text{on} \quad I = (0, \infty) \\
\frac{a_2(x)}{x^2} &= a_1(x) = -x \quad a_2(x) = 1 \\
g(x) &= 0 \\
y(1) &= 3 \\
y'(1) &= -1 \\
1. \quad a_0(x), a_1(x), a_2(x), g(x) \text{ all cont.}
\text{on} \quad I = (0, \infty) \\
2. \quad a_2(x) \neq 0 \quad \text{on} \quad I = (0, \infty) \\
\end{aligned} \]

This IVP has exactly one soln. We can find it by solving for \( c_1 \) and \( c_2 \):

\[ y(x) = c_1 x + c_2 x \ln x \]

\[ y'(x) = c_1 + c_2 \left( x \cdot \frac{1}{x} + 1 \cdot \ln x \right) = c_1 + c_2 \left( 1 + \ln x \right) \]

\[ \begin{align*}
y(1) &= 3 \\
3 &= y(1) = c_1(1) + c_2(1) \ln(1) \\
3 &= c_1 + 0 \\
c_1 &= 3
\end{align*} \]

\[ \begin{align*}
y'(1) &= -1 \\
-1 &= y'(1) = c_1 + c_2 \left( 1 + \ln(1) \right) \\
-1 &= 3 + c_2 \\
c_2 &= -4
\end{align*} \]

\[ \text{Port. soln. is } y(x) = 3 - 4x \ln x \]
2. (HW Exercise 6, p. 107.)

Given that

\[ y = c_1 + c_2 x^2 \]

is a 2-parameter \((c_1, c_2)\) family of solns.

of

\[ xy'' - y' = 0 \quad \text{on the interval} \quad (-\infty, \infty), \]

show that no member of the family satisfies the initial conditions

\[ y(0) = 0, \quad y'(0) = 1. \]

Explain why this does not violate Picard's Theorem.

Let \[ x_0 > 0 \]

Choose interval \( I \) such that (1) \( x_0 \) is contained in \( I \), and (2) \( y = c_1 + c_2 x^2 \) is defined on \( I \):

Choose \[ I = (-\infty, \infty) \]

Then have IVP
\[ \begin{aligned}
    xy'' - y &= 0 \quad \text{on } I = (-\infty, \infty) \\
y(0) &= 0 \\
y'(0) &= 1
\end{aligned} \]

\[ xy'' + 0 \cdot y' - y = 0 \]
\[ a_0(x) = x \quad a_1(x) = 0 \quad a_2(x) = -1 \quad g(x) = 0 \]

1. \( a_0(x), a_1(x), a_2(x), g(x) \) all cont. on \( I = (-\infty, \infty) \)

2. It is not true that \( a_0(x) \neq 0 \) on \( I = (-\infty, \infty) \) since \( a_2(x) = 0 \) when \( x = 0 \), which is contained in \( I \).

Cannot say whether or not this IVP has a solution and, if it does, whether or not the solution is the only one.

Try to solve for \( c_1 \) and \( c_2 \) anyway:

\[ y(x) = c_1 + c_2 x^2 \]
\[ y'(x) = 2c_2 x \]
\[ y(0) = 0 \Rightarrow 0 = y(0) = c_1 + c_2(0)^2 \]
\[ \therefore c_1 = 0 \]

\[ y'(0) = 1 \Rightarrow 1 = y'(0) = 2c_2(0) \]
\[ \therefore 1 = 0 \quad \text{IMPOSSIBLE!} \]

Therefore, there can be no particular solution.
Existence and Uniqueness of a Linear 2nd-Order Boundary Value Problem (BVP)

Linear 2nd-order BVP:

\[ \begin{align*}
    a_2(x) y'' + a_1(x) y' + a_0(x) y &= g(x) \quad \text{on } I \\
    y(a) &= y_0 \\
    y(b) &= y_1
\end{align*} \]

2 boundary conditions

\( x = a \) and \( x = b \) are contained in the interval \( I \)

Solution curve:

\[ y = y(x) \]
Note: The solution $y(x)$ of the DE

$$a_3(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

is composed of 2 parts.

$$y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x)$$

$y_1$ and $y_2$ are each solns. of the DE

$$a_2y'' + a_1y' + a_0y = 0$$

$y_p$ is a specific soln. of the DE

$$a_2y'' + a_1y' + a_0y = g(x)$$

Theorem.

Consider the following BVP on the interval $I$:

$$\begin{cases}
a_3(x)y'' + a_1(x)y' + a_0(x)y = g(x) \\
y(a) = y_0 \\
y(b) = y_1
\end{cases}$$

(*)

Suppose the soln. of the DE is

$$y = c_1y_1 + c_2y_2 + y_p$$
Consider the system of algebraic equations

\[
\begin{align*}
& c_1 y_1(a) + c_2 y_2(a) = y_0 - \gamma(a) \\
& c_1 y_1(b) + c_2 y_2(b) = y_1 - \gamma(b)
\end{align*}
\]

to be solved for \( c_1 \) and \( c_2 \).

Then the BVP (*) has as many solutions as the system (**).

So, one of the following is true:

1. The BVP (*) has NO soln.
2. The BVP (*) has EXACTLY ONE soln.
3. The BVP (*) has MORE THAN ONE soln. (usually infinitely many solns.)
Example.

DE: \(y'' + y = 0\)

General solution: \(y(x) = c_1 \cos x + c_2 \sin x\)

**Case 1:** \(y(0) = 2, \ y(\pi) = 1\)

Then \(c_1 = 2, \ c_1 = -1 \Rightarrow \text{IMPOSSIBLE} \Rightarrow \text{BVP has no soln.}\)

**Case 2:** \(y(0) = 2, \ y(\pi) = 3\)

Then \(c_1 = 2, \ c_2 = 3 \Rightarrow \text{BVP has exactly one soln.}\)

**Case 3:** \(y(0) = 2, \ y(\pi) = -2\)

Then \(c_1 = 2, \ c_1 = 2 \Rightarrow c_1 = 2 \text{ and } c_2 \text{ is arbitrary} \Rightarrow \text{BVP has infinitely many solns.}\)
Case 2: \( y(0) = 2, \quad y'(\frac{\pi}{2}) = 3 \)

\[ 2 = y(0) = c_1 \cos 0 + c_2 \sin 0 \implies 2 = c_1 \]

\[ 3 = y'(\frac{\pi}{2}) = c_1 \cos \frac{\pi}{2} + c_2 \sin \frac{\pi}{2} \implies 3 = c_2 \]