CHAPTER 2
First-Order Differential Equations.

We will now look at DEs like

\[ \frac{dy}{dx} = \frac{y+1}{x}, \]

\[ (2x-1)dx + (3y+7)dy = 0, \]

\[ x^2y' + xy = 1, \]

\[ \frac{dy}{dx} - y = e^x y^2, \]

\[ \frac{dy}{dx} = \tan^2 (x+y) \]

These are all DEs which can be placed in the form

\[ y' = f(x,y) \quad \text{(actually, } y'(x) = f(x, y(x)) \text{)} \]

and whose highest derivative is a first derivative.
We will then learn about 4
SYSTEMATIC ways of solving them,
I.e., finding all functions \( y = \phi(x) \)
such that

\[
\phi'(x) = f(x, \phi(x)).
\]
Section 2.1. Separable Variables.

The simplest possible first-order DE to have is the following:

\[ y' = f(x). \]

This also involves the simplest method of finding a solution (or, simply, simplest method of solution).

Integrate both sides of the equation \( y' = f(x) \) with respect to \( x \):

\[ \int y' \, dx = \int f(x) \, dx \Rightarrow \]

\[ \int y(x) \, dx = \int f(x) \, dx \Rightarrow \]

By the Fundamental Theorem of Calculus,

\[ \int f(x) \, dx = f(x) + C \]

\[ y(x) + C_1 = \int f(x) \, dx \Rightarrow \]

\[ y(x) = \int f(x) \, dx - C_1 \]
\[ y = \int f(x) \, dx - C_1 \implies \]
\[ y = \int f(x) \, dx + \left( -C_1 \right) \implies \]
Replace by \( C_2 \)
\( \text{if } C_1 \text{ represents all reals, so will } C_2 = -C_1 \)
\[ y = \int f(x) \, dx + C_2 \]
\[ = \emptyset(x) \text{ for any particular value of } C_2 \]
We use a similar idea in solving DEs of the form

\[ y' = g(x) h(y), \]

i.e., when \( f(x, y) \) can be separated into the product of

1. a function of \( x \) only
2. a function of \( y \) only

This DE is called a \textit{separable equation}.

Examples of separable equations:

1. \[ y' = 2xy = \frac{(2x)(y)}{g(x) h(y)} \]

2. \[ y' = \frac{4y}{x} = \frac{(4)(y)}{g(x) h(y)} \]

3. \[ y' = y = \frac{(1)(y)}{g(x) h(y)} \]
4. \((y - y^2) y' = (y+1)^2 \implies\)
\[
y' = \frac{(y+1)^2}{y - y^2} = \frac{(y+1)^2}{y} \cdot \frac{1}{1 - x^2} = \frac{g(x)}{h(y)}
\]

5. \((x+3) y' = x - 1 \implies\)
\[
y' = \frac{x-1}{x+3} = \frac{(x-1)(1)}{g(x)} \cdot \frac{1}{h(y)}
\]

6. \(y' = x + y \) NOT a separable equation

A separable equation is said to be separable or have separable variables.
Method of Solution of Separable Equations: Separation of Variables

Example. (From Exercise 7, p. 35.)

Solve the DE

\[ xy' - 4y = 0. \]

**LONG WAY** (Do not solve the DE this way)

\[ xy' - 4y = 0 \implies xy' = 4y \implies \frac{1}{y} y' = 4 \cdot \frac{1}{x} \implies \]

\[ \frac{1}{y(x)} y'(x) = 4 \cdot \frac{1}{x} \implies \int \frac{1}{y(x)} y'(x) \, dx = \int 4 \cdot \frac{1}{x} \, dx \implies \]

Use INTEGRATION BY SUBSTITUTION:

Let \( u = y(x) \) Then \( \frac{du}{dx} = y'(x) \implies du = y'(x) \, dx \).
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\[- \quad \int \frac{1}{u} \, du = -4 \int \frac{1}{x} \, dx \quad \Rightarrow \quad \ln |u| + C_1 = -4 \ln |x| + C_2 \quad \Rightarrow \]

\[- \quad \ln |u| = -4 \ln |x| + (C_2 - C_1) \quad \Rightarrow \quad \ln |y| = -4 \ln |x| + C_3 \quad \Rightarrow \]

\[- \quad 1 \ln |y| = -4 \ln |x| + C_3 \quad \Rightarrow \quad \ln |y| = -4 \ln |x| + C_3 \quad \Rightarrow \]

\[- \quad e \ln |y| = e^{-4 \ln |x| + C_3} \quad \Rightarrow \quad e^{\ln |y|} = e^{C_4 e^{\ln |x|}} \quad \Rightarrow \quad e^{|y|} = x \quad \Rightarrow \quad |y| = C_4 x \quad \Rightarrow \quad y = e^{|y|} \quad \Rightarrow \quad \frac{|y|}{e^{|y|}} \quad \Rightarrow \quad \frac{y}{e^{|y|}} = \frac{1}{e} \quad \Rightarrow \quad y = e^{|y|} \]

\[- \quad |y| = C_4 x \quad \Rightarrow \quad y = e^{|y|} \quad \Rightarrow \quad y = C_4 x \quad (C_4 > 0) \quad \Rightarrow \quad \]

\[- \quad 1 \ln |y| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \quad \Rightarrow \quad \pm y = C_4 x \quad (C_4 > 0) \quad \Rightarrow \quad \]
\[(\pm 1) (\pm 2y) = (\pm 1) \cdot 2x^4 \quad \Rightarrow \quad y = C_5 \neq 0\]

\[y = C_5 x^4 \quad (C_5 \neq 0)\]

Observe that \[y = 0 \quad (\text{or} \quad y(x) = 0)\]
is also a solution of \[xy' - 4y = 0\]
and can be written as \[y = 0 \cdot x^4\].

So, the general solution (which encompasses all possible solutions) is really:

\[y = \begin{cases} 
  C_5 x^4, & C_5 \neq 0 \\
  0 \cdot x^4 & 
\end{cases} = C x^4, \quad C = C_5 \text{ or } 0\]

\[y = C x^4, \quad C \text{ arbitrary}\]
**SHORTCUT (Solve the DE this way)**

\[ xy' - 4y = 0 \implies \]

\[ x \frac{dy}{dx} = 4y \implies \]

\[ \frac{1}{y} \frac{dy}{dx} = 4 \cdot \frac{1}{x} \implies \]

\[ \int \frac{1}{y} \, dy = 4 \int \frac{1}{x} \, dx \implies \]

\[ \ln |y| = 4 \ln |x| + C \implies \]

\[ \ln |y| = \ln |x|^4 + C \implies \]

\[ \ln |y| = \ln |x|^4 + C \implies \]

\[ e^{\ln |y|} = e^{\ln |x|^4 + C} \implies \]

\[ e^{|y|} = e^C e^{\ln |x|^4} \implies \]

\[ |y| = e^C x^4 \implies \]
\[ y = \pm e^x \Rightarrow \]
\[ y = Cx^4, \quad C \text{ arbitrary} \]