Section 3.4: Solutions by Substitution

Three parts:
1. Substitution with a homogeneous DE
   \[ M(x,y) \frac{dx}{dx} + N(x,y) \frac{dy}{dx} = 0 \]
2. " " " Bernoulli DE
   \[ \frac{dy}{dx} + P(x)y = f(x)y^n \]
3. " " " Potentially separable DE
   \[ \frac{dy}{dx} = f(x) + g(y) \]

For an analogy, recall INTEGRATION BY SUBSTITUTION:

\[ \int f(g(x)) \cdot g'(x) \, dx = \int f(u) \, du \]

Let \( u = g(x) \). Then
\[ \frac{du}{dx} = g'(x) \Rightarrow du = g'(x) \, dx \]

\[ \frac{du}{f(u)} = \frac{f'(u)}{f(u)} \, du = \frac{f'(u)}{f(u)} \, du \]

\[ = F(u) + C \]

Where \( F \) is the antiderivative of \( f \)

\[ = F(g(x)) + C \]
We can transform an integral into one that is more easily solved (i.e., solved by already known techniques) by a "change of variable" or "substitution." E.g., \[ \int 2xe^{x^2} \, dx = \int e^u \, du = e^u + C = e^{x^2} + C \]

Let \( u = x^2 \). Then \( \frac{du}{dx} = 2x \Rightarrow du = 2x \, dx \)

We can do something similar with DEs to reduce them to DEs that we already know how to solve.

**Homogeneous DEs**

Defn: \( f(x, y) \) is said to be a homogeneous function of degree \( \alpha \), where \( \alpha \) is some real number, if when we substitute \( cx \) for \( x \) and \( cy \) for \( y \) with \( c \) any constant, we get

\[ f(cx, cy) = c^\alpha f(x, y) \]
E.g., $f(x,y) = 5x^{1/3}y - 3xy^{1/3}$

\[ f(cx, cy) = 5(cx)^{1/3}(cy) - 3(cx)(cy)^{1/3} \]
\[ = 5c^{1/3}x^{1/3}y - 3c^{1/3}cx^{1/3}y^{1/3} \]
\[ = c^{1/3}(5x^{1/3}y - 3xy^{1/3}) = c^{1/3}f(x, y) \]

\[ \text{\ because } f(x, y) \text{ is a homogeneous function of degree } \frac{1}{3}. \]

E.g., $f(x,y) = x^3 + y^2$

\[ f(cx, cy) = (cx)^3 + (cy)^2 \]
\[ = c^3x^3 + c^2y^2 \]
\[ \neq c^3f(x, y) \]

\[ \text{\ because } f(x, y) \text{ is NOT a homogeneous function.} \]

**Roughly speaking,**

A homogeneous function of degree \( \alpha \) is a function of the form

\[ f(x,y) = x^{m_1}y^{n_1} + x^{m_2}y^{n_2} + \cdots + x^{m_c}y^{n_c} \]

where

\[ m_1 + n_1 = \alpha, \quad m_2 + n_2 = \alpha, \quad \ldots, \quad m_c + n_c = \alpha \]
Defn. A first-order DE of the form

\[ M(x, y) \, dx + N(x, y) \, dy = 0 \]

is said to be homogeneous if (1) \( M(x, y) \) and \( N(x, y) \) are both homogeneous functions of the same degree.

\( E(x, y) = \frac{(x-y) \, dx + (x+y) \, dy}{M(x, y) \, N(x, y)} \)

\( M(x, y) = x - y = x' - y' \implies \text{homog., of degree 1} \)

\( N(x, y) = x + y = x' + y' \implies \text{homog., of degree 1} \)

\( \therefore \text{DE is homogeneous.} \)

Method of Solution of Homogeneous DEs
With a homogeneous DE, we will make either of the following substitutions:

(1) \[ y = ux \]  
(i.e., \( y(x) = u(x)x \))

Then \[ \frac{dy}{dx} = \frac{d}{dx}(ux) = u \cdot 1 + \frac{d}{dx} x \Rightarrow \]

\[ \text{PRODUCT RULE: } (fg)' = fg' + f'g \]

\[ \frac{dy}{dx} = u + x \frac{du}{dx} \Rightarrow \frac{dy}{dx} = u dx + x du \]

- Used when \( M \) is simpler than \( N \) (since replacing \( dy \) with a messier expression).
- End up with DE in \( u(x) \) and \( x \)
- Solve for \( u \) in terms of \( x \)
- Substitute back \( u = y/x \)
- Homogeneity allows for factoring and cancelling of \( x \)

(2) \[ x = vy \]  
(i.e., \( x = v(x)y(x) \))

Then \[ \frac{d}{dx}(x) = \frac{d}{dx}(vy) = v \frac{dx}{dx} + \frac{dv}{dx} \cdot y \Rightarrow \]

\[ 1 = v \frac{dv}{dx} + \frac{dx}{dx} \Rightarrow dx = v dv + y dv \]

- Used when \( M \) is simpler than \( N \) (since replacing \( dx \) with a messier expression).
- End up with DE in \( v(x) \) and \( y \)
- Solve for \( y \) in terms of \( v \)
- Substitute back \( v = x/y \)
- Homogeneity allows for factoring and cancelling of \( x \)
Examples.

1. (Exercise 1, p. 57.) Solve the homogeneous equation

\[(x-y)dx + x\,dy = 0\]

by using an appropriate substitution.

Note: DE is homogeneous since

\[M(x,y) = x^1 - y^1 \Rightarrow \text{homog. of degree } 1\]
\[N(x,y) = x^1 \Rightarrow \text{homog. of degree } 1\]

Note: \(N(x,y) = x\) is "simpler" than \(M(x,y) = x - y\) so choose substitution with

\[y = ux\]
\[dy = u\,dx + x\,du\]
\[(x-y)\,dx + x\,dy = 0 \Rightarrow \]
\[(x-ux)\,dx + x(udx+xdu) = 0 \Rightarrow \]
\[x(1-u)\,dx + x(udx+xdu) = 0 \Rightarrow \]
\[\text{Assuming } x \neq 0 \]
\[(1-u)\,dx + udx + xdu = 0 \Rightarrow \]

So the homogeneity of degree 1 of both \(M\) and \(N\) has allowed us to factor out an \(x\) and then cancel the \(x\)

\[(1-u+u)\,dx + xdu = 0 \Rightarrow \]
\[dx + xdu = 0 \Rightarrow \]

Looks like a separable eq

\[x\,du = -dx \Rightarrow \]
\[du = -\frac{1}{x}\,dx \Rightarrow \]

\[\int 1\cdot du = -\int \frac{1}{x}\,dx \Rightarrow \]
\[ u = \frac{-\ln |x| + c}{x} \Rightarrow \]

Substitute back:
\[ y = ux \Rightarrow u = \frac{y}{x} \]

\[ \frac{y}{x} = -\ln |x| + C \Rightarrow \]

\[ y = -x \ln |x| + Cx \]
2. (Exercise 3, p. 54.) Solve the homogeneous equation
\[ x \, dx + (y - 2x) \, dy = 0 \]
by using an appropriate substitution.

Note: DE is homogeneous since
\[ M(x,y) = x' \, \Rightarrow \text{homog. of degree 1? some degree} \]
\[ N(x,y) = y' - 2x' \, \Rightarrow \text{homog. of degree 1} \]

Note: \( M(x,y) = x \) is "simpler" than \( N(x,y) = y - 2x \) choose substitution with
\[ x = vy \]
\[ dx = v \, dy + y \, dv \]
Actually, we are going to use the other inappropriate substitution,
\[ y = ux \]
\[ dy = u \, dx + x \, du \]
\[
xdx + (y - 2x)\, dy = 0 \implies \\
xdx + (ux - 2x)(udx + xdu) = 0 \implies \\
x\, dx + x(u - 2)(udx + xdu) = 0 \implies \\
dx + (u - 2)(udx + xdu) = 0 \implies \\
dx + u^2\, dx + x\, xdu - 2udx - 2xdu = 0 \implies \\
(1 + u^2 - 2u)\, dx + (ux - 2x)\, du = 0 \implies \\
(u^2 - 2u + 1)\, dx + x\, (u - 2)\, du = 0 \implies \\
(u - 1)^2\, dx + x\, (u - 2)\, du = 0 \implies \\
\text{Looks like a separable eq.,} \\
x\, (u - 2)\, du = -(u - 1)^2\, dx \implies \\
\frac{u - 2}{(u - 1)^2}\, du = -\frac{1}{x}\, dx \implies \\
\int \frac{u - 2}{(u - 1)^2}\, du = -\int \frac{1}{x}\, dx \\
\text{TWO WAYS TO HANDLE THIS INTEGRAL:}
TRICK + INTEGRATION BY SUBSTITUTION

\[ \frac{u-2}{(u-1)^2} = \frac{u-1-1}{(u-1)^2} = \frac{u-1}{(u-1)^2} - \frac{1}{(u-1)^2} \]
\[ = \frac{1}{u-1} - \frac{1}{(u-1)^2} \Rightarrow \]
\[ \int \frac{u-2}{(u-1)^2} \, du = \int \frac{1}{u-1} \, du - \int \frac{1}{(u-1)^2} \, du \]

Let \( w = u-1 \). Then \( \frac{dw}{du} = 1 \Rightarrow \)
\( dw = du \).

\[ \int \frac{1}{w} \, dw - \int \frac{1}{w^2} \, dw \]
\[ = \ln |w| - \frac{1}{w} + C \]
\[ = \ln |u-1| + \frac{1}{u-1} + C \]
\[ \frac{u-2}{(u-1)^2} = \frac{A}{u-1} + \frac{B}{(u-1)^2} \]

\[ u-2 = A(u-1) + B \]

\[ u-2 = Au + (-A+B) \]

\[ A = 1 \]

\[ -A + B = -2 \]

\[ \begin{cases} A = 1 \\ B = -1 \end{cases} \]

\[ \int \frac{u-2}{(u-1)^2} \, du = \int \frac{A}{u-1} \, du + \int \frac{B}{(u-1)^2} \, du \]

\[ = \int \frac{1}{u-1} \, du - \int \frac{1}{(u-1)^2} \, du \]

\[ = \ln|u-1| + \frac{1}{u-1} + C \]
\[ \int \frac{u-2}{(u-1)^2} \, du = - \int \frac{1}{x} \, dx \Rightarrow \]

\[ \ln |u-1| + \frac{1}{u-1} = - \ln |x| + C \Rightarrow \]

Substitute back.

\[ y = ux \Rightarrow u = \frac{y}{x} \]

\[ \ln \left| \frac{y}{x} - 1 \right| + \frac{1}{\frac{y}{x} - 1} = - \ln |x| + C \Rightarrow \]

\[ \ln \left| \frac{y-x}{x} \right| + \frac{x}{y-x} = - \ln |x| + C \Rightarrow \]

\[ \ln (y-x) - \ln |y-x| + \frac{x}{y-x} = - \ln |x| + C \Rightarrow \]

\[ (x-y) \ln |x-y| + \frac{x(x-y)}{y-x} = C(x-y) \Rightarrow \]

\[ (x-y) \ln |x-y| - x = C(x-y) \Rightarrow \]

\[ (x-y) \ln |x-y| - x + y - y = C(x-y) \Rightarrow \]

\[ (x-y) \ln |x-y| -(x-y)-y = C(x-y) \Rightarrow \]

\[ (x-y) \ln |x-y| - y = C(x-y) + i(x-y) \Rightarrow \]

\[ (x-y) \ln |x-y| - y = \frac{(C+1)(x-y)}{C} \Rightarrow \]
\[(x-y) \ln |x-y| - y = C(x-y)\]

Solution in SOLUTIONS MANUAL
Bernoulli DEs

Definition: A Bernoulli equation is a DE of the form

\[ \frac{dy}{dx} + P(x)y = -f(x)y^\alpha, \]

where \( \alpha \) is any real number.

Notes:
1. When \( \alpha = 0 \), we have a linear first-order DE

\[ \frac{dy}{dx} + P(x)y = f(x). \]

2. When \( \alpha = 1 \), we still have a linear first-order DE

\[ \frac{dy}{dx} + P(x)y = f(x)y \quad \Rightarrow \quad \frac{dy}{dx} + [P(x) - f(x)]y = 0. \]

Method of Solution of Bernoulli DEs
Assume \( \alpha \neq 0,1 \) in:

\[
\frac{dy}{dx} + P(x)y = f(x)y^{\alpha}
\]

Let \( u = y^{1-(1-\alpha)} \) (where \( u = u(x) \)). Then

\[
y = \frac{1}{1-\alpha} u
\]

\[
\frac{dy}{dx} = \frac{1}{1-\alpha} \left( \frac{1}{1-\alpha} \frac{du}{dx} \right)
\]

**CHAIN RULE:**

\[
\frac{d}{dx} \left[ f(x) \right]^n = n \left[ f(x) \right]^{n-1} \cdot \frac{df}{dx}
\]

Substitute these expressions for \( y \) and \( \frac{dy}{dx} \) in the DE.

The DE will reduce to a LINEAR FIRST-ORDER DE which we solve by the method given in Sect. 2.3.
Example: (Exercise 15, p. 57.) Solve the Bernoulli equation
\[
x \frac{dy}{dx} + \frac{1}{x} y = \frac{1}{y^2}
\]
by using an appropriate substitution.

**Step 1.** First divide through by \(x\):
\[
\frac{dy}{dx} + \frac{1}{x} y = \frac{1}{x} y^{-2}
\]
\[
P(x) = \frac{1}{x} \quad f(x) = \frac{1}{x}
\]

**Step 2.** We have a Bernoulli equation with \(\alpha = -2\). So, let \(u = y^{1-\alpha}\):
\[
u = y^{1-(-2)} = y^3 \quad \Rightarrow \quad u = y^3
\]
\[
\int u^2 \, dy = y \quad \Rightarrow \quad \left[ u^2 \right] = y
\]
\[
\frac{dy}{dx} u^2 = \frac{1}{3} u^{\frac{1}{3}-1} \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{3} u^{-\frac{2}{3}} \frac{dy}{dx}
\]
\[
\frac{dy}{dx} = \frac{1}{3} u^{-\frac{2}{3}} \frac{du}{dx}
\]
STEP 3. Substitute \( y = u^{1/3} \) into the DE:

\[
\frac{dy}{dx} + \frac{1}{x} y = \frac{1}{x} y^{-2} \Rightarrow \\
\frac{1}{3} u^{-2/3} \frac{du}{dx} + \frac{1}{x} u^{1/3} = \frac{1}{x} (u^{1/3})^{-2} \Rightarrow \\
\frac{1}{3} u^{-2/3} \frac{du}{dx} + \frac{1}{x} u^{1/3} = \frac{1}{x} u^{-2/3} \Rightarrow \\
3 u^{2/3} \left( \frac{1}{3} u^{-2/3} \frac{du}{dx} + \frac{1}{x} u^{1/3} \right) = 3 u^{2/3} \left( \frac{1}{x} u^{-2/3} \right) \Rightarrow \\
u^{2/3-2/3} \frac{du}{dx} + \frac{3}{x} u^{2/3+1/3} = \frac{3}{x} u^{3/3-3/3} \Rightarrow \\
\frac{du}{dx} + \frac{3}{x} u = \frac{3}{x} 
\]

LINEAR FIRST-ORDER DE (in u)

\[ u' + P(x)u = f(x) \]

\[ \frac{u}{u^{2/3}} = \frac{3}{x} \]

STEP 4. Solve linear first-order DE in usual way:

\[
\mu(x) = e^{-\int P(x)dx} = e^{-\int \frac{1}{x} dx} = e^{-\ln x} = e^{\ln x^{-1}} = e^{\ln x^3} = x^3 
\]

will assume \( I = (0, \infty) \)
$$x^2 \left( \frac{du}{dx} + \frac{3}{x} u \right) = x^3 \left( \frac{3}{x} \right) \implies$$

$$x^2 \frac{du}{dx} + 3x^2 u = 3x^3 \implies$$

$$\frac{d}{dx}(x^2 u) = x^3 \left( \frac{du}{dx} \right) + \left( \frac{du}{dx} \right) x = x^3 \frac{du}{dx} + 3x^2 u \checkmark$$

$$\frac{d}{dx}(x^3 u) = 3x^2 \implies$$

$$\int \frac{d}{dx}(x^3 u) \, dx = \int -3x^2 \, dx \implies$$

$$x^3 u = x^3 + C \implies$$

$$u = 1 + \frac{C}{x^3}$$

**STEP 5. Substitute back and replace u by y**:  

$$u = y^3 \implies$$

$$y^3 = 1 + \frac{C}{x^3}$$

$$y = \left( 1 + \frac{C}{x^3} \right)^{\frac{1}{3}}$$
Equations of the Form

\[ \frac{dy}{dx} = f(Ax + By + C) \quad B \neq 0 \]

Note: If \( B = 0 \), then we have

\[ \frac{dy}{dx} = f(Ax + C) \]

which is a separable DE.

Method of Solution

Assume \( B \neq 0 \)

Let \( u = Ax + By + C \) (where \( u = Ax + By(x) + C \) so \( u = u(x) \)). Then

\[ \frac{du}{dx} = \frac{d}{dx} (Ax + By + C) = A + B \frac{dy}{dx} \Rightarrow \]

\[ \frac{du}{dx} = A + B \frac{dy}{dx} \Rightarrow \]
\[ \frac{dy}{dx} = \frac{1}{B} \left( \frac{dy}{dx} - A \right) \]

Substitute these expressions for \( y \) and \( \frac{dy}{dx} \) in the DE.

The DE will reduce to a SEPARABLE DE, which we solve by the method given in Sect. 2.1.
Example. (Exercise 27, p. 58.) Solve the DE

$$\frac{dy}{dx} = 2 + \sqrt{y-2x+3}.$$ 

**STEP 1.** This DE is of the form

$$\frac{dy}{dx} = f(Ax + By + C):$$

$$f(w) = 2 + \sqrt{w}$$

$$A = -2, \quad B = 1, \quad C = 3$$

**STEP 2.** Let $$u = Ax + By + C;$$

$$u = y - 2x + 3 \Rightarrow$$

$$\frac{du}{dx} = \frac{dy}{dx} - 2 \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + 2$$

**STEP 3.** Substitute $$u = -2x + y + 3$$ and $$\frac{dy}{dx} = \frac{du}{dx} + 2$$ into the DE:
\[
\frac{dy}{dx} = 2 + \sqrt{y - 2x + 3} \\
\downarrow \\
\frac{du}{dx} + 2 = 2 + \sqrt{u} \\
\downarrow \\
\frac{du}{dx} = \sqrt{u} \\
\uparrow \\
\text{SEPARABLE DE (in u)} \\
\frac{du}{dx} = 1 \cdot \frac{\sqrt{u}}{\int g(x) \, dx} \\
\text{STEP 4. Solve separable DE in usual way:} \\
\frac{1}{\sqrt{u}} \, du = \cdot dx \\
\downarrow \\
u^{1/2} \, du = \cdot dx \\
\int u^{-1/2} \, du = -\int 1 \, dx \\
\downarrow \\
u^{-\frac{1}{2} + 1} = x + C \\
\downarrow \\
2\sqrt{u} = x + C
STEP 5. Substitute back and replace 

\( u \) by \( y - 2x + 3 \):

\[ u = y - 2x + 3 \Rightarrow \]

\[ 2\sqrt{y - 2x + 3} = x + C \Rightarrow \]

\[ \sqrt{y - 2x + 3} = \frac{x}{2} + C \]

\[ \text{WARNING: Do not rewrite this as} \]

\[ y - 2x + 3 = \left( \frac{x}{2} + C \right)^2 \Rightarrow \]

\[ y = \left( \frac{x}{2} + C \right)^2 + 2x - 3 \]

\[ \text{Where} \]

\[ \sqrt{y - 2x + 3} = \left( \frac{x}{2} + C \right)^2 \]

\[ \text{can be negative since this} \]

\[ \text{quantity is squared and made} \]

\[ \text{non-negative} \]

\[ \text{is not equivalent to} \]
\[
\sqrt{y-2x+3} = \frac{x}{2} + C
\]

Cannot be negative since equal to a square root. Cannot be negative for square root to make sense.