Lecture

Section 2.4 – Solutions by Substitution.

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Three parts:

(1) Substitution with a homogeneous DE

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

(2) " " " Bernoulli DE

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

(3) " " " Potentially Separable DE

$$\frac{dy}{dx} = f(Ax + By + C)$$

For an analogy, recall INTEGRATION BY SUBSTITUTION:

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du$$

Let $u = g(x)$. Then $\frac{du}{dx} = g'(x) \Rightarrow du = g'(x)\,dx$

$$= F(u) + C$$

Where F is the antiderivative of f (i.e., $F' = f$)

$$= F(g(x)) + C$$

We can <u>transform</u> an integral into one that is <u>more easily solved</u> (i.e., solved by already known techniques) by a "<u>change of variable</u>" or "<u>substitution</u>."

E.g., $\int 2xe^{x^2}\,dx = \int e^u\,du = e^u + C = \boxed{e^{x^2}+C}$

let $\boxed{u = x^2}$. Then $\frac{du}{dx} = 2x \Rightarrow \boxed{du = 2x\,dx}$

We can do <u>something similar with DEs</u>, to <u>reduce them to DEs</u> that we <u>already know how to solve</u>.

<u>Homogeneous DEs</u>

<u>Defn.</u> $f(x,y)$ is said to be a <u>homogeneous function of degree α</u>, where α is some real number, if, when we substitute cx for x and cy for y with c any constant, we get

$$f(cx, cy) = c^{\alpha} f(x,y)$$

E.g., $f(x,y) = 5x^{1/3}y - 3xy^{1/3}$

$$\begin{aligned} f(cx,cy) &= 5(cx)^{1/3}(cy) - 3(cx)(cy)^{1/3} \\ &= 5c^{1/3}\cdot c\,x^{1/3}y - 3c\cdot c^{1/3}xy^{1/3} \\ &= 5c^{4/3}x^{1/3}y - 3c^{4/3}xy^{1/3} \\ &= c^{4/3}(5x^{1/3}y - 3xy^{1/3}) \\ &= c^{4/3}f(x,y) \end{aligned}$$

$\therefore$ $f(x,y)$ is a homogeneous function of degree $\frac{1}{3}$.

E.g., $f(x,y) = x^3 + y^2$

$$\begin{aligned} f(cx,cy) &= (cx)^3 + (cy)^2 \\ &= c^3x^3 + c^2y^2 \\ &= c^2(cx^3 + y^2) \\ &\neq c^2 f(x,y) \end{aligned}$$

$\therefore$ $f(x,y)$ is NOT a homogeneous function.

Roughly speaking,

A homogeneous function of degree α is a function of the form

$$f(x,y) = x^{m_1}y^{n_1} + x^{m_2}y^{n_2} + \cdots + x^{m_i}y^{n_i}$$

where

$$m_1+n_1 = m_2+n_2 = \cdots = m_i+n_i = \alpha$$

<u>Defn.</u> A first-order DE of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be <u>homogeneous</u> if (1) $M(x,y)$ and $N(x,y)$ are both homogeneous functions of the same degree.

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E.g., $\underbrace{(x-y)}_{M(x,y)}\,dx + \underbrace{(x+y)}_{N(x,y)}\,dy$

$M(x,y) = x-y = x'-y' \Rightarrow$ homog. of degree 1

$N(x,y) = x+y = x'+y' \Rightarrow$ homog. of degree 1

$\therefore$ DE is homogeneous.

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<u>Method of Solution of Homogeneous DEs</u>

With a homogeneous DE, we will make either of the following substitutions:

(1) $\boxed{y = ux}$ (i.e., $y(x) = u(x)\,x$)

Then $\frac{dy}{dx} = \frac{d}{dx}(ux) = u \cdot 1 + \frac{du}{dx} \cdot x \Rightarrow$

PRODUCT RULE $(fg)' = fg' + f'g$

$$\frac{dy}{dx} = u + x\frac{du}{dx} \Rightarrow \boxed{dy = u\,dx + x\,du}$$

- Used when N is simpler than M (since replacing dy with a messier expression).
- End up with DE in $u(x)$ and x
- Solve for u in terms of x
- Substitute back $u = y/x$
- Homogeneity allows for factoring and cancelling of x

(2) $\boxed{x = vy}$ (i.e., $x = v(x)y(x)$)

Then $\frac{d}{dx}(x) = \frac{d}{dx}(vy) = v\frac{dy}{dx} + \frac{dv}{dx} \cdot y \Rightarrow$

$$1 = v\frac{dy}{dx} + \frac{dv}{dx} \cdot y \Rightarrow \boxed{dx = v\,dy + y\,dv}$$

- Used when M is simpler than N (since replacing dx with a messier expression).
- End up with DE in $v(x)$ and y
- Solve for y in terms of v
- Substitute back $v = x/y$
- Homogeneity allows for factoring and cancelling of x

<u>Examples.</u>

1. (Exercise 1, p. 57.) Solve the homogeneous equation

$$(x-y)\,dx + x\,dy = 0$$

by using an appropriate substitution.

<u>Note</u>: DE is homogeneous since

$$\left.\begin{array}{l} M(x,y) = x^1 - y^1 \Rightarrow \text{homog. of degree } 1 \\ N(x,y) = x^1 \Rightarrow \text{homog. of degree } 1 \end{array}\right\} \text{same degree}$$

<u>Note</u>: $N(x,y) = x$ is "simpler" than $M(x,y) = x - y \Rightarrow$ choose substitution with

$$\boxed{y = ux} \qquad \boxed{dy = u\,dx + x\,du}$$

$$(x-y)\,dx + x\,dy = 0 \Rightarrow$$

$$(x-ux)\,dx + x(u\,dx + x\,du) = 0 \Rightarrow$$

$$\cancel{x}(1-u)\,dx + \cancel{x}(u\,dx + x\,du) = 0 \Rightarrow$$

Assuming $x \neq 0$

$$(1-u)\,dx + u\,dx + x\,du = 0 \Rightarrow$$

So the homogeneity of degree 1 of both M and N has allowed us to factor out an x and then cancel the x

$$(1-\cancel{u}+\cancel{u})\,dx + x\,du = 0 \Rightarrow$$

$$dx + x\,du = 0 \Rightarrow$$

Looks like a separable eq.

$$x\,du = -dx \Rightarrow$$

$$du = -\frac{1}{x}\,dx \Rightarrow$$

$$\int 1 \cdot du = -\int \frac{1}{x}\,dx \Rightarrow$$

$$u = -\ln|x| + C \Longrightarrow$$

Substitute back:
$y = ux \Rightarrow u = \frac{y}{x}$

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$$\boxed{\frac{y}{x} = -\ln|x| + C} \Longrightarrow$$

$$\boxed{y = -x\ln|x| + Cx}$$

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2. (Exercise 3, p. 57.) Solve the homogeneous equation

$$x\,dx + (y - 2x)\,dy = 0$$

by using an appropriate substitution.

<u>Note</u>: DE is homogeneous since

$M(x,y) = x^1 \Longrightarrow$ homog. of degree 1
$N(x,y) = y^1 - 2x^1 \Longrightarrow$ homog. of degree 1
} same degree

<u>Note</u>: $M(x,y) = x$ is "simpler" than $N(x,y) = y - 2x \Longrightarrow$ choose substitution with

$$\boxed{x = vy} \qquad \boxed{dx = v\,dy + y\,dv}$$

Actually, we are going to use the other inappropriate substitution,

$$\boxed{y = ux} \qquad \boxed{dy = u\,dx + x\,du} \checkmark$$

$$x\,dx + (y - 2x)\,dy = 0 \Longrightarrow$$

$$x\,dx + (ux - 2x)(u\,dx + x\,du) = 0 \Longrightarrow$$

$$\cancel{x}\,dx + \cancel{x}(u-2)(u\,dx + x\,du) = 0 \Longrightarrow$$

Assuming $x \neq 0$

$$dx + (u-2)(u\,dx + x\,du) = 0 \Longrightarrow$$

$$dx + u^2\,dx + u\,x\,du - 2u\,dx - 2x\,du = 0 \Longrightarrow$$

$$(1 + u^2 - 2u)\,dx + (ux - 2x)\,du = 0 \Longrightarrow$$

$$(u^2 - 2u + 1)\,dx + x(u-2)\,du = 0 \Longrightarrow$$

$$(u-1)^2\,dx + x(u-2)\,du = 0 \Longrightarrow$$

Looks like a separable eq.

$$x(u-2)\,du = -(u-1)^2\,dx \Longrightarrow$$

$$\frac{u-2}{(u-1)^2}\,du = -\frac{1}{x}\,dx \Longrightarrow$$

$$\boxed{\int \frac{u-2}{(u-1)^2}\,du} = -\int \frac{1}{x}\,dx$$

TWO WAYS TO HANDLE THIS INTEGRAL:

① TRICK + INTEGRATION BY SUBSTITUTION

$$\frac{u-2}{(u-1)^2} = \frac{u-1-1}{(u-1)^2} = \frac{\cancel{u-1}}{(u-1)^{\cancel{2}}} - \frac{1}{(u-1)^2}$$

$$= \frac{1}{u-1} - \frac{1}{(u-1)^2} \Rightarrow$$

$$\int \frac{u-2}{(u-1)^2}\,du = \underbrace{\int \frac{1}{u-1}\,du} - \underbrace{\int \frac{1}{(u-1)^2}\,du}$$

Let $\boxed{w = u-1}$. Then $\frac{dw}{du} = 1 \Rightarrow$ $\boxed{dw = du}$.

Let $\boxed{w = u-1}$. Then $\frac{dw}{du} = 1 \Rightarrow$ $\boxed{dw = du}$.

$$= \int \frac{1}{w}\,dw - \int \frac{1}{w^2}\,dw$$

$$= \int \frac{1}{w}\,dw - \int w^{-2}\,dw$$

$$= \ln|w| - \frac{w^{-2+1}}{-2+1} + C$$

$$= \ln|w| + \frac{1}{w} + C$$

$$= \ln|u-1| + \frac{1}{u-1} + C$$

② METHOD OF PARTIAL FRACTIONS + INTEGRATION BY SUBSTITUTION

$$\frac{u-2}{(u-1)^2} = \frac{A}{u-1} + \frac{B}{(u-1)^2} \Rightarrow$$

$$u-2 = A(u-1) + B \Rightarrow$$

$$1 \cdot u - 2 = Au + (-A+B) \Rightarrow$$

$$\left.\begin{array}{l} A = 1 \\ -A+B = -2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} A = 1 \\ B = -1 \end{array}\right\} \Rightarrow$$

$$\int \frac{u-2}{(u-1)^2}\,du = \int \frac{A}{u-1}\,du + \int \frac{B}{(u-1)^2}\,du$$

$$= \int \frac{1}{u-1}\,du - \int \frac{1}{(u-1)^2}\,du$$

$$\vdots$$

$$= \ln|u-1| + \frac{1}{u-1} + C$$

$$\therefore \int \frac{u-2}{(u-1)^2}\,du = -\int \frac{1}{x}\,dx \Rightarrow$$

$$\ln|u-1| + \frac{1}{u-1} = -\ln|x| + C \Rightarrow$$

Substitute back
$y = ux \Rightarrow u = \frac{y}{x}$

$$\boxed{\ln\left|\frac{y}{x}-1\right| + \frac{1}{\frac{y}{x}-1} = -\ln|x| + C} \Rightarrow$$

$$\boxed{\ln\left|\frac{y-x}{x}\right| + \frac{x}{y-x} = -\ln|x| + C} \Rightarrow$$

$$\ln|y-x| - \cancel{\ln|x|} + \frac{x}{y-x} = -\cancel{\ln|x|} + C \Rightarrow$$

$$\boxed{\ln|y-x| + \frac{x}{y-x} = C} \Rightarrow$$

$$(x-y)\ln|x-y| + \frac{x(x-y)}{y-x} = C(x-y) \Rightarrow$$

$$\boxed{(x-y)\ln|x-y| - x = C(x-y)} \Rightarrow$$

$$(x-y)\ln|x-y| - x + \underbrace{y-y}_{=0} = C(x-y) \Rightarrow$$

$$(x-y)\ln|x-y| - (x-y) - y = C(x-y) \Rightarrow$$

$$(x-y)\ln|x-y| - y = C(x-y) + 1\cdot(x-y) \Rightarrow$$

$$(x-y)\ln|x-y| - y = \underbrace{(C+1)}_{=C}(x-y) \Rightarrow$$

$$\boxed{(x-y)\ln|x-y|-y = C(x-y)}$$

Solution in SOLUTIONS MANUAL

Bernoulli DEs

Defn. A Bernoulli equation is a DE of the form

$$\frac{dy}{dx} + P(x)\,y = f(x)\,y^{\alpha},$$

where α is any real number.

Notes: (1) When $\alpha = 0$, we have a linear first-order DE

$$\frac{dy}{dx} + P(x)\,y = f(x).$$

(2) When $\alpha = 1$, we still have a linear first-order DE

$$\frac{dy}{dx} + P(x)\,y = f(x)\,y \implies$$
$$\frac{dy}{dx} + \left[P(x) - f(x)\right]y = 0.$$

Method of Solution of Bernoulli DEs

- Assume $\alpha \neq 0, 1$ in

$$\frac{dy}{dx} + P(x)y = f(x)y^{\alpha}$$

- Let $u = y^{1-\alpha}$ (where $u = u(x)$). Then

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$$\boxed{y = u^{\frac{1}{1-\alpha}}} \Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{1-\alpha} u^{1-\frac{1}{1-\alpha}} \cdot \frac{du}{dx}}$$

CHAIN RULE:

$$\frac{d}{dx}\left[f(x)\right]^n = n\left[f(x)\right]^{n-1} \cdot \frac{df}{dx}$$

- Substitute these expressions for y and $\frac{dy}{dx}$ in the DE.
- The DE will reduce to a LINEAR FIRST-ORDER DE, which we solve by the method given in Sect. 2.3.

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<u>Example</u>: (Exercise 15, p. 57.) Solve the Bernoulli equation

$$x\frac{dy}{dx} + y = \frac{1}{y^2}$$

by using an appropriate substitution.

<u>STEP 1</u>. First divide through by x:

$$\frac{dy}{dx} + \underbrace{\frac{1}{x}}_{P(x)=\frac{1}{x}}\, y = \underbrace{\frac{1}{x}}_{f(x)=\frac{1}{x}}\, y^{-2} \qquad \alpha = -2$$

<u>STEP 2</u>. We have a Bernoulli equation with $\alpha = -2$. So, let $u = y^{1-\alpha}$:

$$u = y^{1-(-2)} = y^3 \Rightarrow \boxed{u = y^3} \Rightarrow$$

$$u^{1/3} = y \Rightarrow \boxed{y = u^{1/3}} \Rightarrow$$

$$\frac{dy}{dx} = \frac{d}{dx}u^{1/3} = \frac{1}{3}u^{\frac{1}{3}-1}\cdot\frac{du}{dx} = \frac{1}{3}u^{-2/3}\frac{du}{dx} \Rightarrow$$

$$\boxed{\frac{dy}{dx} = \frac{1}{3}u^{-2/3}\frac{du}{dx}}$$

STEP 3. Substitute $\boxed{y = u^{1/3}}$ and $\boxed{\frac{dy}{dx} = \frac{1}{3} u^{-2/3} \frac{du}{dx}}$ into the DE:

$$\frac{dy}{dx} + \frac{1}{x} y = \frac{1}{x} y^{-2} \Longrightarrow$$

$$\frac{1}{3} u^{-2/3} \frac{du}{dx} + \frac{1}{x} u^{1/3} = \frac{1}{x} \left(u^{1/3}\right)^{-2} \Longrightarrow$$

$$\frac{1}{3} u^{-2/3} \frac{du}{dx} + \frac{1}{x} u^{1/3} = \frac{1}{x} u^{-2/3} \Longrightarrow$$

$$3u^{2/3}\left(\frac{1}{3} u^{-2/3} \frac{du}{dx} + \frac{1}{x} u^{1/3}\right) = 3u^{2/3}\left(\frac{1}{x} u^{-2/3}\right) \Longrightarrow$$

$$u^{\frac{2}{3}-\frac{2}{3}} \frac{du}{dx} + \frac{3}{x} u^{\frac{2}{3}+\frac{1}{3}} = \frac{3}{x} u^{\frac{2}{3}-\frac{2}{3}} \Longrightarrow$$

$$\boxed{\frac{du}{dx} + \frac{3}{x} u = \frac{3}{x}}$$

LINEAR FIRST-ORDER DE (in u)

$$u' + \underbrace{P(x)}_{=\frac{3}{x}} u = \underbrace{f(x)}_{=\frac{3}{x}}$$

STEP 4. Solve linear first-order DE in usual way:

$$\mu(x) = e^{\int P(x)\,dx} = e^{\int \frac{3}{x}\,dx} = e^{3\ln|x|}$$

$$= e^{3\ln x} = e^{\ln x^3} = x^3 \Longrightarrow$$

will assume $I = (0, \infty)$

$$x^3\left(\frac{du}{dx} + \frac{3}{x}u\right) = x^3\left(\frac{3}{x}\right) \Rightarrow$$

$$\underbrace{x^3\frac{du}{dx} + 3x^2u}_{} = 3x^2 \Rightarrow$$

$$= \frac{d}{dx}(x^3u) = x^3\left(\frac{d}{dx}u\right) + \left(\frac{d}{dx}x^3\right)u = x^3\frac{du}{dx} + 3x^2u \checkmark$$

$$\frac{d}{dx}(x^3u) = 3x^2 \Rightarrow$$

$$\int \frac{d}{dx}(x^3u)\,dx = \int 3x^2\,dx \Rightarrow$$

$$x^3u = x^3 + C \Rightarrow$$

$$\boxed{u = 1 + \frac{C}{x^3}}$$

<u>STEP 5.</u> Substitute back and replace u by y^3:

$$u = y^3 \Rightarrow$$

$$\boxed{y^3 = 1 + \frac{C}{x^3}} \Rightarrow$$

$$\boxed{y = \left(1 + \frac{C}{x^3}\right)^{1/3}}$$

Equations of the Form

$$\frac{dy}{dx} = f(Ax + By + C), \quad B \neq 0$$

Note: If $B = 0$, then we have

$$\frac{dy}{dx} = f(Ax + C),$$

which is a separable DE.

Method of Solution

- Assume $B \neq 0$
- Let $\boxed{u = Ax + By + C}$ (where $u = Ax + By(x) + C$ so $u = u(x)$). Then

$$\frac{du}{dx} = \frac{d}{dx}(Ax + \underset{\substack{\uparrow \\ y = y(x)}}{By} + C) = A + B\frac{dy}{dx} \Longrightarrow$$

$$\frac{du}{dx} = A + B\frac{dy}{dx} \Longrightarrow$$

$$\boxed{\frac{dy}{dx} = \frac{1}{B}\left(\frac{du}{dx} - A\right)}$$

- Substitute these expressions for y and $\frac{dy}{dx}$ in the DE.

- The DE will reduce to a SEPARABLE DE, which we solve by the method given in Sect. 2.1.

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<u>Example</u>. (Exercise 27, p. 58.) Solve the DE

$$\frac{dy}{dx} = 2 + \sqrt{y - 2x + 3}.$$

<u>STEP 1</u>. This DE is of the form

$$\frac{dy}{dx} = f(Ax + By + C):$$

$$f(w) = 2 + \sqrt{w}$$
$$A = -2, \quad B = 1, \quad C = 3$$

<u>STEP 2</u>. Let $u = Ax + By + C$:

$$\boxed{u = y - 2x + 3} \Rightarrow$$

$$\frac{du}{dx} = \frac{d}{dx}(y - 2x + 3) = \frac{dy}{dx} - 2 \Rightarrow$$

$$\frac{du}{dx} = \frac{dy}{dx} - 2 \Rightarrow \boxed{\frac{dy}{dx} = \frac{du}{dx} + 2}$$

<u>STEP 3</u>. Substitute $\boxed{u = -2x + y + 3}$ and $\boxed{\frac{dy}{dx} = \frac{du}{dx} + 2}$ into the DE:

$$\frac{dy}{dx} = 2 + \sqrt{y - 2x + 3} \Rightarrow$$

$$\frac{du}{dx} + \cancel{2} = \cancel{2} + \sqrt{u} \Rightarrow$$

$$\boxed{\frac{du}{dx} = \sqrt{u}}$$

SEPARABLE DE (in u)

$$\frac{du}{dx} = \underset{g(x)}{1} \cdot \underset{h(u)}{\sqrt{u}}$$

STEP 4. Solve separable DE in usual way:

$$\frac{1}{\sqrt{u}}\, du = dx \Rightarrow$$

$$u^{-1/2}\, du = dx \Rightarrow$$

$$\int u^{-1/2}\, du = \int 1 \cdot dx \Rightarrow$$

$$\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = x + C \Rightarrow$$

$$\boxed{2\sqrt{u} = x + C}$$

<u>STEP 5</u>. Substitute back and replace u by $y-2x+3$:

$$u = y-2x+3 \Longrightarrow$$

$$\boxed{2\sqrt{y-2x+3} = x + C} \Longrightarrow$$

$$\boxed{\sqrt{y-2x+3} = \frac{x}{2} + C}$$

<u>WARNING</u>: Do not rewrite this as

$$y-2x+3 = \left(\frac{x}{2}+C\right)^2 \Longrightarrow$$

$$y = \left(\frac{x}{2}+C\right)^2 + 2x - 3$$

where

$$\overbrace{y-2x+3}^{\text{cannot be negative since equal to a square } (\ )^2} = \Big(\underbrace{\frac{x}{2}+C}_{\text{can be negative since this quantity is squared and made nonnegative.}}\Big)^2$$

is not equivalent to

cannot be negative since equal to a square root $\sqrt{\ }$

$$\underbrace{\sqrt{y-2x+3}}_{\text{cannot be negative for square root to make sense}} = \overbrace{\frac{x}{2}+C}^{\text{cannot be negative since equal to a square root}}$$