Determination of the Linearity or Nonlinearity of Differential Equations

Below is a method for testing whether a differential equation is LINEAR or NONLINEAR. We do not go into the origins of this method or why it works. We simply show how to implement it, but we hope that by doing this, it will become clearer why linearity and nonlinearity have been "defined" as they have been in the text and in class. Keep in mind that the linearity or nonlinearity of a differential equation is determined by the "linearity" or "nonlinearity" of y and its derivatives and NOT by the "linearity" or "nonlinearity" of x.

Method

STEP 1. Start with your differential equation in the form

$$F(x, y, y', \ldots, y^{(n)}) = 0. \tag{0.1}$$

STEP 2. Viewing $F(x, y, y', \ldots, y^{(n)})$ as the SUM of terms that are each expressions in x, y, and/or the derivatives of y, keep all those terms with y or any derivative of y in them on the *left-hand side* of the equals sign in Eq.(0.1)and move all those terms with ONLY x in them over to the *right-hand side* of the equals sign in Eq.(0.1). For convenience, we will denote the end result by

$$f(x, y, y', \ldots, y^{(n)}) = g(x). \tag{0.2}$$

STEP 3. Now consider ONLY the *left-hand side* of Eq.(0.2), i.e., $f(x, y, y', \ldots, y^{(n)})$.

STEP 4. REPLACE the function $y(x)$ (at this point viewed only as a function of x and not necessarily as a solution of the differential equation) in $f(x, y, y', \ldots, y^{(n)})$ by the *linear combination* of two functions,

$$ay_1(x) + by_2(x),$$

where y_1, y_2 represent any two arbitrary functions of x and a, b represent any two arbitrary real numbers.

STEP 5. If you can end up writing, and only if you can end up writing,

$$f(x, ay_1 + by_2, (ay_1 + by_2)', \ldots, (ay_1 + by_2)^{(n)})$$
$$= af(x, y_1, y_1', \ldots, y_1^{(n)}) + bf(x, y_2, y_2', \ldots, y_2^{(n)}).$$

then you can say that YOUR DIFFERENTIAL EQUATION IS LINEAR. Otherwise, it is nonlinear.

Examples

1. $y' + 2xy = \cos x$.

   Replace y by $ay_1 + by_2$:

$$\begin{aligned}
&(ay_1 + by_2)' + 2x(ay_1 + by_2) \\
&= ay_1' + by_2' + 2axy_1 + 2bxy_2 \\
&= a(y_1' + 2xy_1) + b(y_2' + 2xy_2) \\
&= a\cdot(\text{the original left-hand side of the DE in } y_1) \\
&+ b\cdot(\text{the original left-hand side of the DE in } y_2).
\end{aligned}$$

   Therefore, the DE $y' + 2xy = \cos x$ is LINEAR.

2. $(y')^2 + 2xy = \cos x$.

   Replace y by $ay_1 + by_2$:

$$\begin{aligned}
&[(ay_1 + by_2)']^2 + 2x(ay_1 + by_2) \\
&= (ay_1' + by_2')^2 + 2axy_1 + 2bxy_2 \\
&= a^2(y_1')^2 + 2aby_1'y_2' + b^2(y_2')^2 + 2axy_1 + 2bxy_2 \\
&= a[a(y_1')^2 + 2xy_1] + b[b(y_2')^2 + 2xy_2] + 2aby_1'y_2' \\
&= a\cdot(\text{NOT the original left-hand side of the DE in } y_1) \\
&+ b\cdot(\text{NOT the original left-hand side of the DE in } y_2) + \text{EXTRA TERM}.
\end{aligned}$$

   Therefore, the DE $(y')^2 + 2xy = \cos x$ is NONLINEAR.

3. $e^{y'} + 2xy = \cos x.$

Replace y by $ay_1 + by_2$:

$$\begin{aligned} &e^{(ay_1+by_2)'} + 2x(ay_1 + by_2) \\ &= e^{ay_1'+by_2'} + 2axy_1 + 2bxy_2 \end{aligned}$$

But now we cannot get rid of the e in $e^{ay_1'} \cdot e^{by_2'}$ because we are not working with an EQUATION but rather an EXPRESSION so we cannot take the ln of $e^{ay_1'} \cdot e^{by_2'}$ (which would not help anyway since then we would have $\ln\left[e^{ay_1'+by_2'} + 2axy_1 + 2bxy_2\right]$, and this would be worse!). So we CANNOT form the sum of two expressions which are

$a\cdot$(the original left-hand side of the DE in y_1)

and

$b\cdot$(the original left-hand side of the DE in y_2).

Therefore, the DE $e^{y'} + 2xy = \cos x$ is NONLINEAR.