Lecture Notes

Chapter 1 Introduction to Differential Equations.

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We will cover terminology here.

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Section 1.1. Definitions and Terminology

Motivation

Recall the following type of problem from integral (second-semester) calculus:

Find the antiderivative $F(x)$ of $f(x) = x+1$ where $F'(x) = f(x)$

Answer is $F(x) = \frac{1}{2}x^2 + x + C$
since $F'(x) = \frac{1}{2}(2x) + 1 + 0 = x + 1 = f(x)$ ✓

Now, instead of using the notation "$F(x)$" and "$f(x)$," write the above problem this way:

Find the function $y = \phi(x)$ such that

$$\frac{dy}{dx} = x+1 \quad (\text{or } y' = x+1)$$

Then $dy/dx = x+1$ is called a

DIFFERENTIAL EQUATION (DE)

because it is an equation (with an equals sign) with a derivative in it.

The function $\phi(x) = \frac{1}{2}x^2 + x + C$ (or $y = \frac{1}{2}x^2 + x + C$) is called the

GENERAL SOLUTION

of the differential equation.
"General" because C is arbitrary.
"Solution" because $y = \frac{1}{2}x^2 + x + C$ satisfies the differential equation, i.e.,

The derivative of y equals $x+1$.

Note: From now on, instead of writing

$$y = \phi(x),$$

we will write

$$y = y(x).$$

We next can place restrictions on the general solution, $y = \frac{1}{2}x^2 + x + C$, of our DE, and obtain particular solutions out of the whole family of solutions represented by $y = \frac{1}{2}x^2 + x + C$:

Let $y(0) = 1$. This then implies that

$$1 = y(0) = \frac{1}{2}(0)^2 + (0) + C \Rightarrow$$
$$1 = C \Rightarrow$$

$$y = \frac{1}{2}x^2 + x + 1$$

In this case,

$$\begin{cases} \frac{dy}{dx} = x + 1 \\ y(0) = 1 \end{cases}$$

is called an

INITIAL VALUE PROBLEM (IVP)

where y is assigned an "initial" value with

INITIAL CONDITION $y(0) = 1$.

Then $y = \frac{1}{2}x^2 + x + 1$ is called the

PARTICULAR SOLUTION

of the initial value problem.

TYPES OF DEs

I. Ordinary DEs (ODEs)
II. Partial DEs (PDEs)

Examples of ODEs. Involving the derivatives we are familiar with in single-variable calculus.

$$y' + xy = 3 \quad \left(\text{or } \frac{dy}{dx} + xy = 3\right)$$

$$y'' + 5y' + 6y = \cos x$$

$$y'' = [1 + (y')^2](x^2 + y^2)$$

$$\sin y^{(4)} + 2e^x y'' + yy' - x^4 = 0$$

<u>Examples of PDEs</u>. Involving the partial derivatives we have not had yet in multivariate calculus (MATH 307).

DIGRESSION : Review of Partial Derivatives

$f(x) = x^2$ is a function f of one variable x

Here, x is the "independent variable" and $y = f(x)$ is the "dependent variable."

$f(x,y) = x^2y^3$ is a function of two variables x and y

Here, x and y are the independent variables and $z = f(x,y)$ is the dependent variable.

We can still take derivatives of functions of more than one variable, but the derivative can only be taken with respect to one variable at a time while holding the rest constant:

derivative of $f(x,y) = x^2y^3$ with respect to x

$$= 2x\,y^3$$

This acts like a constant.

We differentiate x^2 to get this.

derivative of $f(x,y) = x^2y^3$ with respect to y
$= x^2(3y^2) = 3x^2y^2$

We call these derivatives

PARTIAL DERIVATIVES

and we write

$$\frac{\partial f}{\partial x} = 2xy^3 \quad \text{and} \quad \frac{\partial f}{\partial y} = 3x^2y^2$$

(Examples cont'd. $u = f(x,y)$.)

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{(example of wave equation)}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \text{(example of heat equation)}$$

ORDER of DEs

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<u>Definition</u>. The order of a DE is equal to the highest order of all the derivatives involved in the particular DE.

<u>Example</u>. (Exercise 4, p. 8)

$$x^2 dy + (y - xy - xe^x)\, dx = 0$$

Formally divide through by dx:

$$x^2 \frac{dy}{dx} + (y - xy - xe^x) = 0$$

$\therefore$ Order of DE $= 1$

<u>Example</u>

$$(y')^{10} = \frac{1}{1 + y'''}$$

Order of DE $= 3$

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LINEARITY of DEs

DEs are either I. Linear
or II. Nonlinear
with respect to y and its derivatives.

There are 3 ways to check and see whether or not a DE is linear:

(1) An nth order DE is <u>linear</u> if it can be placed in the form

$$a_n y^{(n)} + \cdots + a_1(x) y' + a_0 y = f(x)$$

E.g., $\frac{dy}{dx} = \frac{y}{x} \Rightarrow y' = \frac{y}{x} \Rightarrow$

$$xy' - y = 0 \Rightarrow \underbrace{x}_{=a_1(x)} y' + \underbrace{(-1)}_{=a_0(x)} y = \underbrace{0}_{f(x)}$$

$\therefore$ Linear

E.g., $\frac{dy}{dx} = \frac{x}{y} \Rightarrow y' = \frac{x}{y} \Rightarrow$

$$\underbrace{y}_{\neq a_1(x)} y' = \underbrace{x}_{=f(x)}$$

$\therefore$ Nonlinear

(2) By inspection, a DE is <u>nonlinear</u> if
- y or any derivative of y is taken to any power greater than 1
- there are products involving different derivatives of y as factors like yy', $y''y^{(5)}$, etc.
- y or any derivative of y is the argument of a function

E.g., $x^3 y^{(4)} \underbrace{- x^2 y''}_{} + 4xy' - 3y = \cos x$

only x is taken to powers or is the argument of a function

$\therefore$ Linear

E.g., $\frac{dy}{dx} - y = e^x y^2$

y is to a power > 1

$\therefore$ Nonlinear

E.g., $y' = \ln y$

y is the argument of a function

$\therefore$ Nonlinear

E.g., $y' = \sqrt{1 + (y'')^2}$

y'' is the argument of a function

$\therefore$ Nonlinear

E.g., $\frac{dy}{dx} = y(2-y) \Rightarrow$

$$\frac{dy}{dx} = 2y - y^2 \Rightarrow$$

nonl. y is to a power ≥ 1

(3) Tried and true method: Replace y and all its derivatives in the DE by the "linear combination" of y's, $ay_1 + by_2$, and all its derivatives. If you get back

$$a\,DE(\text{with } y_1) + b\,DE(\text{with } y_2),$$

then the DE is linear.

E.g., $y' + xy - e^x + x^3 = 0 \Longrightarrow$

$$\boxed{\frac{dy}{dx} + xy} = e^x - x^3$$

Take this and replace y by $ay_1 + by_2$

$$\frac{d}{dx}(ay_1 + by_2) + x(ay_1 + by_2)$$

$$= a\frac{dy_1}{dx} + \frac{dy_2}{dx} + axy_1 + bxy_2$$

$$= a\left(\frac{dy_1}{dx} + xy_1\right) + b\left(\frac{dy_2}{dx} + xy_2\right)$$

$$= a(y_1' + xy_1) + b(y_2' + xy_2) \checkmark$$

$\therefore$ Linear

E.g., $y' = y^2 \Rightarrow \boxed{\frac{dy}{dx} - y^2} = 0$

$$\frac{d}{dx}(ay_1 + by_2) - (ay_1 + by_2)^2 = 0$$

$$= a\frac{dy_1}{dx} + b\frac{dy_2}{dx} - a^2y_1^2 - 2aby_1y_2 - b^2y_2^2$$

$$= a\left(\frac{dy_1}{dx} - ay_1^2\right) + b\left(\frac{dy_2}{dx} - by_2^2\right) - 2aby_1y_2$$

$$\neq a\left(\frac{dy_1}{dx} - y_1^2\right) + b\left(\frac{dy_2}{dx} - y_2^2\right)$$

$\therefore$ Nonlinear

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SOLUTIONS OF A DE

I. Explicit Solution (common)
II. Implicit Solution (Sect. 2.2)

Will give examples of these solutions and how to check for them.

<u>Example of an explicit solution</u>. When the solution y can be expressed as a function of x, $y = \phi(x)$ (Exercise 31, p. 9).

DE: $y'' = y$
Explicit solution: $y = \cosh x + \sinh x$

Verify that $y = \cosh x + \sinh x$ is a solution of $y'' = y$.

<u>Notes</u>: ① $\cosh x = \dfrac{e^x + e^{-x}}{2}$, $\sinh x = \dfrac{e^x - e^{-x}}{2}$

② $\dfrac{d}{dx}\cosh x = \sinh x$, $\dfrac{d}{dx}\sinh x = \cosh x$

$$\begin{aligned} y &= \cosh x + \sinh x \\ y' &= \sinh x + \cosh x \\ y'' &= \cosh x + \sinh x \end{aligned}$$

$\Downarrow$

$y'' = y$:

$(\cosh x + \sinh x) \stackrel{?}{=} (\cosh x + \sinh x)$ Yes

$\therefore$ $y = \cosh x + \sinh x$ is a solution of $y'' = y$.

<u>Example of an implicit solution</u>. When the solution y cannot be expressed as a function of x (e.g., $x^2 + y^2 = 1$) (Exercise 18, p. 9).

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DE: $2xy\,dx + (x^2 + 2y)\,dy = 0$

Implicit solution: $x^2y + y^2 = C_1$.

Verify that $x^2y + y^2 = C_1$ is a "solution" of $2xy\,dx + (x^2 + 2y)\,dy = 0$.

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Implicitly differentiate $x^2y + y^2 = C_1$:

$$\frac{d}{dx}(x^2y + y^2) = \frac{d}{dx}(C_1) \Rightarrow$$

$$\frac{d}{dx}(x^2y) + \frac{d}{dx}(y^2) = 0 \Rightarrow$$

Really $\frac{d}{dx}\left[x^2 y(x)\right]$ Really $\frac{d}{dx}\left[(y(x))^2\right]$

$$\left[\left(\frac{d}{dx}x^2\right)y + x^2\left(\frac{d}{dx}y\right)\right] + 2y\frac{dy}{dx} = 0 \Rightarrow$$

PRODUCT RULE CHAIN RULE

$$2xy + x^2 \frac{dy}{dx} + 2y \frac{dy}{dx} = 0 \Longrightarrow$$

$$2xy\,dx + x^2 dy + 2y\,dy = 0 \Longrightarrow$$

$$2xy\,dx + (x^2 + 2y)\,dy = 0 \checkmark$$

$\therefore$ $x^2 y + y^2 = C_1$ is a solution of $2xy\,dy + (x^2 + 2y)\,dy = 0$.

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GENERAL AND PARTICULAR SOLUTIONS OF A DE REVISITED

Example.

Consider the INITIAL VALUE PROBLEM (IVP)

$$\begin{cases} y' + 2y = 0 & \text{(linear, } 1^{st}\text{-order DE)} \\ y(0) = 3 & \text{(initial condition)} \end{cases}$$

STEP 1. Find the general solution of the DE $y' + 2y = 0$:

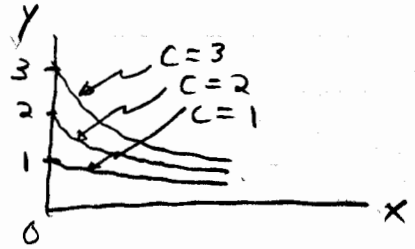
It is given to us as

$$y = Ce^{-2x}$$

general since C is arbitrary at this point

Notice
$$y = Ce^{-2x}$$
$$y' = -2Ce^{-2x}$$

$$(-2Ce^{-2x}) + 2(Ce^{-2x}) \stackrel{?}{=} 0$$
$$0 \stackrel{\checkmark}{=} 0$$



<u>STEP 2</u>. Apply the initial condition $y(0)=3$ to the general solution $y = Ce^{-2x}$ and find C:

$y(0) = 3 \Rightarrow$ When $x=0$, $y=3 \Rightarrow$

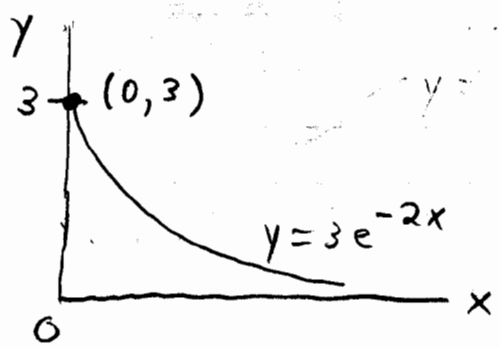
$$3 = y(0) = Ce^{-2(0)} \Rightarrow$$

$$3 = C \cdot 1 \Rightarrow$$

$$C = 3$$

<u>STEP 3</u>. The <u>particular solution</u> to this IVP is then:

$$\boxed{y = 3e^{-2x}}$$



SINGULAR SOLUTION OF A DE

<u>Definition</u>. A <u>singular solution</u> of a DE is one that cannot be obtained from the general solution of the DE (i.e., no value of c will give this solution).

<u>Example</u>. $y' = y^{2/3}$

Later we will see that the general solution is

$$y(x) = \frac{1}{3}(x-c)^3.$$

However,

$$y(x) = 0$$

and

$$y(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \dfrac{x^3}{3} & \text{if } x > 0 \end{cases}$$

are solutions too, but cannot be obtained by specifying the value of c in $y(x) = \frac{1}{3}(x-3)^3$.

SYSTEMS OF DEs

Example.

Let $x = x(t)$, $y = y(t)$. Consider the system of linear (with respect to x and with respect to y) first-order DEs:

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = x \end{cases}$$

It turns out that the general solution is

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$$\begin{cases} x(t) = C_1 e^{t} + C_2 e^{-t} \\ y(t) = C_1 e^{t} - C_2 e^{-t} \end{cases}$$

Determination of the Linearity or Nonlinearity of Differential Equations

Below is a method for testing whether a differential equation is LINEAR or NONLINEAR. We do not go into the origins of this method or why it works. We simply show how to implement it, but we hope that by doing this, it will become clearer why linearity and nonlinearity have been "defined" as they have been in the text and in class. Keep in mind that the linearity or nonlinearity of a differential equation is determined by the "linearity" or "nonlinearity" of y and its derivatives and NOT by the "linearity" or "nonlinearity" of x.

Method

STEP 1. Start with your differential equation in the form

$$F(x, y, y', \ldots, y^{(n)}) = 0. \tag{0.1}$$

STEP 2. Viewing $F(x, y, y', \ldots, y^{(n)})$ as the SUM of terms that are each expressions in x, y, and/or the derivatives of y, keep all those terms with y or any derivative of y in them on the *left-hand side* of the equals sign in Eq.(0.1)and move all those terms with ONLY x in them over to the *right-hand side* of the equals sign in Eq.(0.1). For convenience, we will denote the end result by

$$f(x, y, y', \ldots, y^{(n)}) = g(x). \tag{0.2}$$

STEP 3. Now consider ONLY the *left-hand side* of Eq.(0.2), i.e., $f(x, y, y', \ldots, y^{(n)})$.

STEP 4. REPLACE the function $y(x)$ (at this point viewed only as a function of x and not necessarily as a solution of the differential equation) in $f(x, y, y', \ldots, y^{(n)})$ by the *linear combination* of two functions,

$$ay_1(x) + by_2(x),$$

where y_1, y_2 represent any two arbitrary functions of x and a, b represent any two arbitrary real numbers.

STEP 5. If you can end up writing, and only if you can end up writing,

$$f(x, ay_1 + by_2, (ay_1 + by_2)', \ldots, (ay_1 + by_2)^{(n)})$$
$$= af(x, y_1, y_1', \ldots, y_1^{(n)}) + bf(x, y_2, y_2', \ldots, y_2^{(n)}).$$

then you can say that YOUR DIFFERENTIAL EQUATION IS LINEAR. Otherwise, it is nonlinear.

Examples

1. $y' + 2xy = \cos x$.

   Replace y by $ay_1 + by_2$:

$$(ay_1 + by_2)' + 2x(ay_1 + by_2)$$
$$= ay_1' + by_2' + 2axy_1 + 2bxy_2$$
$$= a(y_1' + 2xy_1) + b(y_2' + 2xy_2)$$
$$= a\cdot(\text{the original left-hand side of the DE in } y_1)$$
$$+b\cdot(\text{the original left-hand side of the DE in } y_2).$$

   Therefore, the DE $y' + 2xy = \cos x$ is LINEAR.

2. $(y')^2 + 2xy = \cos x$.

   Replace y by $ay_1 + by_2$:

$$[(ay_1 + by_2)']^2 + 2x(ay_1 + by_2)$$
$$= (ay_1' + by_2')^2 + 2axy_1 + 2bxy_2$$
$$= a^2(y_1')^2 + 2aby_1'y_2' + b^2(y_2')^2 + 2axy_1 + 2bxy_2$$
$$= a[a(y_1')^2 + 2xy_1] + b[b(y_2')^2 + 2xy_2] + 2aby_1'y_2'$$
$$= a\cdot(\text{NOT the original left-hand side of the DE in } y_1)$$
$$+b\cdot(\text{NOT the original left-hand side of the DE in } y_2) + \text{EXTRA TERM}.$$

   Therefore, the DE $(y')^2 + 2xy = \cos x$ is NONLINEAR.

3. $e^{y'} + 2xy = \cos x$.

Replace y by $ay_1 + by_2$:

$$e^{(ay_1+by_2)'} + 2x(ay_1 + by_2)$$

$$= e^{ay_1'+by_2'} + 2axy_1 + 2bxy_2$$

But now we cannot get rid of the e in $e^{ay_1'} \cdot e^{by_2'}$ because we are not working with an EQUATION but rather an EXPRESSION so we cannot take the ln of $e^{ay_1'} \cdot e^{by_2'}$ (which would not help anyway since then we would have $\ln\left[e^{ay_1'+by_2'} + 2axy_1 + 2bxy_2\right]$, and this would be worse!). So we CANNOT form the sum of two expressions which are

$$a\cdot(\text{ the original left-hand side of the DE in } y_1)$$

and

$$b\cdot(\text{ the original left-hand side of the DE in } y_2).$$

Therefore, the DE $e^{y'} + 2xy = \cos x$ is NONLINEAR.

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$\boxed{\text{CHAPTER 2}}$ First-Order Differential Equations.

We will now look at DEs like

$$\frac{dy}{dx} = \frac{y+1}{x},$$

$$(2x-1)\,dx + (3y+7)\,dy = 0,$$

$$x^2 y' + xy = 1,$$

$$\frac{dy}{dx} - y = e^x y^2,$$

$$\frac{dy}{dx} = \tan^2(x+y)$$

These are all DEs which can be placed in the form

$$y' = f(x,y) \quad (\text{actually, } y'(x) = f(x, y(x)))$$

and whose highest derivative is a <u>first</u> derivative.

We will then learn about 4 SYSTEMATIC ways of <u>solving</u> them, i.e., finding all functions $y = \phi(x)$ such that

$$\phi'(x) = f(x, \phi(x)).$$

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Section 2.1. Separable Variables.

The simplest possible first-order DE to have is the following:

$$y' = f(x).$$

This also involves the simplest method of finding a solution (or, simply, simplest method of solution):

Integrate both sides of the equation $y' = f(x)$ with respect to x:

$$\int y' \, dx = \int f(x) \, dx \Rightarrow$$

$$\int y'(x) \, dx = \int f(x) \, dx \Rightarrow$$

By the FUNDAMENTAL THEOREM OF CALCULUS,

$$\int f'(x) \, dx = f(x) + C$$

$$y(x) + C_1 = \int f(x) \, dx \Rightarrow$$

$$y = \int f(x)\,dx - C_1 \Rightarrow$$

$$y = \int f(x)\,dx + \underbrace{(-C_1)}_{} \Rightarrow$$

Replace by C_2
(if C_1 represents
all reals, so will
$C_2 = -C_1$)

$$y = \underbrace{\int f(x)\,dx + C_2}_{}$$

$= \phi(x)$ for any particular
value of C_2

We use a similar idea in solving DEs of the form

$$y' = g(x)\,h(y),$$

i.e., when $f(x,y)$ can be separated into the product of

(1) a function of x only
(2) a function of y only

This DE is called a

SEPARABLE EQUATION

<u>Examples of separable equations.</u>

1. $y' = 2xy = \underbrace{(2x)}_{g(x)}\underbrace{(y)}_{h(y)}$

2. $y' = \frac{4y}{x} = \underbrace{\left(\frac{4}{x}\right)}_{g(x)}\underbrace{(y)}_{h(y)}$

3. $y' = y = \underbrace{(1)}_{g(x)}\underbrace{(y)}_{h(y)}$

4. $(y - yx^2)y' = (y+1)^2 \Longrightarrow$

$$y' = \frac{(y+1)^2}{y - yx^2} = \underbrace{\left[\frac{(y+1)^2}{y}\right]}_{g(x)} \underbrace{\left(\frac{1}{1-x^2}\right)}_{h(y)}$$

5. $(x+3)y' = x - 1 \Longrightarrow$

$$y' = \frac{x-1}{x+3} = \underbrace{\left(\frac{x-1}{x+3}\right)}_{g(x)} \underbrace{(1)}_{h(y)}$$

6. $y' = x + y$ X NOT a separable equation

A separable equation is said to be <u>separable</u> or have <u>separable variables</u>.

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Method of Solution of Separable Equations: Separation of Variables

Example. (From Exercise 7, p. 35.)

Solve the DE

$$xy' - 4y = 0.$$

LONG WAY (Do not solve the DE this way)

$$xy' - 4y = 0 \implies$$

$$xy' = 4y \implies$$

$$\frac{1}{y} y' = 4 \cdot \frac{1}{x} \implies$$

$$\frac{1}{y(x)} y'(x) = 4 \cdot \frac{1}{x} \implies$$

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$$\underbrace{\int \frac{1}{y(x)} y'(x)\, dx}_{} = \int 4 \cdot \frac{1}{x}\, dx \implies$$

Use INTEGRATION BY SUBSTITUTION:

Let $\boxed{u = y(x)}$ Then $\frac{du}{dx} = y'(x) \implies \boxed{du = y'(x)\, dx}$.

$$\int \frac{1}{u}\,du = 4\int \frac{1}{x}\,dx \Longrightarrow$$

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$$\ln|u| + C_1 = 4\ln|x| + C_2 \Longrightarrow$$

$$\ln|u| = 4\ln|x| + \underbrace{(C_2 - C_1)}_{= C_3}$$

Substitute back in the y

$r\ln x = \ln x^r$

$$\ln|y| = \ln|x|^4 + C_3 \Longrightarrow$$

$|x|^4 = (\pm x)^4 = x^4$

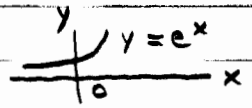
$$\ln|y| = \ln x^4 + C_3 \Longrightarrow$$

$$e^{\ln|y|} = e^{\ln x^4 + C_3} \Longrightarrow$$

$e^{x+y} = e^x e^y \Rightarrow e^{\ln x^4 + C_3} = e^{C_3} e^{\ln x^4}$

$e^{\ln x} = x$

$$|y| = \underbrace{e^{C_3}}_{= C_4 > 0} x^4 \Longrightarrow$$

since 

$$|y| = C_4 x^4 \Longrightarrow$$

$\pm y$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\pm y = C_4 x^4 \quad (C_4 > 0) \Longrightarrow$$

$$(\mp 1)(\pm y) = \underbrace{(\mp 1)\cdot C_4}_{C_5 \neq 0} x^4 \Rightarrow$$

$$y = C_5 x^4 \quad (C_5 \neq 0)$$

Observe that

$$y = 0 \quad (\text{or } y(x) = 0)$$

is also a solution of $xy' - 4y = 0$ and can be written as

$$y = 0 \cdot x^4.$$

So, the general solution (which encompasses all possible solutions) is really

$$y = \left\{ \begin{array}{l} C_5 x^4, \; C_5 \neq 0 \\ 0 \cdot x^4 \end{array} \right\} = Cx^4, \; C = C_5 \text{ or } 0$$

$$\therefore \boxed{y = Cx^4, \; C \text{ arbitrary}}$$

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<u>SHORTCUT</u> (Solve the DE this way)

$$xy' - 4y = 0 \Rightarrow$$

$$x\frac{dy}{dx} = 4y \Rightarrow$$

$$\frac{1}{y}\,dy = 4\cdot\frac{1}{x}\,dx \Rightarrow$$

$$\int \frac{1}{y}\,dy = \int 4\cdot\frac{1}{x}\,dx \Rightarrow$$

$$\int \frac{1}{y}\,dy = 4\int \frac{1}{x}\,dx \Rightarrow$$

$$\ln|y| = 4\ln|x| + C \Rightarrow$$

$$\ln|y| = \ln|x|^4 + C \Rightarrow$$

$$\ln|y| = \ln x^4 + C \Rightarrow$$

$$e^{\ln|y|} = e^{\ln x^4 + C} \Rightarrow$$

$$e^{\ln|y|} = e^{C}e^{\ln x^4} \Rightarrow$$

$$|y| = e^{C}x^4 \Rightarrow$$

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$$y = \pm e^{C} x^{4} \Longrightarrow$$

$$\boxed{y = C x^{4}, \quad C \text{ arbitrary}}$$

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Section 2.2. Exact Equations.

We will consider DEs of the form

(*) $$M(x,y)\,dx + N(x,y)\,dy = 0$$

or, equivalently,

$$M(x,y) + N(x,y)\,y' = 0.$$

E.g., $\underbrace{5x^2y^3}_{=M(x,y)}\,dx + \underbrace{(x+e^y)}_{=N(x,y)}\,dy = 0$

We will assume (and not check) that

$$M,\ N,\ M_x = \frac{\partial M}{\partial x},\ M_y = \frac{\partial M}{\partial y},\ N_x = \frac{\partial N}{\partial x},$$
$$N_y = \frac{\partial N}{\partial y}$$

are all <u>continuous</u> in a "two-dimensional sense (i.e., with respect to x and y at the same time — you will learn about this in MULTIVARIATE or ADVANCED CALCULUS).

Eq. (*) is called an EXACT EQUATION if, in addition to the continuity of

$$M,\ N,\ M_x,\ M_y,\ N_x,\ N_y$$

for certain values of x and y, say,

$$a < x < b,\quad c < y < d,$$

we have

$$\boxed{M_y = N_x}$$

This will mean that we can solve Eq. (*) using a special method that results in an <u>implicit solution</u>.

<u>Examples</u>. (In "differential" form.)

1. (Exercise 8, p. 42.)

$$\underbrace{\left(1 + \ln x + \frac{y}{x}\right)}_{M(x,y)} dx + \underbrace{(\ln x - 1)}_{N(x,y)} dy = 0$$

$$M_y = \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(1 + \ln x + \frac{y}{x}\right) = \frac{1}{x}$$

$$N_x = \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\ln x - 1\right) = \frac{1}{x}$$

$\therefore \quad M_y = N_x \Longrightarrow$ DE is EXACT
(for $x > 0$, $y > 0$)

2. (Exercise 11, p. 43.)

$$\underbrace{\left(y \ln y - e^{-xy}\right)}_{M(x,y)} dx + \underbrace{\left(\frac{1}{y} + x \ln y\right)}_{N(x,y)} dy = 0$$

$$M_y = \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(y \ln y - e^{-xy}\right)$$

$$= \frac{\partial}{\partial y}\left(y \ln y\right) - \frac{\partial}{\partial y}\left(e^{-xy}\right)$$

(1) Hold x constant and differentiate with respect to y
(2) PRODUCT RULE: $(fg)' = f'g + fg'$
(3) CHAIN RULE: $(e^{ax})' = ae^{ax}$

$$= \left[\left(\frac{\partial}{\partial y} y\right)\ln y + y\left(\frac{\partial}{\partial y}\ln y\right)\right] - (-x)e^{-xy}$$

$$= \left(1 \cdot \ln y + y \cdot \frac{1}{y}\right) + xe^{-xy}$$
$$= \ln y + 1 + xe^{-xy}$$

$$N_x = \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y} + x \ln y\right)$$
$$= \ln y$$

$$\therefore \quad M_y = \underbrace{\ln y + 1 + xe^{-xy} \neq \ln y}_{} = N_x \Longrightarrow$$

Actually, can have $\ln y + 1 + xe^{-xy} = \ln y$ since can have $1 + xe^{-xy}$. However, $\ln y + 1 + xe^{-xy} \neq \ln y$ for $x = 0, y = 1$. Hence, can say, in general, that $\ln y + 1 + xe^{-xy} \neq \ln y$.

DE is NOT EXACT (for $-\infty < x < \infty$, $y > 0$).

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Method of Solution of Exact DEs

[Not responsible for]

PRELIMINARY INFO :

① Recall the CHAIN RULE in SINGLE VARIABLE CALCULUS :

$$f(x),\ u(x) \Longrightarrow f(u(x))$$

$$\frac{d}{dx} f(u(x)) = f'(u(x)) \cdot u'(x)$$

$$= \frac{df}{du} \cdot \frac{du}{dx}$$

E.g., $f(x) = \sin x$, $u(x) = x^2$

$f(u(x)) = \sin x^2$

$$\frac{d}{dx} f(u(x)) = \begin{cases} f'(u(x)) \cdot u'(x) = (\cos x^2)(2x) \\ \qquad\qquad = 2x \cos x^2 \\ \frac{df}{du} \cdot \frac{du}{dx} = \left(\frac{d}{du} \sin u\right)\left(\frac{d}{dx} x^2\right) \\ \qquad\qquad = (\cos u)(2x) \\ \qquad\qquad = 2x \cos x^2 \end{cases}$$

(2) The CHAIN RULE can be extended to include functions of more than one variable. In particular, consider

$$f(x,y),\ u(x),\ v(x) \Rightarrow f(u(x), v(x))$$

E.g., $f(x,y) = x + y$, $u(x) = x^2$, $v(x) = x^3$

$$f(u(x), v(x)) = x^2 + x^3$$

Then, $f(u(x), v(x))$ is 2 things:

2/2/01 (1) a function of 2 variables, $f(u,v)$;
F (2) a function of 1 variable,
Lecture $f(u(x), v(x)) = F(x)$.

Thus we can take the <u>ordinary derivative</u> of f with respect to x and that can be equated to an expression involving the <u>partial derivatives</u> of f:

$$\underbrace{\frac{d}{dx} f(u(x), v(x))}_{\text{ordinary derivative}} = \underbrace{\frac{\partial f}{\partial u} \cdot \frac{du}{dx} + \frac{\partial f}{\partial v} \cdot \frac{dv}{dx}}_{\text{expression involving partial derivatives}}$$

E.g., $f(x,y) = x + y$, $u(x) = x^2$, $v(x) = x^3$

$$f(u(x), v(x)) = x^2 + x^3$$

$$f(u,v) = u + v$$

$$\frac{d}{dx} f(u(x), v(x)) \stackrel{?}{=} \frac{\partial f}{\partial u} \cdot \frac{du}{dx} + \frac{\partial f}{\partial v} \cdot \frac{dv}{dx}$$

$$\frac{d}{dx}(x^2 + x^3) \quad \left(\frac{\partial}{\partial u}(u+v)\right)\left(\frac{d}{dx} x^2\right) + \left(\frac{\partial}{\partial v}(u+v)\right)\left(\frac{d}{dx} x^3\right)$$

$$(1+0)(2x) + (0+1)(3x^2)$$

$$2x + 3x^2 \quad \stackrel{\checkmark}{=} \quad 2x + 3x^2$$

For simplicity, we write

$$\boxed{\frac{df}{\partial x} = f_u(u,v) \cdot \frac{du}{dx} + f_v(u,v) \cdot \frac{dv}{dx}}$$

TOTAL DERIVATIVE OF $f(u(x), v(x))$

Now, let

$$u(x) = x \quad \text{and} \quad v(x) = y(x).$$

Then the <u>total derivative of</u> <u>$f(x,y) = f(x, y(x))$</u> is given by

$$\frac{df}{dx} = f_x(x,y) \cdot \frac{dx}{dx} + f_y(x,y) \cdot \frac{dy}{dx} \Rightarrow$$

$$\boxed{\frac{df}{dx} = f_x(x,y) + f_y(x,y) \frac{dy}{dx}}$$

(3) Consider a DE of the form

$$f_x(x,y)\,dx + f_y(x,y)\,dy = 0,$$

Partials of some function $f(x,y)$, yet to be discovered

E.g., $2xy^3\,dx + 3x^2y^2\,dy = 0$, $f(x,y) = x^2y^3$

Rewrite this DE as

$$f_x(x,y) + f_y(x,y)\frac{dy}{dx} = 0 \Longrightarrow$$

$$\underbrace{\frac{df}{dx} = 0}$$

TOTAL DERIVATVE OF $f(x,y) = f(x, y(x))$ is equal to 0

"$\frac{df}{dx} = 0$" is a DE and is analogous to the following (very simple) DE:

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$$\underbrace{\frac{dy}{dx} = 0}$$

ORDINARY DERIVATIVE OF $y = y(x)$ is equal to 0

Solution: $y = C$ (= arbitrary constant)

Verify "$y = C$" as a solution by "differentiating both sides" and getting back DE:

$$\frac{d}{dx}(y) = -\frac{d}{dx}(C) \Rightarrow$$

$$\frac{dy}{dx} = 0 \checkmark$$

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Similarly,

$$f(x,y) = C$$

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is a (IMPLICIT) solution of the DE

$$\frac{df}{dx} = 0,$$

It can be checked by "differentiating both sides" (i.e., IMPLICIT DIFFERENTIATION) and getting back the DE:

$$\underbrace{\frac{d}{dx} f(x, y(x))}_{\text{IMPLICIT DIFF. renders the same result as TOTAL DIFF. (proof omitted)}} = \frac{d}{dx}(C)$$

$$f_x(x,y) + f_y(x,y)\frac{dy}{dx} = 0 \checkmark$$

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(4) The condition for a DE of the form $M\,dx + N\,dy = 0$ to be exact,

$$\underset{\frac{\partial M}{\partial y}}{\underset{\parallel}{M_y}} = \underset{\frac{\partial N}{\partial x}}{\underset{\parallel}{N_x}},$$

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guarantees that there exists a function

$$f(x, y)$$

such that

(1) $M(x,y) = f_x(x,y)$
(2) $N(x,y) = f_y(x,y)$
(3) $f(x,y) = C$ is the IMPLICIT SOLUTION of the DE $M\,dx + N\,dy = 0$

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So, strategy to solve an EXACT EQUATION:

Given an exact DE

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$(M, N, M_x, M_y, N_x, N_y$ all continuous$)$

$$M_y = N_x$$

We will solve it by assuming that THERE EXISTS some $f(x,y)$ such that

$$M(x,y) = f_x(x,y), \quad N(x,y) = f_y(x,y).$$

We will then find $f(x,y)$, set it equal to an arbitrary C,

$$f(x,y) = C,$$

and then call this equation the IMPLICIT SOLUTION of the exact DE.

<u>WARNING</u>: The solution is

$$f(x,y) = C,$$

not just

$$f(x,y)\ !$$

<u>Example</u>. (Exercise 27, p. 43.)

Solve the IVP (initial value problem)

$$\begin{cases} (4y + 2x - 5)\,dx + (6y + 4x - 1)\,dy = 0 \\ y(-1) = 2 \end{cases}$$

I. We first find the general implicit soln. of the DE.

<u>STEP 1</u>. $M_y = N_x$? :

$$\underbrace{(4y+2x-5)}_{M(x,y)}\,dx + \underbrace{(6y+4x-1)}_{N(x,y)}\,dy = 0$$

$$M_y = \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(4y + 2x - 5) = 4 + 0 - 0 = 4$$

$$N_x = \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(6y + 4x - 1) = 0 + 4 - 0 = 4$$

$\therefore\ M_y = N_x \Rightarrow$ DE is EXACT

<u>STEP 2</u>. Set $f_x = M$ (or $f_y = N$) and find f by integration:

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$$f_x(x,y) = M(x,y) \Rightarrow$$

$$\int f_x(x,y)\,dx = \int M(x,y)\,dx \Rightarrow$$

$$f(x,y) = \int (4y + 2x - 5)\,dx$$

$$= 4xy + 2\left(\frac{x^2}{2}\right) - 5x + g(y)$$

$$= 4xy + x^2 - 5x + g(y)$$

= constant (with respect to x) of integration, where y is acting like a constant so function of y is acting like a constant.

check: $\frac{\partial}{\partial x}\left(4xy + x^2 - 5x + g(y)\right)$

$$= 4y + 2x - 5 + 0$$

$$= 4y + 2x - 5 \checkmark$$

(OR $f_y(x,y) = N(x,y) \Rightarrow$

$$\int f_y(x,y)\,dy = \int N(x,y)\,dy \Rightarrow$$

$$f(x,y) = \int (6y + 4x - 1)\,dy$$

$$= 6\left(\frac{y^2}{2}\right) + 4xy - y + h(x)$$

$$= 3y^2 + 4xy - y + h(x)$$)

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STEP 3. Find $g(y)$ (or $h(x)$):

Look at $f_y(x,y)$, which comes from 2 places:

(1) $f_y(x,y) = \frac{\partial}{\partial y}\left(4xy + x^2 - 5x + g(y)\right)$

$= 4x + 0 - 0 + g'(y)$

$= 4x + g'(y)$

(2) $f_y(x,y) = N(x,y)$

$= 6y + 4x - 1$

$\therefore \quad \cancel{4x} + g'(y) = 6y + \cancel{4x} - 1 \implies$

$$g'(y) = \underbrace{6y - 1}$$

This side should be an expression in y only, since $g'(y)$ is a function of y only!

(OR look at $f_x(x,y)$:

(1) $f_x(x,y) = \frac{\partial}{\partial x}\left(3y^2 + 4xy - y + h(x)\right)$

$= 4y + h'(x)$

(2) $f_x(x,y) = M(x,y) = 4y + 2x - 5$

$\therefore \quad \cancel{4y} + h'(x) = \cancel{4y} + 2x - 5 \Rightarrow h'(x) = 2x - 5$)

STEP 4. Determine $g(y)$ from $g'(y)$ (or $h(x)$ from $h'(x)$) by integration:

$$g'(y) = 6y - 1 \Rightarrow \int g'(y)\,dy = \int (6y-1)\,dy$$

$$\Rightarrow g(y) = 6\left(\frac{y^2}{2}\right) - y + C_1$$

$$\Rightarrow g(y) = 3y^2 - y + C_1$$

(OR $h'(x) = 2x - 5 \Rightarrow \int h'(x)\,dx = \int (2x-5)\,dx$

$$\Rightarrow h(x) = 2\left(\frac{x^2}{2}\right) - 5x + C_1$$

$$\Rightarrow h(x) = x^2 - 5x + C_1$$

STEP 5. Substitute the expression for $g(y)$ (or $h(x)$) into the expression for $f(x,y)$:

$$f(x,y) = 4xy + x^2 - 5x + g(y) \Rightarrow$$
$$f(x,y) = 4xy + x^2 - 5x + 3y^2 - y + C_1$$

(OR $f(x,y) = 3y^2 + 4xy - y + h(x) \Rightarrow$
$f(x,y) = 3y^2 + 4xy - y + x^2 - 5x + C_1$)

STEP 8. Set $f(x,y) = C_2$:

$$f(x,y) = C_2 \Longrightarrow$$

$$4xy + x^2 - 5x + 3y^2 - y + C_1 = C_2 \Longrightarrow$$

$$4xy + x^2 - 5x + 3y^2 - y = \underbrace{C_2 - C_1}_{=C} \Longrightarrow$$

$$\boxed{4xy + x^2 - 5x + 3y^2 - y = C}$$

GENERAL IMPLICIT SOLUTION

WARNING:

$$f(x,y) = 4xy + x^2 - 5x + 3y^2 - y + C_1$$

is <u>not</u> the solution of the DE,

$$4xy + x^2 - 5x + 3y^2 + C_1 = C_2$$

or

$$4xy + x^2 - 5x + 3y^2 = C$$

<u>is</u> the solution of the DE.

(OR $f(x,y) = C_2 \Longrightarrow$

$$3y^2 + 4xy - y + x^2 - 5x + C_1 = C_2 \Longrightarrow$$

$$\boxed{3y^2 + 4xy - y + x^2 - 5x = C}$$

GENERAL IMPLICIT SOLUTION)

STEP 9. Apply initial condition $y(-1)=2$ and find particular solution:

$y(-1)=2 \Rightarrow y=2$ when $x=-1$

$$\Rightarrow (4xy + x^2 - 5x + 3y^2 - y = C)$$

$$4(-1)(2) + (-1)^2 - 5(-1) + 3(2)^2 - (2) = C$$

$$\Rightarrow -8+1+5+12-2 = C$$

$$\Rightarrow C = 8$$

$$\Rightarrow \boxed{4xy + x^2 - 5x + 3y^2 - y = 8}$$

PARTICULAR SOLUTION

SKIP HW #2 Exercise 38, p. 43, Sect. 2.2.

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Section 2.3. Linear Equations.

So far : Studied 2 techniques used to solve FIRST-ORDER DEs which may or may not be LINEAR.

(1) Separable Equations

E.g., $\frac{dy}{dx} = \frac{y}{x} \Rightarrow xy' - y = 0$

$a_1(x)y' + a_0(x)y = f(x)$

LINEAR

E.g., $\frac{dy}{dx} = \frac{x}{y} \Rightarrow yy' = x$

$a_1(x)y' + a_0(x)y = f(x)$

NONLINEAR

(2) Exact Equations

E.g., $y\,dx + x\,dy = 0$

$(M_y = 1 = N_x)$

$\Rightarrow xy' + y = 0$

$a_1(x)y' + a_0(x)y = 0$

LINEAR

E.g., $(2x-1)\,dx + (3y+7)\,dy = 0$

$(M_y = 0 = N_x)$

$\Rightarrow \quad (3y+7)y' = 2x-1$

$a_1(x)y' + a_0(x)y = f(x)$

NONLINEAR

<u>Today</u>: We will look at all LINEAR FIRST-ORDER DEs, and the particular technique used to solve them. If they also happen to be separable or exact, then you will have more than one method of solution to choose from.

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Definitions

<u>Defn.</u> A <u>linear first-order DE</u> is a DE of the

$$a_1(x)y' + a_0(x)y = g(x).$$

<u>Defn.</u> A linear first-order DE is in <u>standard form</u> if it is in the form

$$y' + P(x)y = f(x)$$

(where $P(x) = a_0(x)/a_1(x)$ and $f(x) = g(x)/a_1(x)$).

<u>Defn.</u> Consider the linear, first-order DE

$$a_1(x)y' + a_0(x)y = g(x)$$

together with its standard form

$$y' + P(x)y = f(x),$$

$$P(x) = \frac{a_0(x)}{a_1(x)}, \quad f(x) = \frac{g(x)}{a_1(x)}.$$

Let I be any interval of x on which

(1) $a_1(x) \neq 0$
(2) $P(x)$ is continuous
(3) $f(x)$ is continuous

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Then we seek a solution $y = \phi(x)$ of the DE that is valid in I or in some subinterval of I.

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<u>Examples</u> (Unusual but to illustrate the above.)

1. $xy' + y = x \Longrightarrow y' + \frac{1}{x} y = 1$

Divide through by x, so will have to assume $x \neq 0$. Then largest I can be either $(0, \infty)$ $(0 < x < \infty)$ or $(-\infty, 0)$ $(-\infty < x < 0)$.

$P(x) = \frac{1}{x}$ and $f(x) = 1$ both continuous on any interval not containing $x = 0$

Soln. turns out to be

$$y = \frac{1}{2} x,$$

which is valid on $(0, \infty)$ (or $(-\infty, 0)$).

2. $xy' + xy = x \Longrightarrow y' + y = 1$

Divide through by $x \Rightarrow$ assume $x \neq 0 \Rightarrow$ choose $I = (0, \infty)$

$P(x) = 1$ and $f(x) = 1$ both continuous on $(-\infty, \infty)$

Soln. turns out to be

$$y = 1, \quad \text{valid on } (0, \infty).$$

3. $y' = \frac{1}{2}y \Longrightarrow y' - \frac{1}{2}y = 0$

$P(x) = -\frac{1}{2}$ and $f(x) = 0$ both continuous on $(-\infty, \infty)$

Soln. turns out to be

$$y = \sqrt{x}, \quad \text{valid on } [0, \infty)$$

Examples.

1. (Exercise 9, p. 51)

Find general solution of the DE

$$x^2y' + xy = 1$$

and state interval on which general solution is defined (the interval should be connected).

STEP 1. Place DE in STANDARD FORM:

$$\frac{1}{x^2}(x^2y' + xy) = \frac{1}{x^2}(1) \Rightarrow$$

$$y' + \frac{1}{x}y = \frac{1}{x^2}$$ so $x \neq 0$

STEP 2. Find INTEGRATING FACTOR $\mu(x)$:

$$y' + \underbrace{\frac{1}{x}}_{P(x)} y = \frac{1}{x^2}$$

$$P(x) \Rightarrow \mu(x) = e^{\int P(x)\,dx} = e^{\int \frac{1}{x}\,dx}$$

$$= e^{\ln|x|} = e^{\ln x} = x$$

Do not bother adding a "C" here

For convenience, let $x > 0$ ($x \neq 0$, so we need to choose between the intervals $(-\infty, 0)$ and $(0, \infty)$)

STEP 3. Multiply both sides of DE by INTEGRATING FACTOR:

$$x\left(y' + \frac{1}{x}y\right) = x\left(\frac{1}{x^2}\right) \Longrightarrow$$

$$\underbrace{xy' + 1\cdot y}_{} = \frac{1}{x} \Longrightarrow$$

OBSERVE that this is really $\frac{d}{dx}(xy)$ or $\frac{d}{dx}(xy(x))$ where

$$\frac{d}{dx}(xy) = x\left(\frac{d}{dx}y\right) + \left(\frac{d}{dx}x\right)y = xy' + 1\cdot y$$

by the PRODUCT RULE ($(fg)' = fg' + f'g$).

$$\frac{d}{dx}(xy) = \frac{1}{x}.$$

STEP 4. Integrate both sides with respect to x:

$$\underbrace{\int \frac{d}{dx}(xy)\,dx}_{} = \int \frac{1}{x}\,dx \Longrightarrow$$

OBSERVE that $\int \frac{d}{dx} f(x)\,dx = f(x)$

$$xy = \ln|x| + C \Longrightarrow$$

Can eliminate $|\cdot|$ since $x>0$

Can divide through by x since $x>0$

$$\boxed{y = \frac{\ln x}{x} + \frac{C}{x}, \quad x > 0}$$

or $0 < x < \infty$

or $x \in (0, \infty)$

2. (Exercise 48, p. 52):

Solve the DE

$$x\,dy + (xy + 2y - 2e^{-x})\,dx = 0$$

subject to the initial condition

$$y(1) = 0.$$

STEP 1.

$$x\,dy + (xy + 2y - 2e^{-x})\,dx = 0 \Longrightarrow$$

$$xy' + (x+2)y = 2e^{-x} \Longrightarrow$$

$$\frac{1}{x}\left[xy' + (x+2)y\right] = \frac{1}{x}\left(2e^{-x}\right) \Longrightarrow$$

$$y' + \left(1 + \frac{2}{x}\right)y = \frac{2e^{-x}}{x}$$

So let $x \neq 0$

STEP 2.

$$y' + \underbrace{\left(1 + \frac{2}{x}\right)}_{P(x)} y = \frac{2e^{-x}}{x}$$

$$P(x) \Longrightarrow \mu(x) = e^{\int P(x)\,dx} = e^{\int \left(1 + \frac{2}{x}\right)dx}$$

$$= e^{x + 2\ln|x|} = e^{x + \ln x^2}$$

$$= e^{x} e^{\ln x^2} = x^2 e^{x}$$

STEP 3.

$$x^2 e^x \left[y' + \left(1 + \frac{2}{x}\right) y \right] = x^2 e^x \left(\frac{2e^{-x}}{x} \right) \Longrightarrow$$

$$\underbrace{x^2 e^x y' + \underbrace{(x^2 e^x + 2xe^x)}_{= \frac{d}{dx}(x^2 e^x)} y}_{} = 2x \Longrightarrow$$

OBSERVE that this is really $\frac{d}{dx}\left[(x^2 e^x) y\right]$ or $\frac{d}{dx}\left[(x^2 e^x) y(x)\right]$ where

$$\frac{d}{dx}\left[(x^2 e^x) y\right] = x^2 e^x \left(\frac{d}{dx} y\right) + \left(\frac{d}{dx} x^2 e^x\right) y$$
$$= x^2 e^x y' + (x^2 e^x + 2xe^x) y$$

by the PRODUCT RULE ($(fg)' = fg' + f'g$).

$$\frac{d}{dx}\left(x^2 e^x y\right) = 2x.$$

STEP 4.

$$\underbrace{\int \frac{d}{dx}\left(x^2 e^x y\right) dx}_{} = \int 2x \, dx \Longrightarrow$$

OBSERVE that $\int \frac{d}{dx} f(x) dx = f(x)$

$$x^2 e^x y = 2\left(\frac{x^2}{2}\right) + C \Longrightarrow$$

$$x^2 e^x y = x^2 + C \Longrightarrow$$

Can divide through by x^2 since $x \neq 0$. Note that $e^x > 0$ for all x.

$$y = \frac{x^2}{x^2 e^x} + \frac{C}{x^2 e^x} \Longrightarrow$$

$$\boxed{y = e^{-x} + C x^{-2} e^{-x}}$$ GENERAL SOLN.

<u>STEP 5</u>. Apply initial condition to general soln. to find C:

$y(1) = 0 \Rightarrow y = 0$ when $x = 1$

$$\Longrightarrow 0 = e^{-1} + C(1)^{-2} e^{-1}$$

$$\Longrightarrow 0 = \frac{1}{e} + \frac{C}{e}$$

$$\Longrightarrow 0 = 1 + C$$

$$\Longrightarrow C = -1.$$

$\therefore \; y = e^{-x} + (-1)\, x^{-2} e^{-x} \implies$

$$\boxed{y = e^{-x} - x^{-2} e^{-x}}$$ PARTICULAR SOLN.

Interval of validity: Choose from $(-\infty, 0)$ and $(0, \infty)$ (where $x \neq 0$). Interval must contain $x = 1$ where initial condition is given as $y(1) = 0$. So, choose

$$\boxed{I = (0, \infty)}$$

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3. (HW Exercise 55, p. 52.)

Express the solution of the IVP

$$\begin{cases} x^3 \dfrac{dy}{dx} + 2x^2 y = 10 \sin x \\ y(1) = 0 \end{cases}$$

in terms of the <u>sine integral function</u>,

$$Si(x) = \int_0^x \frac{\sin t}{t}\, dt$$

<u>STEP 1.</u>

$$x^3 y' + 2x^2 y = 10 \sin x \Rightarrow$$

$$y' + \frac{2}{x} y = \frac{10 \sin x}{x^3}$$

So let $x \neq 0 \Rightarrow$ choose $I = (0, \infty)$

<u>STEP 2.</u>

$$y' + \underbrace{\frac{2}{x}}_{P(x)} y = \frac{10 \sin x}{x^3}$$

$$P(x) \Rightarrow \mu(x) = e^{\int P(x)\,dx} = e^{\int \frac{2}{x} dx} = e^{2 \ln |x|}$$

$$= e^{2 \ln x} = e^{\ln x^2} = x^2$$

STEP 3.

$$x^2\left(y' + \frac{2}{x}y\right) = x^2\left(\frac{10\sin x}{x^3}\right) \Rightarrow$$

$$\underbrace{x^2y' + 2xy}_{=\frac{d}{dx}(x^2y)} = \frac{10\sin x}{x}$$

STEP 4.

$$\int \frac{d}{dx}(x^2y)\,dx = 10\int \frac{\sin x}{x}\,dx \Rightarrow$$

$$x^2y = 10\int \frac{\sin x}{x}\,dx \Rightarrow$$

$$y = \frac{10}{x^2}\underbrace{\int \frac{\sin x}{x}\,dx}_{} \Rightarrow$$

Can rewrite this as $Si(x) + C$: By "SECOND FUNDAMENTAL THEOREM OF CALCULUS," the function F defined by $F(x) = \int_0^x \frac{\sin t}{t}\,dt$ is an antiderivative of $\frac{\sin x}{x}$. So rename F as Si.

$$y = \frac{10}{x^2}\left(Si(x) + C\right) \Rightarrow$$

$$y = \frac{10\,Si(x)}{x^2} + \frac{10C}{x^2} \Rightarrow$$

$$y = \frac{10\,Si(x)}{x^2} + \frac{C}{x^2}$$ GENERAL SOLN.

STEP 5.

$y(1) = 0 \Rightarrow y = 0$ when $x = 1 \Rightarrow$

$$0 = \frac{10\,Si(1)}{(1)^2} + \frac{C}{(1)^2} \Rightarrow$$

$$C = -10\,Si(1) \Rightarrow$$

$$\boxed{y = \frac{10\,Si(x)}{x^2} - \frac{10\,Si(1)}{x^2}}$$

PARTICULAR SOLN.

Lecture

Section 2.4 – Solutions by Substitution.

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Three parts:

(1) Substitution with a homogeneous DE

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

(2) " " " Bernoulli DE

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

(3) " " " Potentially Separable DE

$$\frac{dy}{dx} = f(Ax + By + C)$$

For an analogy, recall INTEGRATION BY SUBSTITUTION:

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du$$

Let $u = g(x)$. Then $\frac{du}{dx} = g'(x) \Rightarrow du = g'(x)\,dx$

$$= F(u) + C$$

Where F is the antiderivative of f (i.e., $F' = f$)

$$= F(g(x)) + C$$

We can <u>transform</u> an integral into one that is <u>more easily solved</u> (i.e., solved by already known techniques) by a "<u>change of variable</u>" or "<u>substitution</u>."

E.g., $\int 2xe^{x^2}\,dx = \int e^u\,du = e^u + C = \boxed{e^{x^2}+C}$

let $\boxed{u = x^2}$. Then $\frac{du}{dx} = 2x \Rightarrow \boxed{du = 2x\,dx}$

We can do <u>something similar with DEs</u>, to <u>reduce them to DEs</u> that we <u>already know how to solve</u>.

<u>Homogeneous DEs</u>

<u>Defn.</u> $f(x,y)$ is said to be a <u>homogeneous function of degree α</u>, where α is some real number, if, when we substitute cx for x and cy for y with c any constant, we get

$$f(cx, cy) = c^{\alpha} f(x,y)$$

E.g., $f(x,y) = 5x^{1/3}y - 3xy^{1/3}$

$$\begin{aligned} f(cx,cy) &= 5(cx)^{1/3}(cy) - 3(cx)(cy)^{1/3} \\ &= 5c^{1/3}\cdot c\,x^{1/3}y - 3c\cdot c^{1/3}xy^{1/3} \\ &= 5c^{4/3}x^{1/3}y - 3c^{4/3}xy^{1/3} \\ &= c^{4/3}(5x^{1/3}y - 3xy^{1/3}) \\ &= c^{4/3}f(x,y) \end{aligned}$$

$\therefore$ $f(x,y)$ is a homogeneous function of degree $\frac{1}{3}$.

E.g., $f(x,y) = x^3 + y^2$

$$\begin{aligned} f(cx,cy) &= (cx)^3 + (cy)^2 \\ &= c^3x^3 + c^2y^2 \\ &= c^2(cx^3 + y^2) \\ &\neq c^2 f(x,y) \end{aligned}$$

$\therefore$ $f(x,y)$ is NOT a homogeneous function.

Roughly speaking,

A homogeneous function of degree α is a function of the form

$$f(x,y) = x^{m_1}y^{n_1} + x^{m_2}y^{n_2} + \cdots + x^{m_i}y^{n_i}$$

where

$$m_1+n_1 = m_2+n_2 = \cdots = m_i+n_i = \alpha$$

<u>Defn.</u> A first-order DE of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be <u>homogeneous</u> if (1) $M(x,y)$ and $N(x,y)$ are both homogeneous functions of the same degree.

homo

E.g., $\underbrace{(x-y)}_{M(x,y)}\,dx + \underbrace{(x+y)}_{N(x,y)}\,dy$

$M(x,y) = x-y = x'-y' \Rightarrow$ homog. of degree 1

$N(x,y) = x+y = x'+y' \Rightarrow$ homog. of degree 1

$\therefore$ DE is homogeneous.

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<u>Method of Solution of Homogeneous DEs</u>

With a homogeneous DE, we will make either of the following substitutions:

(1) $\boxed{y = ux}$ (i.e., $y(x) = u(x)\,x$)

Then $\frac{dy}{dx} = \frac{d}{dx}(ux) = u \cdot 1 + \frac{du}{dx} \cdot x \Rightarrow$

PRODUCT RULE $(fg)' = fg' + f'g$

$$\frac{dy}{dx} = u + x\frac{du}{dx} \Rightarrow \boxed{dy = u\,dx + x\,du}$$

- Used when N is simpler than M (since replacing dy with a messier expression).
- End up with DE in $u(x)$ and x
- Solve for u in terms of x
- Substitute back $u = y/x$
- Homogeneity allows for factoring and cancelling of x

(2) $\boxed{x = vy}$ (i.e., $x = v(x)y(x)$)

Then $\frac{d}{dx}(x) = \frac{d}{dx}(vy) = v\frac{dy}{dx} + \frac{dv}{dx} \cdot y \Rightarrow$

$$1 = v\frac{dy}{dx} + \frac{dv}{dx} \cdot y \Rightarrow \boxed{dx = v\,dy + y\,dv}$$

- Used when M is simpler than N (since replacing dx with a messier expression).
- End up with DE in $v(x)$ and y
- Solve for y in terms of v
- Substitute back $v = x/y$
- Homogeneity allows for factoring and cancelling of x

<u>Examples.</u>

1. (Exercise 1, p. 57.) Solve the homogeneous equation

$$(x-y)\,dx + x\,dy = 0$$

by using an appropriate substitution.

<u>Note</u>: DE is homogeneous since

$$\left.\begin{array}{l} M(x,y) = x^1 - y^1 \Rightarrow \text{homog. of degree } 1 \\ N(x,y) = x^1 \Rightarrow \text{homog. of degree } 1 \end{array}\right\} \text{same degree}$$

<u>Note</u>: $N(x,y) = x$ is "simpler" than $M(x,y) = x - y \Rightarrow$ choose substitution with

$$\boxed{y = ux} \qquad \boxed{dy = u\,dx + x\,du}$$

$$(x-y)\,dx + x\,dy = 0 \Rightarrow$$

$$(x-ux)\,dx + x(u\,dx + x\,du) = 0 \Rightarrow$$

$$\cancel{x}(1-u)\,dx + \cancel{x}(u\,dx + x\,du) = 0 \Rightarrow$$

Assuming $x \neq 0$

$$(1-u)\,dx + u\,dx + x\,du = 0 \Rightarrow$$

So the homogeneity of degree 1 of both M and N has allowed us to factor out an x and then cancel the x

$$(1-\cancel{u}+\cancel{u})\,dx + x\,du = 0 \Rightarrow$$

$$dx + x\,du = 0 \Rightarrow$$

Looks like a separable eq.

$$x\,du = -dx \Rightarrow$$

$$du = -\frac{1}{x}\,dx \Rightarrow$$

$$\int 1 \cdot du = -\int \frac{1}{x}\,dx \Rightarrow$$

$$u = -\ln|x| + C \Longrightarrow$$

Substitute back:
$y = ux \Rightarrow u = \frac{y}{x}$

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$$\boxed{\frac{y}{x} = -\ln|x| + C} \Longrightarrow$$

$$\boxed{y = -x\ln|x| + Cx}$$

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2. (Exercise 3, p. 57.) Solve the homogeneous equation

$$x\,dx + (y - 2x)\,dy = 0$$

by using an appropriate substitution.

<u>Note</u>: DE is homogeneous since

$M(x,y) = x^1 \Longrightarrow$ homog. of degree 1
$N(x,y) = y^1 - 2x^1 \Longrightarrow$ homog. of degree 1
} same degree

<u>Note</u>: $M(x,y) = x$ is "simpler" than $N(x,y) = y - 2x \Longrightarrow$ choose substitution with

$$\boxed{x = vy} \qquad \boxed{dx = v\,dy + y\,dv}$$

Actually, we are going to use the other inappropriate substitution,

$$\boxed{y = ux} \qquad \boxed{dy = u\,dx + x\,du} \checkmark$$

$$x\,dx + (y - 2x)\,dy = 0 \Longrightarrow$$

$$x\,dx + (ux - 2x)(u\,dx + x\,du) = 0 \Longrightarrow$$

$$\cancel{x}\,dx + \cancel{x}(u-2)(u\,dx + x\,du) = 0 \Longrightarrow$$

Assuming $x \neq 0$

$$dx + (u-2)(u\,dx + x\,du) = 0 \Longrightarrow$$

$$dx + u^2\,dx + u\,x\,du - 2u\,dx - 2x\,du = 0 \Longrightarrow$$

$$(1 + u^2 - 2u)\,dx + (ux - 2x)\,du = 0 \Longrightarrow$$

$$(u^2 - 2u + 1)\,dx + x(u-2)\,du = 0 \Longrightarrow$$

$$(u-1)^2\,dx + x(u-2)\,du = 0 \Longrightarrow$$

Looks like a separable eq.

$$x(u-2)\,du = -(u-1)^2\,dx \Longrightarrow$$

$$\frac{u-2}{(u-1)^2}\,du = -\frac{1}{x}\,dx \Longrightarrow$$

$$\boxed{\int \frac{u-2}{(u-1)^2}\,du} = -\int \frac{1}{x}\,dx$$

TWO WAYS TO HANDLE THIS INTEGRAL:

① TRICK + INTEGRATION BY SUBSTITUTION

$$\frac{u-2}{(u-1)^2} = \frac{u-1-1}{(u-1)^2} = \frac{\cancel{u-1}}{(u-1)^{\cancel{2}}} - \frac{1}{(u-1)^2}$$

$$= \frac{1}{u-1} - \frac{1}{(u-1)^2} \Rightarrow$$

$$\int \frac{u-2}{(u-1)^2}\,du = \underbrace{\int \frac{1}{u-1}\,du} - \underbrace{\int \frac{1}{(u-1)^2}\,du}$$

Let $\boxed{w = u-1}$. Then $\frac{dw}{du} = 1 \Rightarrow$ $\boxed{dw = du}$.

Let $\boxed{w = u-1}$. Then $\frac{dw}{du} = 1 \Rightarrow$ $\boxed{dw = du}$.

$$= \int \frac{1}{w}\,dw - \int \frac{1}{w^2}\,dw$$

$$= \int \frac{1}{w}\,dw - \int w^{-2}\,dw$$

$$= \ln|w| - \frac{w^{-2+1}}{-2+1} + C$$

$$= \ln|w| + \frac{1}{w} + C$$

$$= \ln|u-1| + \frac{1}{u-1} + C$$

② METHOD OF PARTIAL FRACTIONS + INTEGRATION BY SUBSTITUTION

$$\frac{u-2}{(u-1)^2} = \frac{A}{u-1} + \frac{B}{(u-1)^2} \Rightarrow$$

$$u-2 = A(u-1) + B \Rightarrow$$

$$1 \cdot u - 2 = Au + (-A+B) \Rightarrow$$

$$\left.\begin{array}{l} A = 1 \\ -A+B = -2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} A = 1 \\ B = -1 \end{array}\right\} \Rightarrow$$

$$\int \frac{u-2}{(u-1)^2}\,du = \int \frac{A}{u-1}\,du + \int \frac{B}{(u-1)^2}\,du$$

$$= \int \frac{1}{u-1}\,du - \int \frac{1}{(u-1)^2}\,du$$

$$\vdots$$

$$= \ln|u-1| + \frac{1}{u-1} + C$$

$$\therefore \int \frac{u-2}{(u-1)^2}\,du = -\int \frac{1}{x}\,dx \Rightarrow$$

$$\ln|u-1| + \frac{1}{u-1} = -\ln|x| + C \Rightarrow$$

Substitute back
$y = ux \Rightarrow u = \frac{y}{x}$

$$\boxed{\ln\left|\frac{y}{x}-1\right| + \frac{1}{\frac{y}{x}-1} = -\ln|x| + C} \Rightarrow$$

$$\boxed{\ln\left|\frac{y-x}{x}\right| + \frac{x}{y-x} = -\ln|x| + C} \Rightarrow$$

$$\ln|y-x| - \cancel{\ln|x|} + \frac{x}{y-x} = -\cancel{\ln|x|} + C \Rightarrow$$

$$\boxed{\ln|y-x| + \frac{x}{y-x} = C} \Rightarrow$$

$$(x-y)\ln|x-y| + \frac{x(x-y)}{y-x} = C(x-y) \Rightarrow$$

$$\boxed{(x-y)\ln|x-y| - x = C(x-y)} \Rightarrow$$

$$(x-y)\ln|x-y| - x + \underbrace{y-y}_{=0} = C(x-y) \Rightarrow$$

$$(x-y)\ln|x-y| - (x-y) - y = C(x-y) \Rightarrow$$

$$(x-y)\ln|x-y| - y = C(x-y) + 1\cdot(x-y) \Rightarrow$$

$$(x-y)\ln|x-y| - y = \underbrace{(C+1)}_{=C}(x-y) \Rightarrow$$

$$\boxed{(x-y)\ln|x-y|-y = C(x-y)}$$

Solution in SOLUTIONS MANUAL

Bernoulli DEs

Defn. A Bernoulli equation is a DE of the form

$$\frac{dy}{dx} + P(x)\,y = f(x)\,y^{\alpha},$$

where α is any real number.

Notes: (1) When $\alpha = 0$, we have a linear first-order DE

$$\frac{dy}{dx} + P(x)\,y = f(x).$$

(2) When $\alpha = 1$, we still have a linear first-order DE

$$\frac{dy}{dx} + P(x)\,y = f(x)\,y \implies$$
$$\frac{dy}{dx} + \left[P(x) - f(x)\right]y = 0.$$

Method of Solution of Bernoulli DEs

- Assume $\alpha \neq 0, 1$ in

$$\frac{dy}{dx} + P(x)y = f(x)y^{\alpha}$$

- Let $u = y^{1-\alpha}$ (where $u = u(x)$). Then

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$$\boxed{y = u^{\frac{1}{1-\alpha}}} \Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{1-\alpha} u^{1-\frac{1}{1-\alpha}} \cdot \frac{du}{dx}}$$

CHAIN RULE:

$$\frac{d}{dx}\left[f(x)\right]^n = n\left[f(x)\right]^{n-1} \cdot \frac{df}{dx}$$

- Substitute these expressions for y and $\frac{dy}{dx}$ in the DE.
- The DE will reduce to a LINEAR FIRST-ORDER DE, which we solve by the method given in Sect. 2.3.

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<u>Example</u>: (Exercise 15, p. 57.) Solve the Bernoulli equation

$$x\frac{dy}{dx} + y = \frac{1}{y^2}$$

by using an appropriate substitution.

<u>STEP 1</u>. First divide through by x:

$$\frac{dy}{dx} + \underbrace{\frac{1}{x}}_{P(x)=\frac{1}{x}}\, y = \underbrace{\frac{1}{x}}_{f(x)=\frac{1}{x}}\, y^{-2} \qquad \alpha = -2$$

<u>STEP 2</u>. We have a Bernoulli equation with $\alpha = -2$. So, let $u = y^{1-\alpha}$:

$$u = y^{1-(-2)} = y^3 \Rightarrow \boxed{u = y^3} \Rightarrow$$

$$u^{1/3} = y \Rightarrow \boxed{y = u^{1/3}} \Rightarrow$$

$$\frac{dy}{dx} = \frac{d}{dx}u^{1/3} = \frac{1}{3}u^{\frac{1}{3}-1}\cdot\frac{du}{dx} = \frac{1}{3}u^{-2/3}\frac{du}{dx} \Rightarrow$$

$$\boxed{\frac{dy}{dx} = \frac{1}{3}u^{-2/3}\frac{du}{dx}}$$

STEP 3. Substitute $\boxed{y = u^{1/3}}$ and $\boxed{\frac{dy}{dx} = \frac{1}{3} u^{-2/3} \frac{du}{dx}}$ into the DE:

$$\frac{dy}{dx} + \frac{1}{x} y = \frac{1}{x} y^{-2} \Longrightarrow$$

$$\frac{1}{3} u^{-2/3} \frac{du}{dx} + \frac{1}{x} u^{1/3} = \frac{1}{x} \left(u^{1/3}\right)^{-2} \Longrightarrow$$

$$\frac{1}{3} u^{-2/3} \frac{du}{dx} + \frac{1}{x} u^{1/3} = \frac{1}{x} u^{-2/3} \Longrightarrow$$

$$3u^{2/3}\left(\frac{1}{3} u^{-2/3} \frac{du}{dx} + \frac{1}{x} u^{1/3}\right) = 3u^{2/3}\left(\frac{1}{x} u^{-2/3}\right) \Longrightarrow$$

$$u^{\frac{2}{3}-\frac{2}{3}} \frac{du}{dx} + \frac{3}{x} u^{\frac{2}{3}+\frac{1}{3}} = \frac{3}{x} u^{\frac{2}{3}-\frac{2}{3}} \Longrightarrow$$

$$\boxed{\frac{du}{dx} + \frac{3}{x} u = \frac{3}{x}}$$

LINEAR FIRST-ORDER DE (in u)

$$u' + \underbrace{P(x)}_{=\frac{3}{x}} u = \underbrace{f(x)}_{=\frac{3}{x}}$$

STEP 4. Solve linear first-order DE in usual way:

$$\mu(x) = e^{\int P(x)\,dx} = e^{\int \frac{3}{x}\,dx} = e^{3\ln|x|}$$

$$= e^{3\ln x} = e^{\ln x^3} = x^3 \Longrightarrow$$

will assume $I = (0, \infty)$

$$x^3\left(\frac{du}{dx} + \frac{3}{x}u\right) = x^3\left(\frac{3}{x}\right) \Rightarrow$$

$$\underbrace{x^3\frac{du}{dx} + 3x^2u}_{} = 3x^2 \Rightarrow$$

$$= \frac{d}{dx}(x^3u) = x^3\left(\frac{d}{dx}u\right) + \left(\frac{d}{dx}x^3\right)u = x^3\frac{du}{dx} + 3x^2u \checkmark$$

$$\frac{d}{dx}(x^3u) = 3x^2 \Rightarrow$$

$$\int \frac{d}{dx}(x^3u)\,dx = \int 3x^2\,dx \Rightarrow$$

$$x^3u = x^3 + C \Rightarrow$$

$$\boxed{u = 1 + \frac{C}{x^3}}$$

<u>STEP 5.</u> Substitute back and replace u by y^3:

$$u = y^3 \Rightarrow$$

$$\boxed{y^3 = 1 + \frac{C}{x^3}} \Rightarrow$$

$$\boxed{y = \left(1 + \frac{C}{x^3}\right)^{1/3}}$$

Equations of the Form

$$\frac{dy}{dx} = f(Ax + By + C), \quad B \neq 0$$

Note: If $B = 0$, then we have

$$\frac{dy}{dx} = f(Ax + C),$$

which is a separable DE.

Method of Solution

- Assume $B \neq 0$
- Let $\boxed{u = Ax + By + C}$ (where $u = Ax + By(x) + C$ so $u = u(x)$). Then

$$\frac{du}{dx} = \frac{d}{dx}(Ax + \underset{\substack{\uparrow \\ y = y(x)}}{By} + C) = A + B\frac{dy}{dx} \Longrightarrow$$

$$\frac{du}{dx} = A + B\frac{dy}{dx} \Longrightarrow$$

$$\boxed{\frac{dy}{dx} = \frac{1}{B}\left(\frac{du}{dx} - A\right)}$$

- Substitute these expressions for y and $\frac{dy}{dx}$ in the DE.

- The DE will reduce to a SEPARABLE DE, which we solve by the method given in Sect. 2.1.

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<u>Example</u>. (Exercise 27, p. 58.) Solve the DE

$$\frac{dy}{dx} = 2 + \sqrt{y - 2x + 3}.$$

<u>STEP 1</u>. This DE is of the form

$$\frac{dy}{dx} = f(Ax + By + C):$$

$$f(w) = 2 + \sqrt{w}$$
$$A = -2, \quad B = 1, \quad C = 3$$

<u>STEP 2</u>. Let $u = Ax + By + C$:

$$\boxed{u = y - 2x + 3} \Rightarrow$$

$$\frac{du}{dx} = \frac{d}{dx}(y - 2x + 3) = \frac{dy}{dx} - 2 \Rightarrow$$

$$\frac{du}{dx} = \frac{dy}{dx} - 2 \Rightarrow \boxed{\frac{dy}{dx} = \frac{du}{dx} + 2}$$

<u>STEP 3</u>. Substitute $\boxed{u = -2x + y + 3}$ and $\boxed{\frac{dy}{dx} = \frac{du}{dx} + 2}$ into the DE:

$$\frac{dy}{dx} = 2 + \sqrt{y - 2x + 3} \Rightarrow$$

$$\frac{du}{dx} + \cancel{2} = \cancel{2} + \sqrt{u} \Rightarrow$$

$$\boxed{\frac{du}{dx} = \sqrt{u}}$$

SEPARABLE DE (in u)

$$\frac{du}{dx} = \underset{g(x)}{1} \cdot \underset{h(u)}{\sqrt{u}}$$

STEP 4. Solve separable DE in usual way:

$$\frac{1}{\sqrt{u}}\, du = dx \Rightarrow$$

$$u^{-1/2}\, du = dx \Rightarrow$$

$$\int u^{-1/2}\, du = \int 1 \cdot dx \Rightarrow$$

$$\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = x + C \Rightarrow$$

$$\boxed{2\sqrt{u} = x + C}$$

<u>STEP 5</u>. Substitute back and replace u by $y-2x+3$:

$$u = y-2x+3 \Longrightarrow$$

$$\boxed{2\sqrt{y-2x+3} = x + C} \Longrightarrow$$

$$\boxed{\sqrt{y-2x+3} = \frac{x}{2} + C}$$

<u>WARNING</u>: Do not rewrite this as

$$y-2x+3 = \left(\frac{x}{2}+C\right)^2 \Longrightarrow$$

$$y = \left(\frac{x}{2}+C\right)^2 + 2x - 3$$

where

$$\overbrace{y-2x+3}^{\text{cannot be negative since equal to a square } (\ )^2} = \Big(\underbrace{\frac{x}{2}+C}_{\text{can be negative since this quantity is squared and made nonnegative.}}\Big)^2$$

is not equivalent to

cannot be negative since equal to a square root $\sqrt{\ }$

$$\underbrace{\sqrt{y-2x+3}}_{\text{cannot be negative for square root to make sense}} = \overbrace{\frac{x}{2}+C}^{\text{cannot be negative since equal to a square root}}$$

Lecture

CHAPTER 3. Modeling with First-Order DEs

In this chapter, we will see how the techniques that we learned about in Chapter 2 apply to DEs that represent or model real-life situations.

Note: Section 1.3, which we skipped, is not a prerequisite for this chapter.

Lecture

Section 3.1. Linear Equations.

We will look at four models

$\boxed{\text{MODEL 1}}$ Population Growth.

For example, suppose we want to come up with a model giving

# of bacteria in a petri dish at any time t (after $t=0$).

Simplest approach:

population $\propto$ rate of growth of population

So, let

$N = N(t)$
= # of members in the population at any time t

$N_0 = N(0)$
= initial (i.e., at time $t=0$) # of members

k = growth constant

Then we have the following IVP:

$$\begin{cases} \dfrac{dN}{dt} = kN, \quad k>0, & \text{(DE)} \\ N(0) = N_0 & \text{(INITIAL CONDITION)} \end{cases}$$

Solution:

$$\frac{dN}{dt} = kN$$

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LINEAR

$$N' \underbrace{- kN}_{} = 0$$

$$P(t) = -k \Rightarrow \mu(t) = e^{\int P(t)\,dt}$$

$$= e^{\int -k\,dt} = e^{-kt} \Rightarrow$$

$$e^{-kt}(N' - kN) = e^{-kt}(0) \Rightarrow$$

$$\underbrace{e^{-kt}N' - ke^{-kt}N}_{= \frac{d}{dt}(e^{-kt}N)} = 0 \Rightarrow$$

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$$\frac{d}{dt}\left(e^{-kt}N\right) = 0 \Rightarrow$$

$$\int \frac{d}{dt}\left(e^{-kt}N\right)dt = \int 0\,dt \Rightarrow$$

$$e^{-kt}N = C \Rightarrow$$

$$e^{kt}e^{-kt}N = Ce^{kt} \Rightarrow$$

$$N = Ce^{kt}$$ GEN. SOLN.

SEPARABLE

$$\frac{1}{N}dN = k\,dt \Rightarrow$$

$$\int \frac{1}{N}dN = \int k\,dt \Rightarrow$$

$$\ln|N| = kt + C \Rightarrow$$

$$e^{\ln|N|} = e^{kt+C} \Rightarrow$$

$$|N| = e^{C}e^{kt} \Rightarrow$$

$$N = \pm e^{C}e^{kt} \Rightarrow$$

$$N = Ce^{kt}$$

GEN. SOLN.

Then apply the initial condition $N(0) = N_0$ to the gen. soln.:

$N(0) = N_0 \Rightarrow N = N_0$ when $t = 0 \Rightarrow$

$N_0 = N(0) = Ce^{k(0)} \Rightarrow N_0 = C \cdot 1 \Rightarrow$
$C = N_0 \Rightarrow$

$\boxed{N = N_0 e^{kt}}$ PART. SOLN.

<u>Example</u>. (SEE HW Exercise 4, p. 68.)

A population of rabbits in a village grows at a rate proportional to the number of rabbits present at any time (there are no predators of the rabbits in the village).

After 10 days, it is observed that there are 20 rabbits present. After 100 days, there are 1500 rabbits present.

What was the initial number of rabbits present?

Let t = time (days)

$N(t)$ = # of rabbits present at time t

$N(0) = N_0$ = initial # of rabbits present

Then, by Example 1 on pp. 61-62 of the text, we have

$$\begin{cases} \dfrac{dN}{dt} = kN, & k > 0, \\ N(0) = N_0 \end{cases} \Longrightarrow$$

$$N(t) = N_0 e^{kt}$$

We need to find this, but first we need to find k, the GROWTH CONSTANT

$$\left.\begin{aligned} 20 = N(10) &= N_0 e^{k(10)} \\ 1500 = N(100) &= N_0 e^{k(100)} \end{aligned}\right\} \Longrightarrow$$

$$\frac{1500}{20} = \frac{\cancel{N_0} e^{100k}}{\cancel{N_0} e^{10k}} \Longrightarrow 75 = e^{100k - 10k} \Longrightarrow$$

$$75 = e^{90k} \Longrightarrow \ln 75 = \ln e^{90k} \Longrightarrow$$

$$\ln 75 = 90k \Longrightarrow k = \frac{\ln 75}{90} \approx \boxed{0.04797}$$

Now we can find N_0, given $N(10) = 20$ and $k \approx 0.04797$:

$$20 = N(10) = N_0 e^{(0.04797)(10)} \Rightarrow$$

$$20 = N_0 e^{0.4797} \Rightarrow$$

$$N_0 = \frac{20}{e^{0.4797}} \approx \boxed{12}$$

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$\boxed{\text{MODEL 2}}$ Radioactive Decay.

The DE representing or modeling radioactive decay is of the same form as the DE modeling population growth, EXCEPT THAT $k<0$ WITH DECAY.

The model or IVP is

$$\begin{cases} \dfrac{dA}{dt} = kA, \quad k<0, & \text{(DE)} \\ A(0) = A_0 & \text{(INITIAL CONDITION)} \end{cases}$$

A = amount of substance <u>present</u> at time t

N_0 = initial amount <u>present</u>

k = decay constant

Solution: $A(t) = A_0 e^{kt}$

Along with the decay of a substance goes the idea of HALF-LIFE of that substance:

t_h = half-life
= amount of time it takes for ½ of a substance to disappear

<u>Example</u>. (SEE HW Exercise 6, p. 68.)

Initially, a glass is full of 16 ounces.

After 8 hours, the amount of water has decreased by 1.5%.

If the rate of decay is proportional to the amount of water present at any time, find the amount of water <u>remaining</u> after 3 days.

Let t = time (hours)
$A(t)$ = amount of water present (ounces) at time t
$A(0) = A_0$ = initial amount of water present

Then, by Example 2 on pp. 62–63 of the text, we have

$$\begin{cases} \frac{dA}{dt} = kA, & k < 0, \\ A(0) = A_0 \end{cases} \Longrightarrow$$

$$A(t) = A_0 e^{kt}$$

We need to find $A(72)$ (3 days = 72 hours), but first we need to find k, the DECAY CONSTANT

To find k, we use the fact that

$A(8)$ = amount present after 8 hours
= initial amount − amount that disappears
$= A_0 - \underbrace{0.015 A_0}_{\text{initial amount of water is reduced by } 1.5\%}$

$= 0.985 A_0$

So,

$$0.985 A_0 = A(8) = A_0 e^{k(8)} \Longrightarrow$$

$$0.985 \cancel{A_0} = \cancel{A_0} e^{8k} \Longrightarrow$$

$$0.985 = e^{8k} \Rightarrow \ln 0.985 = \ln e^{8k} \Rightarrow$$

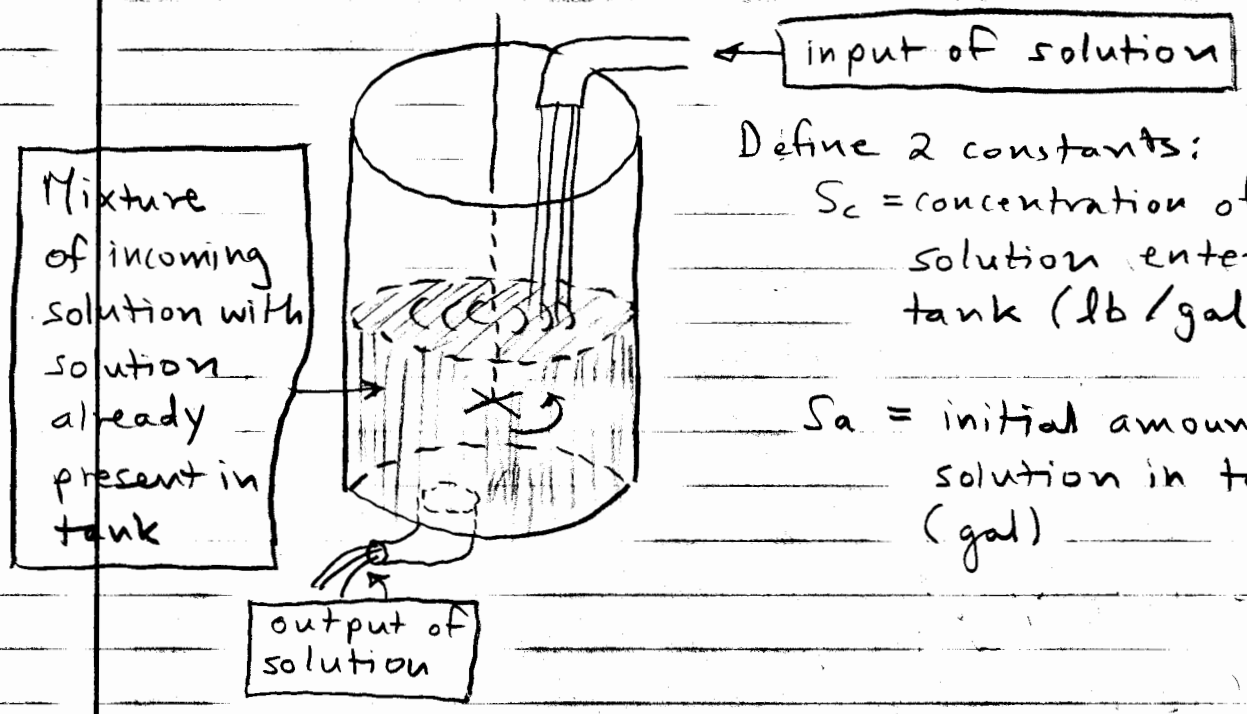
$$\ln 0.985 = 8k \Rightarrow k = \frac{\ln 0.985}{8} \approx -0.001889$$

Then

$$A(72) = A_0 e^{-0.001889(72)} \approx 0.873 A_0$$

$$= 0.873(16) \approx \boxed{14 \text{ ounces}}$$

MODEL 3 Mixture of Two Salt Solutions.



Define 2 constants:

S_c = concentration of solution entering tank (lb/gal)

S_a = initial amount of solution in tank (gal)

Solution (gal) = solute (lb) + solvent (gal)
e.g., salt — e.g., water

Also let

A = amt of substance (lb) in the tank at any time t (min)

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R_1 = rate of solution entering tank (gal/min)

R_2 = rate of solution leaving tank (gal/min)

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Then

$$\frac{dA}{dt} = \text{net rate of change of amt of substance (lb/min)}$$

$$= \left(\text{rate of substance entering}\right) - \left(\text{rate of substance leaving}\right)$$

$$= \underbrace{R_1 S_c}_{\frac{gal}{min}\cdot\frac{lb}{gal} = \frac{lb}{min}} - \underbrace{R_2}_{\frac{gal}{min}} \underbrace{\left[\frac{A}{S_a + (R_1 - R_2)t}\right]}_{\frac{lb}{gal + \frac{gal}{min}\cdot min}}$$

$$\frac{\text{amt of solution}}{\text{amt of time}} \times \frac{\text{amt of solute}}{\text{amt of solution}}$$

$$\frac{\text{amt of solution}}{\text{amt of time}} \times \frac{\text{amt of solute}}{\text{amt of solution}}$$

where $(R_1 - R_2)t$ is the amt of solution accumulating or disappearing from the tank, if $R_1 \neq R_2$

$$\therefore \boxed{\frac{dA}{dt} = (R_1 S_c) - \left[\frac{R_2}{S_a + (R_1 - R_2)t}\right] A}$$

LINEAR DE of the form

$$A' = f(t) - P(t)\,A \implies$$

$$A' + P(t)\,A = f(t)$$

only now $f(t) \neq 0$.

<u>Example</u>. (Exercise 25, p. 70.)

- Tank filled with 10 gal of fluid with 10 lbs of salt dissolved in it.
- Solution of ½ lb salt per gal pumped in at rate 6 gal/min.
- Solution well mixed
- Solution pumped out at rate 4 gal/min.

How much salt (in lbs) is left in the tank after 30 min?

Let t = time (min)

$A = A(t)$
= amt of salt (lbs) in tank at time t

R_1 = rate of soln. entering tank
= 6 gal/min

R_2 = rate of soln. leaving tank
= 4 gal/min

S_a = amt of soln. in tank
= 100 gal

S_c = conc. of soln. in tank
= 0.5 lb/gal

Then, by Example 5 on pp. 65-66 of the text, we have

$$\frac{dA}{dt} = (R_1 S_c) - \left[\frac{R_2}{S_a + (R_1 - R_2)t}\right] A$$

$$= (6)(0.5) - \left[\frac{4}{100 + (6-4)t}\right] A$$

$$= 3 - \left(\frac{2}{50+t}\right) A$$

<u>and</u>

$$A(0) = 10$$

$\therefore$ We have the following IVP to solve:

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$$\begin{cases} \dfrac{dA}{dt} = 3 - \left(\dfrac{2}{50+t}\right) A, \\ A(0) = 10 \end{cases}$$

$$A' = 3 - \left(\frac{2}{50+t}\right) A \Rightarrow A' + \underbrace{\left(\frac{2}{50+t}\right)}_{P(t)} A = 3$$

$$P(t) \Rightarrow \mu(t) = e^{\int P(t)\,dt}$$

$$= e^{\int \frac{2}{50+t}\,dt}$$

$$\int \frac{1}{t+a}\,dt = \ln|\ \ -$$

$$= e^{2\ln|50+t|}$$

$$= e^{\ln(50+t)^2}$$

$$= (50+t)^2 \Rightarrow$$

$$(50+t)^2 \left[A' - \left(\frac{2}{50+t}\right) A\right] = 3(50+t)^2 \Rightarrow$$

$$(50+t)^2 A' - 2(50+t) A = 3(50+t)^2 \Rightarrow$$

$$\int \frac{d}{dt}\left[(50+t)^2 A\right] dt = \underbrace{\int -3(50+t)^2\,dt}_{\text{INTEGRATION BY SUBSTITUTION}} \Rightarrow$$

product is always $\mu(t)\,A$

$w = 50+t$ then

Let $\boxed{w = 50 + t}$. Then
$\frac{dw}{dt} = 1 \Rightarrow \boxed{dw = dt}$.

$$(50+t)^2 A = 3\int w^2\,dw \Rightarrow$$

$$(50+t)^2 A = \cancel{3}\left(\frac{w^3}{\cancel{3}}\right) + C \Rightarrow$$

$$(50+t)^2 A = w^3 + C \Rightarrow$$

$$(50+t)^2 A = (50+t)^3 + C \Rightarrow$$

$$A(t) = 50 + t + \frac{C}{(50+t)^2}$$ GEN. SOLN.

$A(0) = 10$

$$10 = A(0) = 50 + 0 + \frac{C}{(50+0)^2} \Rightarrow$$

$$10 = 50 + \frac{C}{2500} \Rightarrow$$

$$25{,}000 = 125{,}000 + C \Rightarrow$$

$$C = -100{,}000 \Rightarrow$$

$$A(t) = 50 + t - \frac{100{,}000}{(50+t)^2}$$ PART. SOLN.

So,

$$A(30) = 50 + 30 - \frac{100,000}{(50+30)^2}$$

$$= \boxed{64.375 \text{ lbs}}$$

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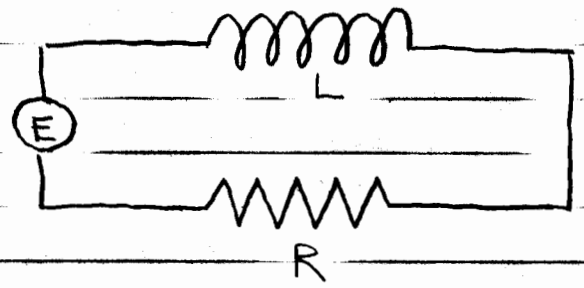
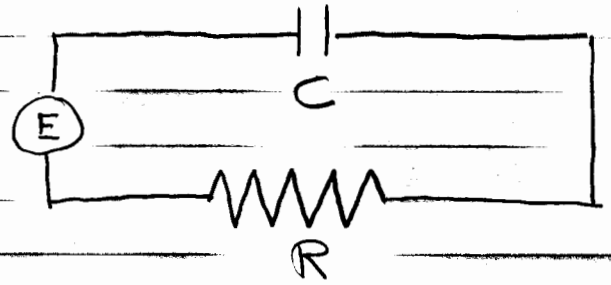
MODEL 4 Electrical Series Circuits.



LR Series Circuit



RC Series Circuit

Let t = time } variables
q = charge
i = current $= \frac{dq}{dt}$

C = capacitance
L = inductance } constants
R = resistance
$E(t)$ = impressed voltage

LR Series Circuit DE:

$$L\frac{di}{dt} + Ri = E(t)$$

LINEAR, FIRST-ORDER

RC Series Circuit DE:

2/21/01? $R\frac{dq}{dt} + \frac{1}{C}q = E(t)$ — LINEAR, FIRST-ORDER

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<u>Example</u>. (SEE HW Exercise 15, p. 69.)

- A 100-volt electromotive force is applied to an <u>LR series circuit</u>
- Inductance = 0.5 henries
- Resistance = 300 ohms

Find current, $i(t)$, if $i(0) = 0$. Determine current as $t \to \infty$.

From Example 6, our IVP is

$$\begin{cases} L\frac{di}{dt} + Ri = E(t), \\ i(0) = i_0 \end{cases} \Longrightarrow$$

$$\begin{cases} 0.5\,\dfrac{di}{dt} + 300\,i = 100, \\ i(0) = 0. \end{cases}$$

Then the solution of the DE is obtained using the METHOD OF SOLUTION OF FIRST-ORDER LINEAR DEs :

$$0.5\,\frac{di}{dt} + 300\,i = 100 \Rightarrow$$

$$i' + 600\,i = 200$$

$$e^{\int 600\,dt} = e^{600t}$$

$$e^{600t}(i' + 600\,i) = e^{600t}(200) \Rightarrow$$

$$\underbrace{e^{600t}\,i' + 600\,e^{600t}\,i}_{= \frac{d}{dt}\left(e^{600t}\cdot i\right)} = 200\,e^{600t} \Rightarrow$$

$$\int \frac{d}{dt}\left(e^{600t}\cdot i\right) dt = 200 \int e^{600t}\,dt \Rightarrow$$

$$e^{600t}\cdot i = 200\left(\frac{e^{600t}}{600}\right) + C \Rightarrow$$

$$e^{600t} \cdot i = \frac{1}{3} e^{600t} + C \Rightarrow$$

$$\cancel{e^{-600t}} \left(\cancel{e^{600t}} \cdot i \right) = \cancel{e^{-600t}} \left(\frac{1}{3} \cancel{e^{600t}} + C \right)$$

$$i(t) = \frac{1}{3} + C e^{-600t}$$

$i(0) = 0$

$$0 = i(0) = \frac{1}{3} + C e^{-600(0)} \Rightarrow$$

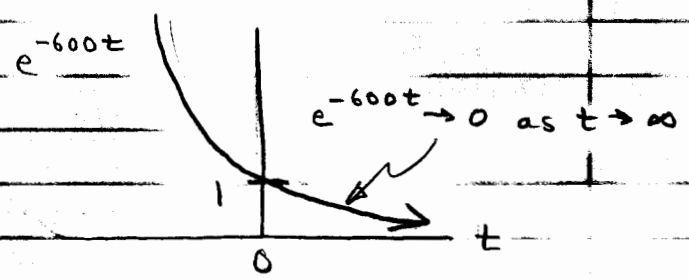
$$0 = \frac{1}{3} + C \Rightarrow C = -\frac{1}{3} \Rightarrow$$

$$\boxed{i(t) = \frac{1}{3} - \frac{1}{3} e^{-600t}}$$

The current, $i(t)$, as $t \to \infty$, is the following:

$$\lim_{t \to \infty} i(t) = \lim_{t \to \infty} \left(\frac{1}{3} - \frac{1}{3} e^{-600t} \right)$$

$$= \frac{1}{3} - \frac{1}{3} \cdot \underbrace{\lim_{t \to \infty} e^{-600t}}$$



or

$$t \to \infty \Rightarrow e^{600t} \to e^{600 \cdot \infty} = \infty$$

$$\Rightarrow e^{-600t} = \frac{1}{e^{600t}} \to \frac{1}{\infty} = 0$$

$$= \frac{1}{3} - \frac{1}{3} \cdot 0$$

$$= \boxed{\frac{1}{3}}$$

$$\therefore \; i(t) = \underbrace{\frac{1}{3}}_{\text{STEADY-STATE CURRENT}} - \underbrace{\frac{1}{3} e^{-600t}}_{\text{TRANSIENT TERM}}$$

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MODEL 5 Drug Dissemination in the Bloodstream (Like Temperature Cooling or Diffusion of a Substance in a Medium)

Suppose a person is given a drug intravenously and after that point the concentration of the drug in the person's bloodstream is measured periodically.

A simple model for this, which assumes that the liver and kidney are <u>not</u> functioning to metabolize and remove the drug from the person, is the following IVP:

$$\begin{cases} \dfrac{dx}{dt} = r - kx, & r, k > 0 \\ x(0) = 0 \end{cases}$$

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t = time

$x = x(t)$

= concentration of drug in bloodstream at any time t

Solution:

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$$\frac{dx}{dt} = r - kx$$

SEPARABLE

$$\frac{dx}{dt} = -k\left(x - \frac{r}{k}\right) \Rightarrow \frac{1}{x - \frac{r}{k}}\,dx = -k\,dt \ldots$$

LINEAR

$$x' + \underbrace{k}_{P(t)}x = r$$

$$P(t) \Rightarrow \mu(t) = e^{\int P(t)dt} = e^{\int k\,dt} = e^{kt} \Rightarrow$$

$$e^{kt}(x' + kx) = e^{kt}(r) \Rightarrow \underbrace{e^{kt}x' + ke^{kt}x}_{= \frac{d}{dt}(e^{kt}x)} = re^{kt} \Rightarrow$$

$$\int \underbrace{\frac{d}{dt}(e^{kt}x)}_{\frac{d}{dt}(\mu(t)x)}\,dt = r\int e^{kt}\,dt \Rightarrow$$

$$e^{kt}x = r\left(\frac{e^{kt}}{k}\right) + C \Rightarrow e^{-kt}e^{kt}x = e^{-kt}\left(\frac{re^{kt}}{k} + C\right) \Rightarrow$$

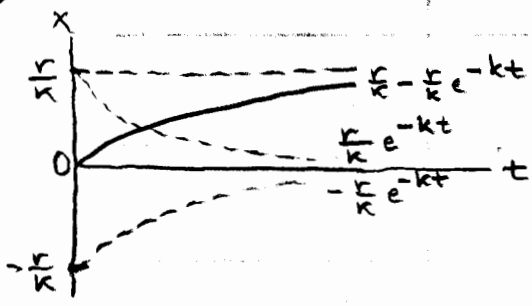
$$x(t) = \frac{r}{k} + Ce^{-kt}$$ GEN. SOLN.

$x(0) = 0 \Rightarrow x = 0$ when $t = 0$

$$0 = x(0) = \frac{r}{k} + Ce^{-k(0)} \Rightarrow$$

$$0 = \frac{r}{k} + C \cdot 1 \Rightarrow C = -\frac{r}{k} \Longrightarrow$$

$$\boxed{x(t) = \frac{r}{k} - \frac{r}{k} e^{-kt}}$$ PART. SOLN.



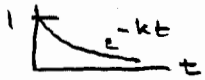
Then, in this type of situation, one is interested in the

LIMITING VALUE or STEADY STATE VALUE or EQUILIBRIUM VALUE

of $x(t)$ (as $t \to \infty$):

$$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} \left(\frac{r}{k} - \frac{r}{k} e^{-kt} \right)$$

$$= \frac{r}{k} - \frac{r}{k} \lim_{t \to \infty} \underbrace{e^{-kt}}$$



$$= \frac{r}{k} - \frac{r}{k} \cdot 0$$

$$= \boxed{\frac{r}{k}} > 0$$

(SEE HW Exercise 28, p. 71.)

Lecture

Section 3.2. Nonlinear Equations.

Introduction

Look at NONLINEAR DE or IVP models mainly of the form

$$(*) \qquad \frac{dP}{dt} = (a - bP)(c - dP)$$

Eq. (*) is nonlinear (with respect to P, not t!) since rearrangement of it gives

$$P' + \left[(ad + bc)P - \underbrace{bd\,P^2}_{\text{NONLINEAR TERM with } P^2}\right] = ac$$

Eq. (*) describes population or substance with LIMITS on its growth or accumulation:

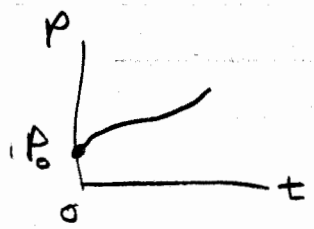
1) If $P(0)$ is small enough (i.e., if $P(t)$ starts out small enough), $P(t)$ increases but eventually levels off
2) If $P(0)$ is large enough, $P(t)$ decreases but eventually levels off

E.g., $\frac{dP}{dt} = (\underset{\substack{\downarrow \\ 0}}{a} - \underset{\substack{\downarrow \\ -1}}{b}P)(\underset{\substack{\downarrow \\ 2}}{c} - \underset{\substack{\downarrow \\ 1}}{d}P) \Longrightarrow$

$$\frac{dP}{dt} = P(2-P).$$

We can plot the graph of any particular solution of the IVP

$$\begin{cases} \frac{dP}{dt} = P(2-P) \\ P(0) = P_0 \end{cases}$$



<u>Defn.</u> The graph of a particular soln. (i.e., P versus t) is called a <u>solution curve</u>

<u>Note</u>: $P(t) = 2$ is a particular solution of

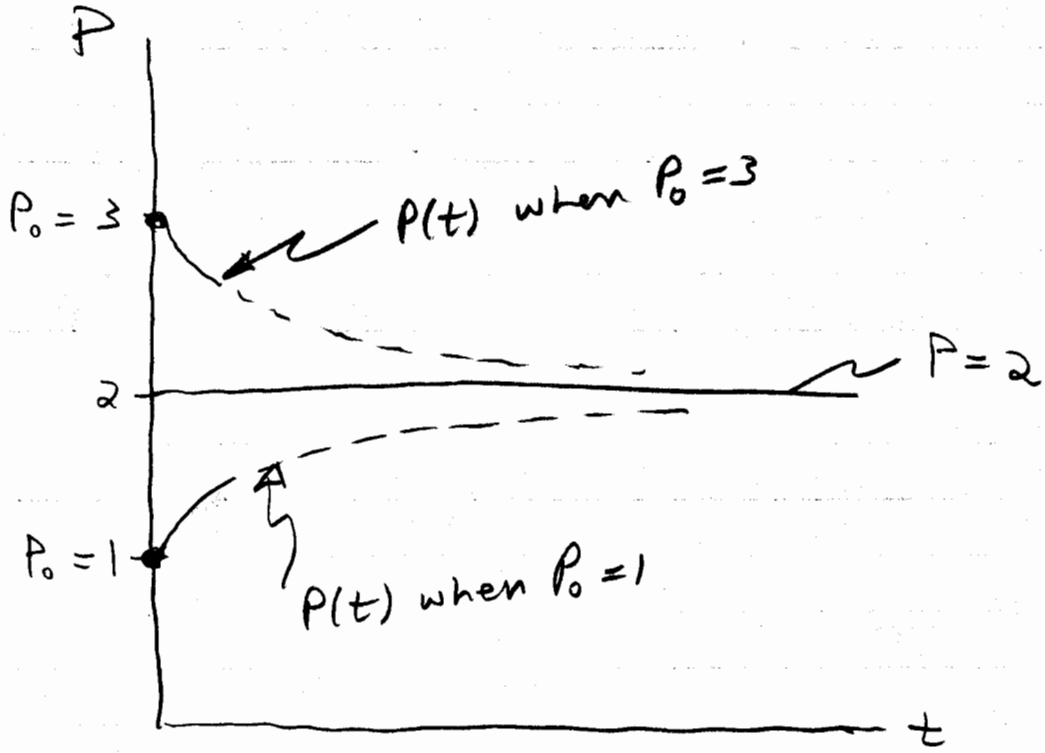
$$\begin{cases} \frac{dP}{dt} = P(2-P) \\ P(0) = 2 \end{cases}$$

<u>Rule of Thumb</u>: (Proof outside scope of course)

No two solution curves can cross.

So, rewrite DE as

$$P'(t) = P(t)\,[2 - P(t)]$$



$P_0 = 1$:

$P'(0) = P(0)[2 - P(0)] = 1 \cdot (2-1) = 1 \Rightarrow$
$P'(0) > 0 \Rightarrow$
solution curve initially increasing but below $P = 2$ curve since cannot cross $P = 2$ curve $\Rightarrow$
$P'(t) > 0$ always, since keep plugging into DE a $P(t) < 2 \Rightarrow$
solution curve approaches $P = 2$ curve asymptotically from below.

$P_0 = 3$:

$P'(0) = P(0)\,[2 - P(0)] = 3 \cdot (2-3) = -1 \Longrightarrow$
$P'(0) < 0 \Longrightarrow$
soln. curve initially decreasing but above $P=2$ curve $\Longrightarrow$
$P'(t) < 0$ always, since keep plugging into DE a $P(t) < 2 \Longrightarrow$
soln. curve approaches $P=2$ curve asymptotically from above.

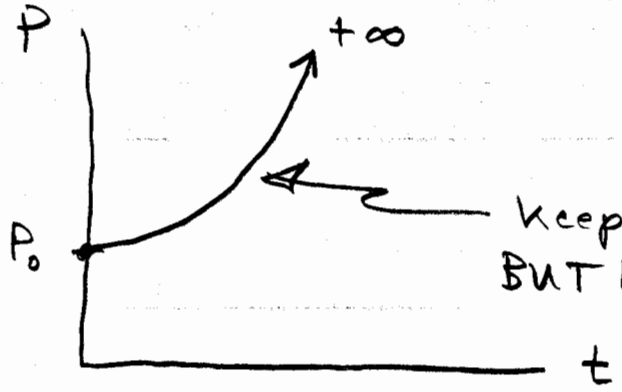
This is in contrast to the

$$\begin{cases} \dfrac{dP}{dt} = P, \\ P(0) = P_0 \end{cases}$$

Soln. is

$$P(t) = P_0 e^t$$

and soln. curve is



keeps increasing BUT NEVER LEVELS OFF, So "unrestricted growth."

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Section 3.2. Nonlinear Equations.

$\boxed{\text{MODELS}}$

Now look at the following DE MODELS:

(1) Population Models
  (a) Logistic Eq.
  (b) Gompertz Eq.
(2) Chemical Reaction Models

Population Models

In general, of the form

$$P' = P f(P).$$

Can rewrite as

$$\underbrace{\frac{P'}{P}}_{} = f(P)$$

This ratio called GROWTH RATE PER INDIVIDUAL
(where P' is the growth rate of the <u>whole</u> population)

$f(P)$ can be various things: In Sect. 3.1 it was the following:

$$\frac{P'}{P} = r \quad (f(P) = r), \quad r > 0$$

CONSTANT FUNCTION OF P

- Called "MALTHUSIAN GROWTH RATE PER INDIVIDUAL" (due to Thomas R. Malthus)
- Soln. is $P(t) = P_0 e^{rt}$
- Population growth exponential and unrestricted; r depends only on environment
- True of populations <u>so small</u> that there is no competition between its members.

In this section, we will look at 2 other forms of $f(P)$.

I. Logistic Equation

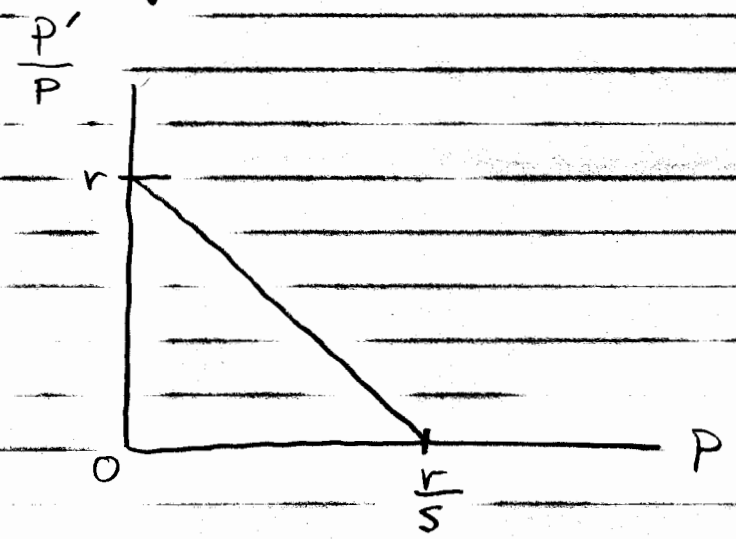
A. Introduction

$$\frac{P'}{P} = r - sP \quad (f(P) = r - sP), \quad r, s > 0$$

LINEAR FUNCTION OF P $(mP + b)$

Called "LOGISTIC GROWTH RATE PER INDIVIDUAL" (due to Verhulst and Pearl)

$\Downarrow$

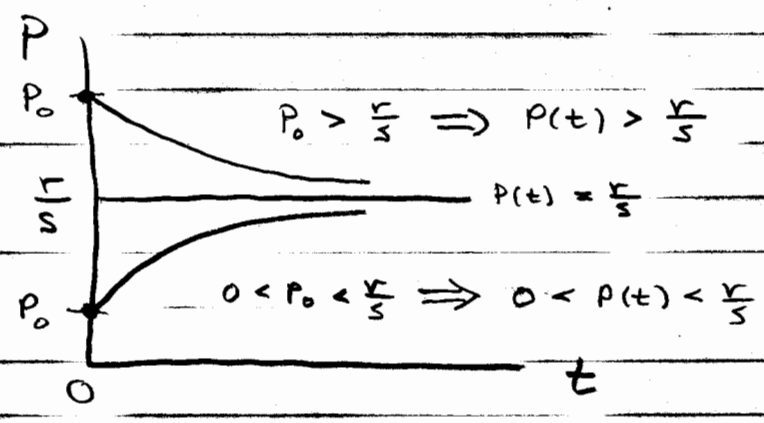


r = INTRINSIC RATE OF NATURAL INCREASE

– the "exponential part" of P'/P

$\frac{r}{s}$ = CARRYING CAPACITY
or
SATURATION LEVEL
or
LIMITING NUMBER OF THE POPULATION

= maximum # of individuals in a popul. that the enviroment is able to sustain



$$\lim_{t \to \infty} P(t) = \frac{r}{s}$$

B. $\boxed{\text{Solution}}$

$$\frac{dP}{dt} = P(r - sP), \qquad r, s > 0$$

Use SEPARATION OF VARIABLES:

$$\underbrace{\int \frac{1}{P(r-sP)}\,dP}_{} = \int 1 \cdot dt$$

Must use METHOD OF PARTIAL FRACTIONS to evaluate:

<u>Goal</u> When you evaluate something like $\int \frac{1}{(x-a)(x-b)}\,dx$, you want to be able to rewrite the integrand as $\frac{A}{x-a} + \frac{B}{x-b}$

where then $\int \frac{1}{(x-a)(x-b)}\,dx = \int \frac{A}{x-a}\,dx + \int \frac{B}{x-b}\,dx$

$= A \ln|x-a| + B \ln|x-b| + C.$

① $$\frac{1}{P(r-st)} = \frac{1}{-sP(P-\frac{r}{s})} = -\frac{1}{s}\,\frac{1}{P(P-\frac{r}{s})}$$

② $$\frac{1}{P(P-\frac{r}{s})} = \frac{A}{P} + \frac{B}{P-\frac{r}{s}} \Longrightarrow$$

$$\cancel{P}(P-\cancel{\tfrac{r}{s}}) \cdot \frac{1}{\cancel{P}(\cancel{P-\tfrac{r}{s}})} = P(P-\tfrac{r}{s})\left(\frac{A}{P} + \frac{B}{P-\frac{r}{s}}\right) \Longrightarrow$$

$$1 = \left(P - \frac{r}{s}\right)A + PB \Longrightarrow$$

$$1 = AP - \frac{r}{s}A + BP \Longrightarrow$$

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$$0 \cdot P + 1 = (A+B)P - \frac{r}{s}A \Longrightarrow$$

$$\left.\begin{aligned} A + B &= 0 \\ -\frac{r}{s}A &= 1 \end{aligned}\right\} \Longrightarrow \boxed{\begin{aligned} A &= -\frac{s}{r} \\ B &= \frac{s}{r} \end{aligned}}$$

$$\therefore \int \frac{1}{P(r-sP)}\,dP = -\frac{1}{s}\int \frac{1}{P\left(P-\frac{r}{s}\right)}\,dP$$

$$= -\frac{1}{s}\int \left(\frac{A}{P} + \frac{B}{P-\frac{r}{s}}\right) dP$$

$$= \left(-\frac{1}{\cancel{s}}\right)\left(-\frac{\cancel{s}}{r}\right)\int \frac{1}{P}\,dP + \left(-\frac{1}{\cancel{s}}\right)\left(\frac{\cancel{s}}{r}\right)\int \frac{1}{P-\frac{r}{s}}\,dP$$

$$= \boxed{\frac{1}{r}\ln|P| - \frac{1}{r}\ln\left|P - \frac{r}{s}\right| + C}$$

$$\therefore \int \frac{1}{P\left(P-\frac{r}{s}\right)}\,dP = \int 1 \cdot dt \Longrightarrow$$

$$\frac{1}{r}\ln|P| - \frac{1}{r}\ln\left|P - \frac{r}{s}\right| = t + C_1 \Longrightarrow$$

$$\ln|P| - \ln\left|P - \frac{r}{s}\right| = rt + \underbrace{rC_1}_{=C_2} \Longrightarrow$$

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$$\ln\left|\frac{P}{P-\frac{r}{s}}\right| = rt + C_2 \Longrightarrow$$

$$e^{\ln\left|\frac{P}{P-\frac{r}{s}}\right|} = e^{rt + C_2} \Longrightarrow$$

$$\left|\frac{P}{P-\frac{r}{s}}\right| = \underbrace{e^{C_2}}_{=C_3} e^{rt} \Longrightarrow$$

$$\pm\left(\frac{P}{P-\frac{r}{s}}\right) = C_3 e^{rt} \Longrightarrow$$

$$\frac{P}{P-\frac{r}{s}} = \underbrace{\pm C_3}_{=C_4} e^{rt} \Longrightarrow$$

$$\frac{P}{P-\frac{r}{s}} = C_4 e^{rt} \Longrightarrow$$

$$-\frac{1}{s}\cdot\frac{P}{P-\frac{r}{s}} = \underbrace{-\frac{1}{s}C_4}_{=C} e^{rt} \Longrightarrow$$

$$\frac{P}{r-sP} = Ce^{rt}$$

Now <u>solve for P</u>:

$$\frac{P}{r-sP} = Ce^{rt} \Longrightarrow$$

$$P = Ce^{rt}(r-sP) \Longrightarrow$$

$$P = \underbrace{rC}e^{rt} - \underbrace{sC}e^{rt}P \Longrightarrow$$

WARNING: DO NOT REPLACE both of these by a C like so:

$$Ce^{rt} - Ce^{rt}P.$$

This is because <u>generally</u>

$$r \neq s \Longrightarrow rC \neq sC.$$

So, LEAVE rC AND sC ALONE or replace rC by C_1 and sC by C_2, where C_1, C_2 are 2 arbitrary constants.

$$P + sCe^{rt}P = rCe^{rt} \Longrightarrow$$

$$(1+sCe^{rt})P = rCe^{rt} \Longrightarrow$$

$$P = \frac{rCe^{rt}}{1+sCe^{rt}} \Longrightarrow$$

$$P = \frac{rCe^{rt}}{1+sCe^{rt}} \cdot \frac{e^{-rt}}{e^{-rt}} \Longrightarrow$$

so exponential expression only seen in denominator

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$$P(t) = \frac{rC}{e^{-rt}+sC}$$ GEN. SOLN.

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Now set $P(0) = P_0$ and solve for C:

$$P_0 = P(0) = \frac{rC}{e^{-r(0)}+sC} \Rightarrow P_0 = \frac{rC}{1+sC} \Rightarrow$$

$$(1+sC)P_0 = rC \Rightarrow P_0 + sCP_0 = rC \Rightarrow$$

$$sCP_0 - rC = -P_0 \Rightarrow (sP_0 - r)P_0 = -P_0 \Rightarrow$$

$$C = \frac{P_0}{r-sP_0} \Rightarrow$$

$$P(t) = \frac{rC}{e^{-rt}+sC} = \frac{r\left(\frac{P_0}{r-sP_0}\right)}{e^{-rt}+s\left(\frac{P_0}{r-sP_0}\right)} \Longrightarrow$$

$$P(t) = \frac{r\left(\frac{P_0}{\cancel{r-sP_0}}\right)}{e^{-rt}+s\left(\frac{P_0}{\cancel{r-sP_0}}\right)} \cdot \frac{\cancel{r-sP_0}}{\cancel{r-sP_0}} \Longrightarrow$$

$$\boxed{P(t) = \frac{rP_0}{(r - sP_0)e^{-rt} + sP_0}}$$ PART. SOLN.

C. $\boxed{\text{Example}}$

1. (HW Exercise 2, p. 78.)

<u>Given</u>: t = time (no units given)
$N(t)$ = # of people in a community who see a particular ad
$N(0) = N_0 = 500$
$N(1) = 1000$
LIMITING # OF PEOPLE IN COMMUNITY WHO WILL SEE AD $= 50{,}000$

$N(t)$ governed by LOGISTIC EQ.

<u>Find</u>: $N(t)$ at any time t.

LOGISTIC EQ.: $N' = N(r - sN)$

LIMITING # of $N(t)$: $\frac{r}{s} = 50{,}000$

SOLUTION:

$$N(t) = \frac{rN_0}{(r - sN_0)e^{-rt} + sN_0}$$

$$= \frac{rN_0}{(r - sN_0)e^{-rt} + sN_0} \cdot \frac{\frac{1}{s}}{\frac{1}{s}}$$

$$= \frac{\frac{r}{s} N_0}{\left(\frac{r}{s} - N_0\right) e^{-rt} + N_0}$$

$$= \frac{(50{,}000)(500)}{(50{,}000 - 500) e^{-rt} + 500} \Rightarrow$$

(*) $$N(t) = \frac{25{,}000{,}000}{49{,}500 e^{-rt} + 500}$$

To find r, apply $N(1) = 1000$ to Eq. (*):

$$1000 = N(1) = \frac{25{,}000{,}000}{49{,}500 e^{-r(1)} + 500} \Rightarrow$$

$$1000 = \frac{25{,}000{,}000}{49{,}500 e^{-r} + 500} \Rightarrow$$

$$49{,}500{,}000 e^{-r} + 500{,}000 = 25{,}000{,}000 \Rightarrow$$

$$e^{-r} = \frac{24{,}500{,}000}{49{,}500{,}000} = \frac{245}{495} \Rightarrow$$

$$\ln e^{-r} = \ln\left(\frac{245}{495}\right) \Rightarrow -r = \ln\left(\frac{245}{495}\right) \Rightarrow$$

$$r = -\ln\left(\frac{245}{495}\right) \approx 0.7033.$$

$$\therefore\ N(t) = \frac{25{,}000{,}000}{49{,}500\,e^{-0.7033t} + 500} \cdot \frac{\frac{1}{500}}{\frac{1}{500}} \Rightarrow$$

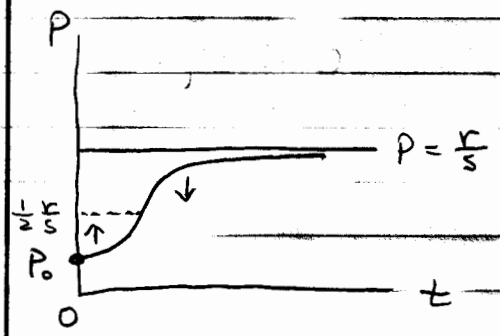
$$\boxed{N(t) = \frac{50{,}000}{99\,e^{-0.7033t} + 1}}$$

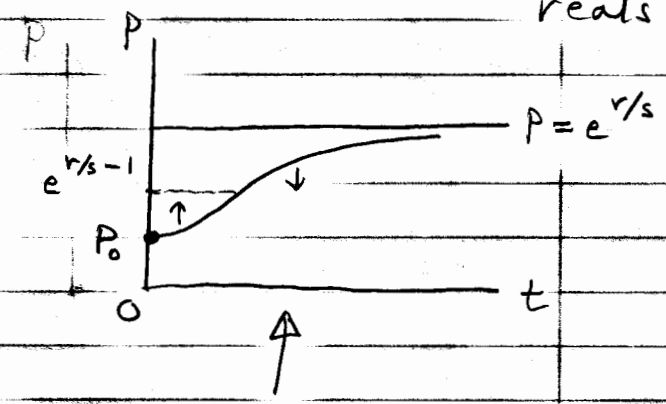
II. Gompertz Equation

A. Introduction

The GOMPERTZ EQUATION is a variant of the LOGISTIC EQUATION:

Logistic Eq.	vs.	Gompertz Eq.
$P' = P(r - sP)$, $r, s > 0$		$P' = P(r - s\ln P)$, r, s: any reals





Soln. curve has FLATTER S-SHAPE due to the ln

Gompertz eq. models growth of anything from population size to finances.

E.g., Gompertz eq. used in actuarial science for specifying various mortality laws (like the probability a newborn will achieve a certain age).

<u>WARNING</u>: Since there is an "$\ln P$" term in the DE, <u>must have</u>

$$P(0) = P_0 > 0.$$

Then it can be shown that

$$P(t) > 0 \text{ for all } t > 0,$$

which we want.
(Not really necessary to have $P(0) = P_0 > 0$ for the logistic DE.)

B. $\boxed{\text{Solution}}$ (Easier than with Logistic Eq.)

$$\frac{dP}{dt} = P(r - s\ln P), \quad r, s \text{ any reals}$$

Use SEPARATION OF VARIABLES:

$$\underbrace{\int \frac{1}{P(r - s\ln P)}\, dP}_{} = \int 1 \cdot dt$$

Use INTEGRATION BY SUBSTITUTION to evaluate:

Let $w = r - \ln P$. Then $\frac{dw}{dP} = -s \cdot \frac{1}{P} \Rightarrow$

$$-\frac{1}{s}\, dw = \frac{1}{P}\, dP.$$

$$-\frac{1}{s}\int \frac{1}{w}\, dw = \int 1 \cdot dt \Rightarrow$$

$$-\frac{1}{s}\ln|w| = t + C_1 \Rightarrow$$

$$\ln|w| = -st \underbrace{- sC_1}_{= C_2} \Rightarrow$$

$w = r - s\ln P$

$$\ln|r - s\ln P| = -st + C_2 \Rightarrow$$

$$e^{\ln|r - s\ln P|} = e^{-st + C_2} \Rightarrow$$

$$|r - s\ln P| = \underbrace{e^{C_2}}_{=C_3} e^{-st} \Rightarrow$$

$$\pm(r - s\ln P) = C_3 e^{-st} \Rightarrow$$

$$r - s\ln P = \underbrace{\pm C_3}_{=C_4} e^{-st} \Rightarrow$$

$$r - s\ln P = C_4 e^{-st}$$

Now <u>solve for P</u>:

$$r - s\ln P = C_4 e^{-st} \Rightarrow$$

$$-s\ln P = C_4 e^{-st} - r \Rightarrow$$

$$\ln P = -\underbrace{\left(\tfrac{1}{s} C_4\right)}_{=C} e^{-st} + \frac{r}{s} \Rightarrow$$

<u>NOTE</u>: Having C_4 "absorb" the $1/s$ but not the minus sign in order to remain consistent with the text.

$$\ln P = -Ce^{-st} + \frac{r}{s} \Rightarrow$$

$$e^{\ln P} = e^{-Ce^{-st} + \frac{r}{s}} \Rightarrow$$

$$\boxed{P(t) = e^{r/s}\, e^{-Ce^{-st}}}$$

GENERAL SOLUTION

We will stop here and not determine the PARTICULAR SOLUTION.

<u>WARNING</u>: Leave the "$e^{r/s}$" alone and do not attempt to replace it by a "C." It is a <u>specific</u>, not an arbitrary, constant.

r, s = PARAMETERS
- constants that vary from one type of population to another

C = ARBITRARY CONSTANT
- constant that varies with the different situations a single population can go through

OBSERVATIONS:

(1) $\underbrace{e^{r/s}}_{>0}\,\underbrace{e^{-Ce^{-st}}}_{>0} > 0$ for all $t \geq 0 \Longrightarrow$

$P(t) > 0$ for all $t \geq 0$.

We need this because of the "$\ln P$" term in the DE.

(2) $$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} e^{r/s}\, e^{-Ce^{-st}}$$

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$$= e^{r/s} \lim_{t\to\infty} \left(\frac{1}{e^{Ce^{-st}}} \right)$$

$$= e^{r/s} \lim_{t\to\infty} \left(\frac{1}{e^{C\left(\frac{1}{e^{st}}\right)}} \right)$$

Assume $C > 0$. Then:

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Case (a) $s > 0$.

$$\lim_{t\to\infty} P(t) = e^{r/s} \lim_{t\to\infty} \left(\frac{1}{e^{C\left(\frac{1}{e^{st}}\right)}} \right)$$

$$= e^{r/s}\left(\frac{1}{e^{c\left(\frac{1}{e^{s\cdot\infty}}\right)}}\right) = e^{r/s}\left(\frac{1}{e^{c\left(\frac{1}{e^{\infty}}\right)}}\right)$$

$$= e^{r/s}\left(\frac{1}{e^{c\left(\frac{1}{\infty}\right)}}\right) = e^{r/s}\left(\frac{1}{e^{c\cdot 0}}\right)$$

$$= e^{r/s}\left(\frac{1}{1}\right) = e^{r/s}$$

<u>Case (b)</u> $s < 0$,

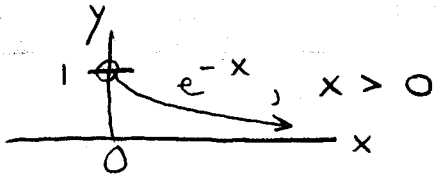
$$\lim_{t\to\infty} P(t) = e^{r/s} \lim_{t\to\infty}\left(\frac{1}{e^{c\left(\frac{1}{e^{st}}\right)}}\right)$$

$$= e^{r/s}\left(\frac{1}{e^{c\left(\frac{1}{e^{s\cdot\infty}}\right)}}\right) = e^{r/s}\left(\frac{1}{e^{c\left(\frac{1}{e^{-\infty}}\right)}}\right)$$

$$= e^{r/s}\left(\frac{1}{e^{c(e^{\infty})}}\right) = e^{r/s}\left(\frac{1}{e^{c\cdot\infty}}\right)$$

$$= e^{r/s}\left(\frac{1}{e^{\infty}}\right) = e^{r/s}\cdot 0 = 0$$

(3) $e^{r/s}\underbrace{e^{-Ce^{-st}}}_{<1 \text{ as long as } C>0}$
(and true for all $t \geq 0$)

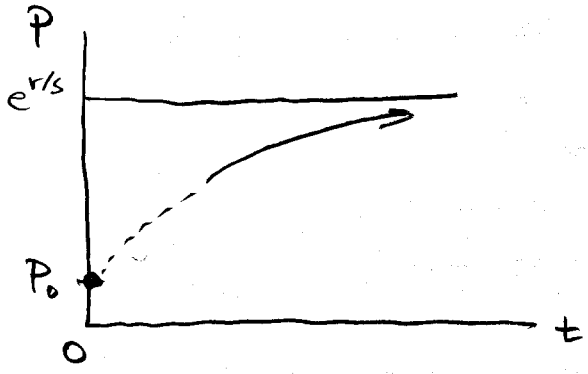


$\therefore\ e^{r/s}\, e^{-Ce^{-st}} < e^{r/s} \cdot 1 = e^{r/s}$ for all $t \geq 0 \Rightarrow$

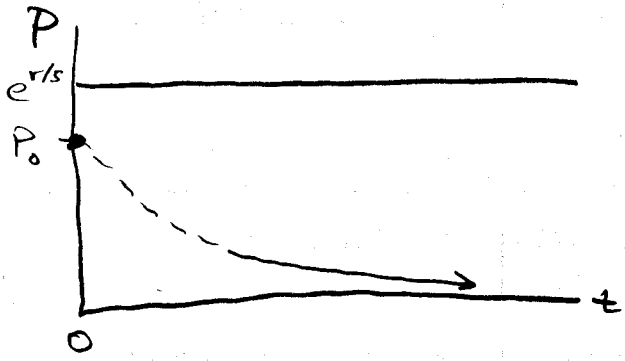
$$\boxed{P(t) < e^{r/s} \quad \text{for all } t \geq 0}$$

(4) (1) + (3) $\Rightarrow$ $\boxed{0 < P(t) < e^{r/s} \text{ for all } t \geq 0 \text{ (as long as } C > 0)}$

(5) (2) + (4) $\Longrightarrow$



$C > 0,\ s > 0$



$C > 0,\ s < 0$

($C > 0 \Leftrightarrow P_0 < e^{r/s}$ since $P_0 = P(0) = e^{r/s} e^{-Ce^{-s \cdot 0}}$
$= e^{r/s} e^{-C \cdot 1} = e^{r/s} e^{-C} < e^{r/s} \Leftrightarrow C > 0$.)

<u>Example</u>. (Exercise 6, p. 78.)

Assuming $0 < P_0 < e^{r/s}$ (so $C > 0$) and $r > 0$, use the Gompertz equation

$$P' = P(r - s \ln P)$$

to find the ordinate (P) of the point of inflection ((t, P)) for a Gompertz (solution) curve.

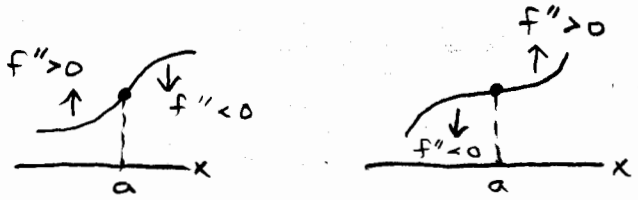
<u>Recall</u>:

<u>Defn</u>. A point $x = a$ is a <u>pt of inflection</u> of a function $f(x)$ if

(1) $f''(a) = 0$

(2) $f''(x)$ changes sign as x passes through a

E.g.,



We will find INFLECTION POINT using only the DE and not the general soln. :

Differentiate both sides of $P' = P(r - s\ln P)$ to get

$$\frac{d}{dt}\left(\frac{dP}{dt}\right) = \frac{d}{dt}\left[P(r - s\ln P)\right] \Longrightarrow$$

$$P'' = P\frac{d}{dt}(r - s\ln P) + \left(\frac{dP}{dt}\right)(r - s\ln P) \Longrightarrow$$

$$P'' = \cancel{P}\left(-s \cdot \frac{P'}{\cancel{P}}\right) + P'(r - s\ln P) \Longrightarrow$$

$$P'' = (-s + r - s\ln P)\underset{\substack{\parallel \\ P(r - s\ln P)}}{P'} \Longrightarrow$$

$$P'' = (r - s - \ln P)\,P\,(r - s\ln P).$$

Then $P'' = 0 \Longrightarrow$

(1) $r - s - \ln P = 0 \Longrightarrow \ln P = r - s \Longrightarrow$

$$\boxed{P = e^{r-s}}$$

X(2) $P = 0$ IMPOSSIBLE where $P(t) > 0$ for all $t \geq 0$ and $P = 0$ makes $\ln P$ undefined

X(3) $r - s \ln P = 0 \Rightarrow \ln P = \frac{r}{s} \Rightarrow$

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$P = e^{r/s}$ IMPOSSIBLE where $P_0 < e^{r/s} \Rightarrow C > 0 \Rightarrow$ $P(t) < e^{r/s}$ for all $t \geq 0$.

Chemical Reaction Model

We will SKIP this, along with HW Exercise 7, p. 78, Sect. 3.2, HW # 4.

Lecture

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Section 3.3. Systems of Linear and Nonlinear DEs.

System of DEs = more than one DE in more than one unknown function, where the DEs are to be solved simultaneously for all unknown functions

Examples of SYSTEMS OF LINEAR DEs.

1. (Example, p. 8, text)

$$\left.\begin{aligned} \frac{dx}{dt} &= 3x - 4y \\ \frac{dy}{dt} &= x + y \end{aligned}\right\}$$

- Must solve for $x = x(t)$ and $y = y(t)$ simultaneously
- Both first-order DEs

2. (Exercise 50, p. 10, text)

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= 4y + e^t \\ \frac{d^2y}{dt^2} &= 4x - e^t \end{aligned}\right\}$$

- Soln.: $x(t) = \cos 2t + \sin 2t + \frac{1}{5}e^t$
  $y(t) = -\cos 2t - \sin 2t - \frac{1}{5}e^t$
- Both second-order DEs

∴ System of two LINEAR FIRST-ORDER DEs is of the form

$$\frac{dx}{dt} = ax + by + f(t)$$ e.g., $\sin t$

$$\frac{dy}{dt} = cx + dy + g(t)$$ e.g., $t^2 + 3t - 1$

<u>Example of SYSTEM OF NONLINEAR DE.</u>

(HW Exercise 9, p. 92)

$$\frac{dx}{dt} = k_1 x(\alpha - x) = k_1 \alpha x - k_1 x^2$$ NONLINEAR TERM

$$\frac{dy}{dt} = k_2 xy$$ NONLINEAR TERM

<u>Note.</u> A single DE is called a

SCALAR DE.

<u>Examples of SCALAR DEs.</u>

1. Most everything we have seen up to now.

2. $\frac{dy}{dx} = \frac{y}{x}$

3. $\frac{dN}{dt} = kN$

4. $xy'' + (x^2+1)y' + 3y = \cos x$

We will now take a look at <u>one model</u> from this section.

$\boxed{\text{MODEL}}$ Radioactive Series.

$$A \xrightarrow{\text{decays}} B \xrightarrow{\text{decays}} C$$

A	B	C
• radioactive	• radioactive	• not radioactive
• unstable	• unstable	• stable

Let x = amt of A present (remaining) at time t
y = amt of B " " " " "
z = amt of C " " " " "

Suppose the following:

(1) The decay of A is described by

$$\frac{dx}{dt} = -\lambda_1 x, \qquad \lambda_1 > 0 \text{ (\underline{decay constant} for A)}$$

(2) The decay of B, WITHOUT ANY INPUT FROM A (i.e., when isolated from A) is described by

$$\frac{dy}{dt} = -\lambda_2 y, \quad \lambda_2 > 0 \quad \text{(\underline{decay constant} for B)}$$

(3) The decay of C, WITHOUT ANY INPUT FROM B, is described by

$$\frac{dz}{dt} = 0$$

since C does not decay.

However, when we do not view B in isolation of A and C in isolation of B and instead view A, B, and C as "linked up" like so

$$A \xrightarrow[\text{decaying into}]{} B \xrightarrow[\text{decaying into}]{} C,$$

then we have, simultaneously,

$$(*)\quad \begin{cases} \dfrac{dx}{dt} = \underbrace{-\lambda_1 x}, & \lambda_1 > 0 \\[2ex] \dfrac{dy}{dt} = \underbrace{-\lambda_2 y} + \overbrace{\lambda_1 x}, & \lambda_2 > 0 \\[2ex] \dfrac{dz}{dt} = 0 + \overbrace{\lambda_2 y} \end{cases}$$

Amt of A lost per unit time (where A is transformed into B) becomes amt of B gained per unit time

Amt of B lost per unit time (where B is transmuted into C) becomes amt of C gained per unit time

<u>Example</u>. (HW Exercise 1, p. 87.)

Find a solution of (*) subject to the initial conditions

$$x(0) = x_0, \quad y(0) = 0, \quad z(0) = 0.$$

We will solve the SYSTEM (*) using familiar techniques for scalar DEs.

① First consider the IVP

$$\begin{cases} \dfrac{dx}{dt} = -\lambda_1 x, \\ x(0) = x_0 \end{cases}$$

alone. Can do that since everything is in terms of x (and t). So, have:

$$\frac{dx}{dt} = -\lambda_1 x \Rightarrow x' + \underbrace{\lambda_1}_{P(t)=\lambda_1} x = 0 \Rightarrow \qquad \text{INTEG. FACTOR } e^{\int P(t)\,dt} = e^{\int \lambda_1 dt} = e^{\lambda_1 t}$$

$$e^{\lambda_1 t}(x' + \lambda_1 x) = e^{\lambda_1 t}(0) \Rightarrow \int \frac{d}{dt}(e^{\lambda_1 t} x)\,dt = \int 0\,dt$$

$$\Rightarrow e^{\lambda_1 t} x = C \Rightarrow x(t) = Ce^{-\lambda_1 t}$$

$$x(0) = x_0 \Rightarrow x_0 = x(0) = Ce^{-\lambda_1 (0)} \Rightarrow x_0 = C \cdot 1 \Rightarrow$$

$$C = x_0 \Rightarrow \boxed{x(t) = x_0 e^{-\lambda_1 t}}$$

Part of the soln. of (*)

② Next, consider the IVP

$$\begin{cases} \dfrac{dy}{dt} = -\lambda_2 y + \lambda_1 x, \\ y(0) = 0 \end{cases}$$

and replace the x by $x_0 e^{-\lambda_1 t}$:

$$\frac{dy}{dt} = -\lambda_2 y + \lambda_1 x_0 e^{-\lambda_1 t} \Rightarrow$$

$$y' + \underbrace{\lambda_2}_{P(t)=\lambda_2} y = \lambda_1 x_0 e^{-\lambda_1 t} \text{ (in the form } y' + P(t)y = f(t)) \Rightarrow$$

$$e^{\int P(t)\,dt} = e^{\int \lambda_2 dt} = e^{\lambda_2 t} \text{ INTEG. FACTOR}$$

$$e^{\lambda_2 t}(y' + \lambda_2 y) = e^{\lambda_2 t}(\lambda_1 x_0 e^{-\lambda_1 t}) \Rightarrow$$

$$e^{\lambda_2 t} y' + \lambda_2 e^{\lambda_2 t} y = \lambda_1 x_0 e^{\lambda_2 t - \lambda_1 t} \Rightarrow$$

$$\frac{d}{dt}(e^{\lambda_2 t} y) = \lambda_1 x_0 e^{(\lambda_2 - \lambda_1)t} \Rightarrow$$

$$\int \frac{d}{dt}(e^{\lambda_2 t} y)\,dt = \lambda_1 x_0 \int e^{(\lambda_2 - \lambda_1)t}\,dt \Rightarrow$$

$$\int e^{at}\,dt = \frac{e^{at}}{a} + C$$

$$e^{\lambda_2 t} y = \lambda_1 x_0 \left[\frac{e^{(\lambda_2 - \lambda_1)t}}{\lambda_2 - \lambda_1}\right] + C \Rightarrow$$

MULTIPLY BOTH SIDES BY $e^{-\lambda_2 t}$

$$y = \frac{\lambda_1 x_0 e^{\lambda_2 t - \lambda_1 t} e^{-\lambda_2 t}}{\lambda_2 - \lambda_1} + C e^{-\lambda_2 t} \Rightarrow$$

$$y(t) = \frac{\lambda_1 x_0 e^{-\lambda_1 t}}{\lambda_2 - \lambda_1} + C e^{-\lambda_2 t}$$

$$y(0) = 0 \Rightarrow 0 = y(0) = \frac{\lambda_1 x_0 e^{-\lambda_1 (0)}}{\lambda_2 - \lambda_1} + C e^{-\lambda_2 (0)}$$

$$\Rightarrow \quad 0 = \frac{\lambda_1 x_0}{\lambda_2 - \lambda_1} + C \Rightarrow C = -\frac{\lambda_1 x_0}{\lambda_2 - \lambda_1}$$

$$\boxed{y(t) = \frac{\lambda_1 x_0 e^{-\lambda_1 t}}{\lambda_2 - \lambda_1} - \frac{\lambda_1 x_0 e^{-\lambda_2 t}}{\lambda_2 - \lambda_1}} \Longrightarrow$$

$$\boxed{y(t) = \frac{\lambda_1 x_0}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)}$$

Another part of the soln. of (*)

(3) Finally, consider the IVP

$$\begin{cases} \dfrac{dz}{dt} = \lambda_2 y, \\ z(0) = 0 \end{cases}$$

and replace the y by $\dfrac{\lambda_1 x_0}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$:

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$$\frac{dz}{dt} = \frac{\lambda_1 \lambda_2 x_0}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) \Longrightarrow$$

in the form $\dfrac{dz}{dt} = g(t) \Longrightarrow$

can use SEPARATION OF VARIABLES

$$dz = \frac{\lambda_1 \lambda_2 x_0}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) dt \Longrightarrow$$

$$\int 1 \cdot dz = \frac{\lambda_1 \lambda_2 x_0}{\lambda_2 - \lambda_1} \int \left(e^{-\lambda_1 t} - e^{-\lambda_2 t} \right) dt \Longrightarrow$$

$$z = \frac{\lambda_1 \lambda_2 x_0}{\lambda_2 - \lambda_1} \left(\frac{e^{-\lambda_1 t}}{-\lambda_1} - \frac{e^{-\lambda_2 t}}{-\lambda_2} \right) + C \Longrightarrow$$

$$z = \frac{\lambda_1 \lambda_2 x_0}{\lambda_2 - \lambda_1} \left(\frac{e^{-\lambda_2 t}}{\lambda_2} - \frac{e^{-\lambda_1 t}}{\lambda_1} \right) + C \Longrightarrow$$

$$z(t) = \frac{x_0}{\lambda_2 - \lambda_1} \left(\lambda_1 e^{-\lambda_2 t} - \lambda_2 e^{-\lambda_1 t} \right) + C$$

$$z(0) = 0 \Longrightarrow 0 = z(0) = \frac{x_0}{\lambda_2 - \lambda_1} \left(\lambda_1 e^{-\lambda_2 (0)} - \lambda_2 e^{-\lambda_1 (0)} \right) + C$$

$$\Longrightarrow 0 = \frac{x_0}{\lambda_2 - \lambda_1} (\lambda_1 - \lambda_2) + C \Longrightarrow$$

$$0 = \frac{-x_0 \cancel{(\lambda_2 - \lambda_1)}}{\cancel{\lambda_2 - \lambda_1}} + C \Longrightarrow 0 = -x_0 + C \Longrightarrow$$

$$C = x_0 \Longrightarrow$$

$$\boxed{z(t) = \frac{x_0}{\lambda_2 - \lambda_1} \left(\lambda_1 e^{-\lambda_2 t} - \lambda_2 e^{-\lambda_1 t} \right) + x_0}$$

Remaining part of soln. of (*)

$\therefore$ PARTICULAR SOLUTION of (*) is

$$\boxed{\begin{aligned} x(t) &= x_0 e^{-\lambda_1 t} \\ y(t) &= \frac{\lambda_1 x_0}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) \\ z(t) &= \frac{x_0}{\lambda_2 - \lambda_1}\left(\lambda_1 e^{-\lambda_2 t} - \lambda_2 e^{-\lambda_1 t}\right) + x_0 \end{aligned}}$$

Note: We were able to solve SYSTEM (*) by solving each DE in the system separately since the system is an essentially

UNCOUPLED SYSTEM

We will SKIP (1) "Solution Mixtures" Model
(2) "Electrical Networks" Model
(3) "Predator-Prey/Competition" Models

along with HW Exercises 5 and 12,

pp. 87-89, Sect. 3.3; HW #4.

<u>Suggestions for HW Exercise 9, p. 92, REVIEW of Ch. 3, HW #4</u>

To solve for $x = x(t)$ and $y = y(t)$ in the SYSTEM

$$\begin{cases} \dfrac{dx}{dt} = k_1 x(\alpha - x), \\ \dfrac{dy}{dt} = k_2 xy \end{cases}$$

do the following.

(1) First solve the DE (which alone is a LOGISTIC DE)

$$\frac{dx}{dt} = k_1 x(\alpha - x)$$

using METHOD OF SEPARATION OF VARIABLES and METHOD OF PARTIAL FRACTIONS:

$$\frac{dx}{dt} = k_1 x(\alpha - x) \Rightarrow \int \underbrace{\frac{1}{x(\alpha-x)}}\, dx = \int k_1\, dt \Rightarrow$$

$$\frac{1}{x(\alpha-x)} = \frac{A}{x} + \frac{B}{\alpha - x} \Rightarrow 1 = A(\alpha - x) + Bx \Rightarrow$$

$$0 \cdot x + 1 = (-A+B)x + \alpha A \Rightarrow \left.\begin{array}{l} -A+B=0 \\ \alpha A = 1 \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} A = \frac{1}{\alpha} \\ B = \frac{1}{\alpha} \end{array}}$$

$$\int \left(\frac{A}{x} + \frac{B}{\alpha - x} \right) dx = \int k_1\, dt \Rightarrow$$

$$\frac{1}{\alpha} \int \frac{1}{x}\, dx + \frac{1}{\alpha} \int \underbrace{\frac{1}{\alpha - x}}\, dx = \int k_1\, dt \Longrightarrow$$

Use INTEGRATION BY SUBSTITUTION:

$$\boxed{w = \alpha - x} \Rightarrow \frac{dw}{dx} = -1 \Rightarrow \boxed{-dw = dx}$$

$$\frac{1}{\alpha} \ln|x| + \frac{1}{\alpha} \int \frac{-1}{w}\, dw = k_1 t + C_1 \Rightarrow$$

$$\frac{1}{\alpha} \ln|x| - \frac{1}{\alpha} \ln|w| = k_1 t + C_1 \Longrightarrow$$

$$\frac{1}{\alpha} \ln|x| - \frac{1}{\alpha} \ln|\alpha - x| = k_1 t + C_1 \Longrightarrow$$

$$\ln|x| - \ln|\alpha - x| = \alpha k_1 t + \underbrace{\alpha C_1}_{= C_1} \Rightarrow$$

$$\ln\left|\frac{x}{\alpha - x}\right| = \alpha k_1 t + C_1 \Longrightarrow$$

$$e^{\ln\left|\frac{x}{\alpha - x}\right|} = e^{\alpha k_1 t + C_1} \Longrightarrow$$

$$\left|\frac{x}{\alpha - x}\right| = e^{C_1} e^{\alpha k_1 t} \Longrightarrow$$

$$\frac{x}{\alpha - x} = \underbrace{\pm e^{c_1}}_{=C_1} e^{\alpha k_1 t} \Rightarrow$$

$$\frac{x}{\alpha - x} = C_1 e^{\alpha k_1 t} \Rightarrow$$

$$x = C_1 e^{\alpha k_1 t}(\alpha - x) \Rightarrow$$

$$x = \alpha C_1 e^{\alpha k_1 t} - C_1 e^{\alpha k_1 t} x \Rightarrow$$

$$x + C_1 e^{\alpha k_1 t} x = \alpha C_1 e^{\alpha k_1 t} \Rightarrow$$

$$x(1 + C_1 e^{\alpha k_1 t}) = \alpha C_1 e^{\alpha k_1 t} \Rightarrow$$

$$\boxed{x(t) = \frac{\alpha C_1 e^{\alpha k_1 t}}{1 + C_1 e^{\alpha k_1 t}}}$$

(2) Then solve the DE

$$\frac{dy}{dt} = k_2 x y$$

by first replacing the x by $\frac{\alpha C_1 e^{\alpha k_1 t}}{1 + C_1 e^{\alpha k_1 t}}$, and then using the METHOD OF SEPARATION OF VARIABLES:

$$\frac{dy}{dt} = k_2\, xy \Rightarrow \frac{dy}{dt} = k_2\left(\frac{\alpha C_1 e^{\alpha k_1 t}}{1+C_1 e^{\alpha k_1 t}}\right) y \Rightarrow$$

in the form $\frac{dy}{dt} = \underbrace{g(t)}_{\frac{\alpha k_2 C_1 e^{\alpha k_1 t}}{1+C_1 e^{\alpha k_1 t}}}\ \underbrace{h(y)}_{y}$

$$\frac{1}{y}\, dy = \frac{\alpha k_2 C_1 e^{\alpha k_1 t}}{1+C_1 e^{\alpha k_1 t}}\, dt \Rightarrow$$

$$\int \frac{1}{y}\, dy = \alpha k_2 C_1 \int \underbrace{\frac{e^{\alpha k_1 t}}{1+C_1 e^{\alpha k_1 t}}}\, dt \Rightarrow$$

use INTEGRATION BY SUBSTITUTION:

$$\boxed{w = 1 + C_1 e^{\alpha k_1 t}} \Rightarrow \frac{dw}{dt} = \alpha k_1 C_1 e^{\alpha k_1 t} \Rightarrow$$

$$\frac{d}{dt} e^{at} = a e^{at}$$

$$\boxed{\frac{1}{\alpha k_1 C_1}\, dw = e^{\alpha k_1 t}\, dt}$$

$$\ln|y| = \alpha k_2 C_1 \int \frac{\frac{1}{\alpha k_1 C_1}}{w}\, dw \Rightarrow$$

$$\ln|y| = \frac{\cancel{\alpha} k_2 \cancel{C_1}}{\cancel{\alpha} k_1 \cancel{C_1}} \ln|w| + C \Rightarrow$$

$$\boxed{\ln|y| = \frac{k_2}{k_1} \ln|1 + C_1 e^{\alpha k_1 t}| + C}$$

Lecture

$\boxed{\text{CHAPTER 4}}$ DEs of Higher Order

We will look at specific methods of solution of

(1) SCALAR nth-order DEs

primarily 2nd-order DEs for simplicity

(2) SYSTEMS of n 1st-order DEs

We will SKIP Sections 4.5 (an alternative method to the one given in Section 4.4) and 4.7 (on a specific type of DE

Lecture

Section 4.1. Preliminary Theory: Linear Equations.

I will primarily state RESULTS WITHOUT PROOF.

This section made up of 3 subsections:

I. Existence and uniqueness of solutions of

A. IVPs (initial value problems)
B. BVPs (boundary value problems)

II. Can have 0, 1, or >1 soln.

0: Soln. does not exist
1: Soln. exists and is unique
>1: Soln. is not unique

II. Homogeneous (linear) DEs ($\neq$ DE in 2.4 where have $Mdx + Ndy = 0$ and can make substitution $y = ux$): $a_1 y^{(n)} + \cdots + a_0 y = \underline{\underline{0}}$

A. Superposition principle:
Sum of solns. of a linear DE is also a soln.

B. Linear dependence/independence:
nth-order DE has n DISTINCT (i.e., linearly independent) solns.

C. General soln. of a homogeneous linear nth-order DE:

"Linear combination" of n
- linearly independent solns.
of DE or

$$\underbrace{y(x) = c_1 y_1(x) + \cdots + c_2 y_2(x)}_{\text{GEN. SOLN.}}$$

III. Nonhomogeneous (linear) DEs :

$$a_1 y^{(n)} + \cdots + a_0 y = \underbrace{g(x)}_{\text{not } \underbrace{\text{identically equal to } 0}_{\text{means equal to } 0 \text{ for all } x}}$$

A. General soln. :
soln. of homogeneous version +
particular soln.

$$a_1 y^{(n)} + \cdots + a_0 y = 0 \leftarrow y_c$$

$$a_1 y^{(n)} + \cdots + a_0 y = g(x) \leftarrow y_p$$

$$y(x) = y_c(x) + y_p(x)$$

B. Superposition principle :
Sum of particular solns. is also
a particular soln.

Lecture

Subsection 4.1.1. Initial Value and Boundary Value Problems (IVPs and BVPs)

Linear nth-order IVP:

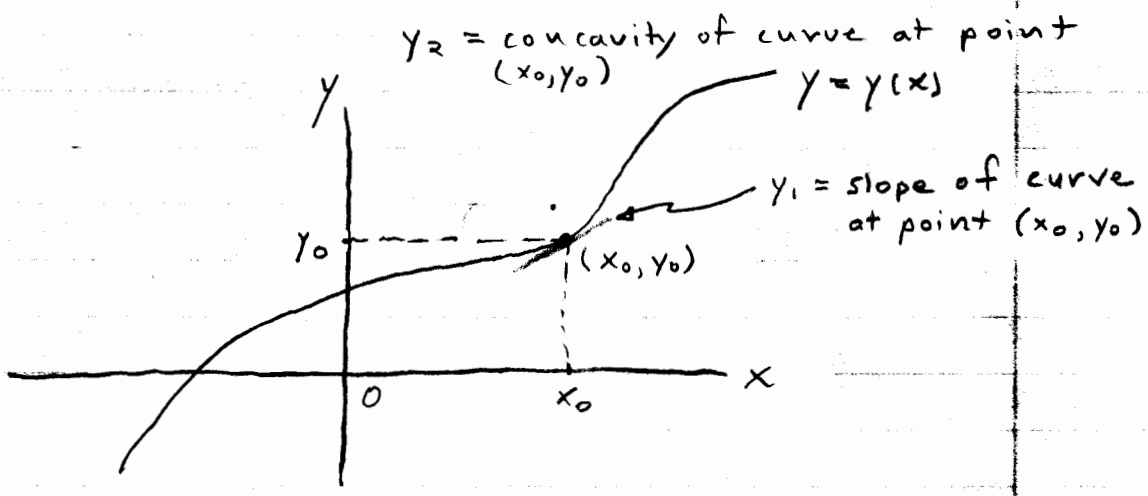
$$\begin{cases} a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x) \\ \left.\begin{array}{l} y(x_0) = y_0 \\ y'(x_0) = y_1 \\ \vdots \\ y^{(n-1)}(x_0) = y_{n-1} \end{array}\right\} \text{ n INITIAL CONDITIONS} \end{cases}$$

General Soln.: Will end up with n arbitrary constants $C_1, C_2, \ldots, C_n$. In a sense, these constants are "constants of integration" which come from the "n integrations" of $y^{(n)} = d^n y/dx^n$ to obtain y:

$$y^{(n)} = -\frac{a_{n-1}(x)}{a_n(x)}y^{(n-1)} - \cdots - \frac{a_1(x)}{a_n(x)}y' - \frac{a_0(x)}{a_n(x)} + \frac{g(x)}{a_n(x)}$$

Particular Soln.: Need n initial conditions to solve for $C_1, C_2, \ldots, C_n$.

Solution Curve of IVP:



$y_3 - y_{n-1}$ further describe curve and its curvature at point (x_0, y_0)

<u>Note</u>. When we say

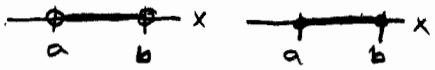
CONSIDER THE IVP ON THE INTERVAL I

we will mean

CONSIDER THE PART. SOLN. $y = y(x)$ AS HOLDING FOR ALL x IN I (AND I ONLY)

I will be of the form

(a, b) $\quad$ $[a, b]$



(a, ∞) $\quad$ $[a, \infty)$

$(-\infty, a)$ $\quad$ $(-\infty, a]$

open intervals $\quad$ closed intervals

Existence and Uniqueness of a Linear nth-Order IVP

PICARD'S EXISTENCE AND UNIQUENESS THEOREM (for Linear nth-Order IVPs).

Consider the following IVP on the interval I:

$$(*)\begin{cases} a_n(x)y^{(n)} + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y' + a_0(x)y = g(x) \\ y(x_0) = y_0 \\ y'(x_0) = y_1 \\ \vdots \\ y^{(n-1)}(x_0) = y_{n-1} \end{cases}$$

IF (1) $a_0(x), a_1(x), \ldots, a_n(x), g(x)$ are all continuous on I

(2) $a_n(x) \neq 0$ on I

THEN $(*)$ has exactly one particular solution (i.e., there exists a part. soln. and it is unique or the only solution that works).

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<u>Examples.</u>

1. (HW Exercise 5, p. 107.)

Given that

$$y = c_1 x + c_2 x \ln x$$

is a 2-parameter (c_1, c_2) family of solns. of

$$x^2 y'' - x y' + y = 0 \quad \text{on the interval } (-\infty, \infty),$$

Find a member of the family satisfying the initial conditions

$$y(1) = 3, \quad y'(1) = -1.$$

Let $\boxed{x_0 = 1}$.

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Replace $(-\infty, \infty)$ by an interval I such that (1) x_0 is contained in I, and (2) $y = c_1 x + c_2 x \ln x$ or $\ln x$ is defined on I:

Choose $\boxed{I = (0, \infty)}$

Then have IVP

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$$\begin{cases} x^2 y'' - x y' + y = 0 & \text{on } I = (0, \infty) \\ \quad a_2(x) = x^2 \quad a_1(x) = -x \quad a_0(x) = 1 \quad g(x) = 0 \\ y(1) = 3 \\ y'(1) = -1 \end{cases}$$

1. $a_0(x)$, $a_1(x)$, $a_2(x)$, $g(x)$ all cont. on $I = (0, \infty)$

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2. $a_2(x) \neq 0$ on $I = (0, \infty)$

$\therefore$ This IVP has exactly one soln. We can find it by solving for c_1 and c_2:

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$$y(x) = c_1 x + c_2 x \ln x$$

$$y'(x) = c_1 + c_2\left(x \cdot \frac{1}{x} + 1 \cdot \ln x\right) = c_1 + c_2(1 + \ln x)$$

$$y(1) = 3 \Rightarrow 3 = y(1) = c_1(1) + c_2(1)\overbrace{\ln(1)}^{=0}$$
$$\Rightarrow 3 = c_1 + 0$$
$$\Rightarrow c_1 = 3$$

$$y'(1) = -1 \Rightarrow -1 = y'(1) = \overbrace{c_1}^{=3} + c_2(1 + \overbrace{\ln(1)}^{=0})$$
$$\Rightarrow -1 = 3 + c_2$$
$$\Rightarrow c_2 = -4$$

$\therefore$ Part. soln. is $\boxed{y(x) = 3 - 4x\ln x}$

2. (HW Exercise 6, p. 107.)

Given that

$$y = c_1 + c_2 x^2 \quad .$$

is a 2-parameter (c_1, c_2) family of solns. of

$$xy'' - y' = 0 \quad \text{on the interval } (-\infty, \infty),$$

show that no member of the family satisfies the initial conditions

$$y(0) = 0, \quad y'(0) = 1.$$

Explain why this does not violate PICARD'S THEOREM.

Let $\boxed{x_0 = 0}$

Choose interval I such that (1) x_0 is contained in I, and (2) $y = c_1 + c_2 x^2$ is defined on I:

$$\text{Choose } \boxed{I = (-\infty, \infty)}$$

Then have IVP

$$\begin{cases} xy'' - y = 0 & \text{on } I = (-\infty, \infty) \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

$$\underbrace{x}_{a_2(x)=x}y'' + \underbrace{0}_{a_1(x)=0}\cdot y' \underbrace{-}_{a_0(x)=-1} y = \underbrace{0}_{g(x)=0}$$

1. $a_0(x), a_1(x), a_2(x), g(x)$ all cont. on $I = (-\infty, \infty)$
2. It is <u>not</u> true that $a_2(x) \neq 0$ on $I = (-\infty, \infty)$ since $a_2(x) = 0$ when $x = 0$, which is contained in I.

$\therefore$ Cannot say whether or not this IVP has a solution and, if it does, whether or not the solution is the only one.

Try to solve for c_1 and c_2 anyway:

$$y(x) = c_1 + c_2 x^2$$

$$y'(x) = 2c_2 x$$

$$y(0) = 0 \Rightarrow 0 = y(0) = c_1 + c_2(0)^2$$
$$\Rightarrow c_1 = 0$$

$$y'(0) = 1 \Rightarrow 1 = y'(0) = 2c_2(0)$$
$$\Rightarrow 1 = 0 \quad \text{IMPOSSIBLE!}$$

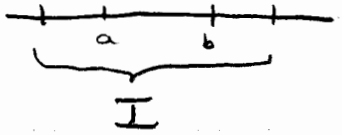
$\therefore$ There can be no particular solution. ✓

Existence and Uniqueness of a Linear 2nd-Order Boundary Value Problem (BVP)

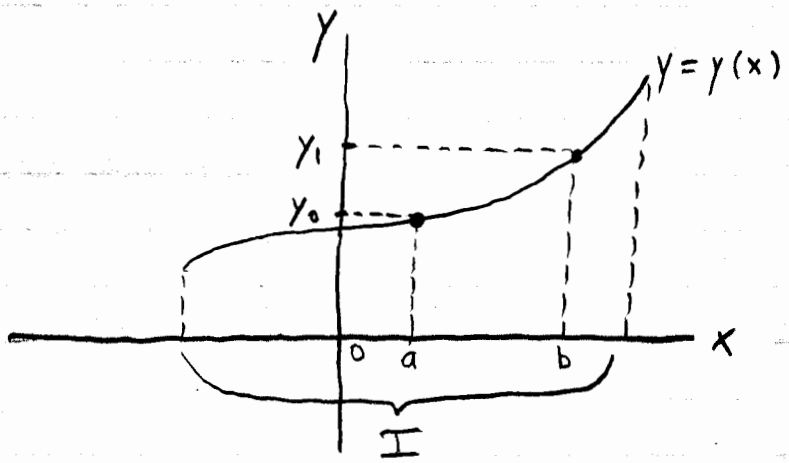
Linear 2nd-order BVP :

$$\begin{cases} a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \quad \text{on} \quad I \\ \left.\begin{array}{l} y(a) = y_0 \\ y(b) = y_1 \end{array}\right\} \text{2 BOUNDARY CONDITIONS} \end{cases}$$

$x = a$ and $x = b$ are contained in the interval I



Solution curve :



<u>Note</u>: The solution $y(x)$ of the DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

is composed of 2 parts.

$$y(x) = \underbrace{c_1 y_1(x) + c_2 y_2(x)}_{} + \underbrace{y_p(x)}_{}$$

y_1 and y_2 are each solns. of the DE

$a_2 y'' + a_1 y' + a_0 y = 0$

y_p is a specific soln. of the DE

$a_2 y'' + a_1 y' + a_0 y = g(x)$

<u>Theorem</u>.

Consider the following BVP on the interval I:

$$(*) \begin{cases} a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \\ y(a) = y_0 \\ y(b) = y_1 \end{cases}$$

Suppose the soln. of the DE is

$$y = c_1 y_1 + c_2 y_2 + y_p$$

Consider the system of algebraic equations

$$(**)\quad \begin{cases} c_1 y_1(a) + c_2 y(a) = y_0 - y_p(a) \\ c_1 y_1(b) + c_2 y(b) = y_1 - y_p(b) \end{cases}$$

to be solved for c_1 and c_2.

THEN the BVP $(*)$ has as many solutions as the system $(**)$.

So, one of the following is true:

(1) The BVP $(*)$ has NO soln.
(2) The BVP $(*)$ has EXACTLY ONE soln.
(3) The BVP $(*)$ has MORE THAN ONE soln. (usually infinitely many solns.)

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<u>Example.</u>

DE: $y'' + y = 0$

General solution: $y(x) = c_1 \cos x + c_2 \sin x$

$\boxed{\text{Case 1}}$: $y(0) = 2, \quad y(\pi) = 1$

Then $c_1 = 2$, $c_1 = -1$ $\Rightarrow$ IMPOSSIBLE $\Rightarrow$ BVP has <u>no soln.</u>

$\boxed{\text{Case 2}}$: $y(0) = 2$, $y(\frac{\pi}{2}) = 3$

Then $c_1 = 2$, $c_2 = 3$ $\Rightarrow$ BVP has <u>exactly one soln.</u>

3/21/01 W lecture $\boxed{\text{Case 3}}$: $y(0) = 2$, $y(\pi) = -2$

Then $c_1 = 2$, $c_1 = 2$ $\Rightarrow$ $c_1 = 2$ and c_2 is arbitrary $\Rightarrow$ BVP has <u>infinitely many solns.</u>

$\boxed{\text{Case 2}}$: $y(0) = 2$, $y(\frac{\pi}{2}) = 3$

$$2 = y(0) = c_1 \overbrace{\cos 0}^{=1} + c_2 \overbrace{\sin 0}^{=0} \Longrightarrow$$

$$2 = c_1$$

$$3 = y(\tfrac{\pi}{2}) = c_1 \overbrace{\cos \tfrac{\pi}{2}}^{=0} + c_2 \overbrace{\sin \tfrac{\pi}{2}}^{=1} \Longrightarrow$$

$$3 = c_2$$

Lecture

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Subsection 4.1.2. Homogeneous Equations.

Definition of Homogeneous Equation

We will look at LINEAR homogeneous DEs.

Write a linear nth-order DE in standard form:

$$(*) \quad \underbrace{a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y}_{\text{All terms with } y \text{ or its derivatives are placed on the left}} = \underbrace{g(x)}_{\text{Any expression in } x \text{ only is placed on the right}}$$

<u>Defn.</u> Consider Eq. (*) on an interval I. If $g(x) = 0$ for ALL x in I, then Eq. (*) is called <u>homogeneous</u>. Otherwise, Eq. (*) is called <u>nonhomogeneous</u>.

<u>Examples</u>,

1. $x^2y'' - xy' + y = 0$ homogeneous

2. $x^2y'' - xy' + y = x^2 + 2x + 1$ nonhomogeneous

3. $y'' = 2y' - 2y \Rightarrow y'' - 2y' + 2y = 0$ homogeneous

4. $(x-2)y'' + 3y - 1 = 0 \Rightarrow (x-2)y'' + 3y = 1$ nonhomogeneous

WARNING: "Homogeneous" means one thing in Sect. 2.4, and another thing here.

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Differential Operators

SKIP this part.
We will also SKIP Sects. 4.5 and 4.8 which involve differential operators.

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Superposition Principle for Homogeneous Linear nth-Order DEs

Superposition Principle. If $y_1(x), y_2(x), \ldots, y_k(x)$ are solutions of a homogeneous linear nth-order DE on an interval I, then

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

is also a solution on I, where $c_1, c_2, \ldots, c_k$ are arbitrary constants.

Example. (taken from HW Exercise 5, p. 107)

$$x^2 y'' - x y' + y = 0$$

Observe: $y_1(x) = x$ and $y_2(x) = x \ln x$ are each solns. of the DE

$$\left.\begin{array}{l} y_1 = x \\ y_1' = 1 \\ y_1'' = 0 \end{array}\right\} \quad x^2(0) - x(1) + x \overset{\checkmark}{=} 0$$

$$\left.\begin{array}{l} y_2 = x \ln x \\ y_2' = 1 + \ln x \\ y_2'' = \frac{1}{x} \end{array}\right\} \quad x^2\left(\frac{1}{x}\right) - x(1 + \ln x) + x \ln x \overset{\checkmark}{=} 0$$

Claim: $y(x) = c_1 y_1(x) + c_2 y_2(x)$ is ALSO a soln. of the DE

Proof:
$$y = c_1 y_1 + c_2 y_2 = c_1 x + c_2 x \ln x$$
$$y' = c_1 y_1' + c_2 y_2' = c_1 + c_2(1 + \ln x)$$
$$y'' = c_1 y_1'' + c_2 y_2'' = c_2\left(\tfrac{1}{x}\right)$$

$$x^2 c_2\left(\tfrac{1}{x}\right) - x\left(c_1 + c_2(1+\ln x)\right) + \left(c_1 x + c_2 x \ln x\right) \stackrel{?}{=} 0$$

$$\cancel{c_2 x} - \cancel{c_1 x} - \cancel{c_2 x} - \cancel{c_2 x \ln x} + \cancel{c_1 x} + \cancel{c_2 x \ln x} \stackrel{?}{=} 0$$

$$0 \stackrel{\checkmark}{=} 0$$

<u>NOTES</u>:

1. "$c_1 x + c_2 x \ln x$" is called a <u>linear combination</u> of the two solutions $y_1 = x$ and $y_2 = x \ln x$

LINEAR COMB. OF FUNCTIONS
= take all the functions, multiply each by a constant, sum the resulting terms

2. $y = c_1 x + c_2 x \ln x$ is called the

GENERAL SOLN.

of the DE since it "contains" all

possible solns. of the DE, i.e., every soln. of the DE is obtained by setting c_1 and c_2 equal to specific values.

The reason why $y = c_1 x + c_2 x \ln x$ is the GENERAL SOLN. is because

(1) x and $x \ln x$ are "distinct from each other" (there is a way to characterize this "distinctness" mathematically).

(2) The number of "distinct" solns., which is 2, is equal to the order of the DE, which is 2.

$\therefore$ The GENERAL SOLN. of a homogeneous linear nth-order DE

= linear combination of n "distinct" solns. of the DE

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Mathematical Way to Describe "Distinct" Functions

<u>Defn.</u> Let $\{f_1(x), \ldots, f_n(x)\}$ be a set of functions. Form the linear combination and set it equal to 0:

(*) $$c_1 f_1(x) + \cdots + c_n f_n(x) = 0.$$

Then: (i) $\{f_1(x), \ldots, f_n(x)\}$ is <u>linearly dependent</u> on an interval I if there exist one set of constants, $\{c_1, \ldots, c_n\}$, NOT ALL 0, such that Eq. (*) holds for ALL x in I.

(ii) $\{f_1(x), \ldots, f_n(x)\}$ is <u>linearly independent</u> on an interval I if the only set of constants, $\{c_1, \ldots, c_n\}$, for which Eq. (*) holds for ALL x in I is the set $\{0, \ldots, 0\}$. (In this case, $f_1(x), \ldots, f_n(x)$ are "distinct.")

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A Set of Two Functions

RESULT 1. $\{f_1(x), f_2(x)\}$ is <u>linearly dependent on I</u> if

$$\frac{f_1(x)}{f_2(x)} = c = \text{constant} \quad \text{for all } x \text{ in } I$$

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(Note: Do not worry about $f_1(x)$ or $f_2(x)$ being 0 for some x in I. That does not affect the above result.)

PROOF. For all x in I,

$$\frac{f_1(x)}{f_2(x)} = c \Rightarrow f_1(x) = c f_2(x)$$

$$\Rightarrow \underbrace{1}_{=c_1} \cdot f_1(x) + \underbrace{(-c)}_{=c_2} f_2(x) = 0$$

$\therefore$ There exist $\{c_1, c_2\}$ with at least $c_1 = 1 \neq 0$ such that $c_1 f_1(x) + c_2 f_2(x) = 0$ for all x in I.

Examples.

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1. $f_1(x) = x$, $f_2(x) = 3x$ on $(-\infty, \infty)$

$$\frac{f_1(x)}{f_2(x)} = \frac{\cancel{x}}{3\cancel{x}} = \frac{1}{3} \Rightarrow \text{lin. dep.}$$

2. $f_1(x) = \sin x$, $f_2(x) = \cos x$ on $(-\infty, \infty)$

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$$\frac{f_1(x)}{f_2(x)} = \frac{\sin x}{\cos x} = \tan x \neq \text{constant for all } x \text{ in } (-\infty, \infty)$$

$\Longrightarrow$ lin. indep.

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3. $f_1(x) = x$, $f_2(x) = x \ln x$

$$\frac{f_1(x)}{f_2(x)} = \frac{\cancel{x}}{\cancel{x}\ln x} = \frac{1}{\ln x} \neq \text{constant for all } x \text{ in } (-\infty, \infty)$$

$\Longrightarrow$ lin. indep.

4. $f_1(x) = x + 1$, $f_2(x) = x$ on $(-\infty, \infty)$

$$\frac{f_1(x)}{f_2(x)} = \frac{x+1}{x} = 1 + \frac{1}{x} \neq \text{constant for all } x \text{ in } (-\infty, \infty)$$

$\Longrightarrow$ lin. indep.

5. $f_1(x) = x^2$, $f_2(x) = x$ on $(-1, 1)$

$$\frac{f_1(x)}{f_2(x)} = \frac{x^2}{x} = x \neq \text{constant for all } x \text{ in } (-1, 1)$$

$\Rightarrow \{f_1, f_2\}$ lin. indep.

6. $f_1(x) = |x|$, $f_2(x) = x$ on $(-\infty, \infty)$

$$\frac{f_1(x)}{f_2(x)} = \frac{|x|}{x} = \begin{cases} \frac{x}{x} = 1 & \text{for } x > 0 \\ \frac{-x}{x} = -1 & \text{for } x < 0 \end{cases}$$

$$= \begin{cases} 1 & \text{on } (0, \infty) \\ -1 & \text{on } (-\infty, 0) \end{cases}$$

$\neq$ constant on $(-\infty, \infty) \Longrightarrow$

$\{f_1, f_2\}$ lin. indep. on $(-\infty, \infty)$

A Set of Greater Than or Equal to Two Functions

(SEE MATH 310 LINEAR ALGEBRA)

We move from dealing with linear combinations to dealing with

DETERMINANTS

RECALL: Cramer's rule, which makes use of 2×2 determinants.

Consider 2 homogeneous algebraic eqs. in 2 unknowns:

$$\begin{cases} ax + by = 0 \\ cx + dy = 0 \end{cases}$$

Then the solution(s) come(s) from

$x =$

$$x = \frac{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{0 \cdot d - 0 \cdot b}{ad - cb} = \frac{0}{ad - cb}$$

2 x 2 DETERMINANTS

$$y = \frac{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{a \cdot 0 - c \cdot 0}{ad - cb} = \frac{0}{ad - cb}$$

REVIEW OR NEW:

Evaluation of 2 x 2 determinant;

$$\begin{vmatrix} a & \overset{(-)}{b} \\ c & \underset{(+)}{d} \end{vmatrix} = ad - cb$$

e.g., $\begin{vmatrix} 2 & 4 \\ 9 & 18 \end{vmatrix} = 2 \cdot 18 - 9 \cdot 4 = 0$

Evaluation of 3x3 determinant:

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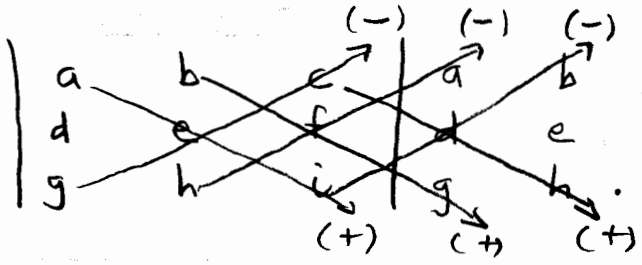

$$= aei + bfg + cdh - gec - hfa - idb$$

How This All Relates To Solving DEs

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Defn. Wronskian of $f_1, \ldots, f_n$ on I:

$$W(f_1, \ldots, f_n) = \begin{vmatrix} f_1(x) & \cdots & f_n(x) \\ \vdots & & \\ f_1^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

Wronskian of $f_1, \ldots, f_n$ at a point x_0 in I:

$$W(f_1, \ldots, f_n)(x_0) = \begin{vmatrix} f_1(x_0) & \cdots & f_n(x_0) \\ \vdots & & \\ f_1^{(n-1)}(x_0) & \cdots & f_n^{(n-1)}(x_0) \end{vmatrix}$$

<u>RESULT 2</u>. Let $y_1, \ldots, y_n$ be n solns. of a homogeneous linear nth-order DE on I. Then <u>either</u>

$W(y_1, \ldots, y_n) \neq 0$ for <u>all</u> x in I, in which case $\{y_1, \ldots, y_n\}$ is lin. indep. on I

<u>or</u>

$W(y_1, \ldots, y_n) = 0$ for <u>all</u> x in I, in which case $\{y_1, \ldots, y_n\}$ is lin. dep. on I

What does this mean for us?

To see if $\{y_1, \ldots, y_n\}$ are lin. dep. or indep. on I, choose any x_0 in I. Then

$$W(y_1, \ldots, y_n)(x_0) \neq 0 \Longrightarrow \{y_1, \ldots, y_n\} \text{ lin. indep. on } I$$

$$W(y_1, \ldots, y_n)(x_0) = 0 \Longrightarrow \{y_1, \ldots, y_n\} \text{ lin. dep. on } I$$

<u>Defn.</u> A lin. indep. set of solns., $\{y_1, \ldots, y_n\}$, of a homogeneous linear nth-order DE on I is called a

FUNDAMENTAL SET OF SOLUTIONS ON I

<u>RESULT 3</u>. Let $\{y_1, \ldots, y_n\}$ be a fund. set of solns. of a homog. lin. nth-order DE on I. Then lin. comb. of $y_1, \ldots, y_n$,

$$y = c_1 y_1 + \cdots + c_n y_n$$

is the GENERAL SOLN. of the DE.

<u>Example</u>. (HW Exercise 29, p. 108.)

Verify x, x^{-2}, $x^{-2}\ln x$ forms a fundamental set of solutions of the DE

$$x^3y''' + 6x^2y'' + 4xy' - 4y = 0 \text{ on } I = (0, \infty).$$

Form the general solution.

<u>STEP 1</u>. Verify that x, x^{-2}, and $x^{-2}\ln x$ are each solns. of the DE.

<u>STEP 2</u>. Note that

$$\underset{\substack{\parallel \\ 3}}{\#\text{ of solns.}} = \underset{\substack{\parallel \\ 3}}{\text{order of DE}}$$

<u>STEP 3</u>. Show $\{x, x^{-2}, x^{-2}\ln x\}$ lin. indep. using the Wronskian at <u>any</u> x_0 in I.

Let $y_1 = x$, $y_2 = x^{-2}$, $y_3 = x^{-2}\ln x$ and let $x_0 = 1$, where

(1) $x_0 = 1$ is in $I = (0, \infty)$

(2) $\ln x_0 = \ln 1 = \ln 0 \Rightarrow$ so many entries will simplify in the Wronskian at $x_0 = 1$.

$$W(x, x^{-2}, x^{-2}\ln x)(1) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}_{x=1}$$

$$= \begin{vmatrix} x & x^{-2} & x^{-2}\ln x \\ 1 & -2x^{-3} & -2x^{-3}\ln x + x^{-3} \\ 0 & 6x^{-4} & 6x^{-4}\ln x - 2x^{-4} - 3x^{-4} \end{vmatrix}_{x=1}$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 6 & -5 \end{vmatrix}$$

$$= \left|\begin{array}{ccc|cc} 1 & 1 & 0 & 1 & 1 \\ 1 & -2 & 1 & 1 & -2 \\ 0 & 6 & -5 & 0 & 6 \end{array}\right|$$

(−) (−) (−)

(+) (+) (+)

$$= (1)(-2)(-5) + (1)(1)(0) + (0)(1)(6)$$
$$- (0)(-2)(0) - (6)(1)(1) - (-5)(1)(1)$$

$$= 10 - 6 + 5$$

$$= 9 \neq 0 \ !$$

$\therefore$ $\{x, x^{-2}, x^{-2}\ln x\}$ lin. indep. on $I = (0, \infty) \Longrightarrow \{x, x^{-2}, x^{-2}\ln x\}$ is a FUND. SET OF SOLNS. ON I. (3 solns. + lin. indep.).

$\therefore$ GEN. SOLN. = lin. comb. of x, x^{-2}, and $x^{-2}\ln x$

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$$\boxed{y = c_1 x + c_2 x^{-2} + c_3 x^{-2} \ln x}$$

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Subsection 4.1.3. Nonhomogeneous Equations.

A nonhomogeneous linear nth-order DE is of the form

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = g(x) \text{ on } I.$$

$g(x) \not\equiv 0$ on I (i.e., $g(x)$ is not identically equal to zero on I) $\Rightarrow$ $g(x)$ is not equal to zero for all x in I $\Rightarrow$ there is at least one x in I such that $g(x) \neq 0$.

Its general solution (i.e., the solution giving all possible solutions of the DE) is of the form

$$y(x) = y_c(x) + y_p(x)$$

complementary soln.
= general soln. of the corresponding homogeneous eq.
$a_n(x)y^{(n)} + \cdots + a_0(x)y = 0$
$= c_1 y_1(x) + \cdots + c_n y_n(x)$

particular soln.
– any particular soln. (i.e., without an arbitrary const. multiplying it) of the nonhomogeneous eq.

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[PROOF omitted - SEE text for proof using "differential operators."]

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Examples.

1. (Exercise 38(a), p. 109.)

By inspection [i.e., by making an educated guess], determine a particular solution of

$$y'' + 2y = 10$$

SUGGESTION: y_p will "look like" the function $g(x) = 10$. In other words, y_p will be some constant function. So, let $y_p(x) = A$.

Let $y_p(x) = A$. Substitute y_p and its derivatives into the DE $y'' + 2y = 10$ to find A:

$$\left.\begin{aligned} y_p(x) &= A \\ y_p'(x) &= 0 \\ y_p''(x) &= 0 \end{aligned}\right\} \Longrightarrow$$

$$y_p'' + 2y_p = 10 \Rightarrow$$

$$0 + 2A = 10 \Rightarrow$$

$$A = 5 \Rightarrow \boxed{y_p(x) = 5}$$

2. (Exercise 38(d), p.109.)

By inspection, determine a particular solution of

$$y'' + 2y = 8x + 5.$$

SUGGESTION: y_p will "look like" the function $g(x) = 8x + 5$. In other words, y_p will be some linear function. So, let $y_p(x) = Ax + B$.

Let $y_p(x) = Ax + B$. Substitute y_p and its derivatives into the DE $y'' + 2y = 8x + 5$ to find A and B:

$$\left.\begin{aligned} y_p(x) &= Ax + B \\ y_p'(x) &= A \\ y_p''(x) &= 0 \end{aligned}\right\} \Longrightarrow$$

$$y_p'' + 2y_p = 8x + 5 \Longrightarrow$$

$$0 + 2(Ax + B) = 8x + 5 \Longrightarrow$$

$$2A + 2B = 8x + 5 \Longrightarrow$$

$$\left.\begin{array}{l} 2A = 8 \\ 2B = 5 \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} A = 4 \\ B = \frac{5}{2} \end{array}\right\} \Rightarrow$$

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$$\boxed{y_p(x) = 4x + \frac{5}{2}}$$

3. (HW Exercise 33, p. 109.)

Verify that

$$y = c_1 e^{2x} + c_2 e^{5x} + 6e^{x}$$

is the general solution of

$$y'' - 7y' + 10y = 24e^{x} \quad \text{on } (-\infty, \infty)$$

$$y = \underbrace{c_1 e^{2x} + c_2 e^{5x}}_{= y_c} + \underbrace{6e^{x}}_{= y_p}$$

① Show y_c is a soln. of $y'' - 7y' + 10y = 0$:

$$\left.\begin{aligned} y_c &= c_1 e^{2x} + c_2 e^{5x} \\ y_c' &= 2c_1 e^{2x} + 5c_2 e^{5x} \\ y_c'' &= 4c_1 e^{2x} + 25c_2 e^{5x} \end{aligned}\right\} \quad \frac{d}{dx} e^{ax} = a e^{ax}$$

$$y_c'' - 7y_c' + 10y_c \stackrel{?}{=} 0$$

$$(4c_1 e^{2x} + 25c_2 e^{5x}) - 7(2c_1 e^{2x} + 5c_2 e^{5x}) + 10(c_1 e^{2x} + c_2 e^{5x}) \stackrel{?}{=} 0$$

$$4c_1 e^{2x} + 25c_2 e^{5x} - 14c_1 e^{2x} - 35c_2 e^{5x} + 10c_1 e^{2x} + 10c_2 e^{5x} \stackrel{?}{=} 0$$

$$-10c_1e^{2x} - 10c_2e^{5x} + 10c_1e^{2x} + 10c_2e^{5x} \stackrel{?}{=} 0$$

$$0 \stackrel{\checkmark}{=} 0$$

② Show y_p is a soln. of $y'' - 7y' + 10y = 24e^x$:

$y_p = 6e^x$

$y_p' = 6e^x$

$y_p'' = 6e^x$

$$y_p'' - 7y_p' + 10y_p \stackrel{?}{=} 24e^x$$

$$6e^x - 7(6e^x) + 10(6e^x) \stackrel{?}{=} 24e^x$$

$$6e^x - 42e^x + 60e^x \stackrel{?}{=} 24e^x$$

$$24e^x \stackrel{\checkmark}{=} 24e^x$$

③ Show $y_c + y_p$ is a soln. of $y'' - 7y' + 10y = 24e^x$:

$y_c + y_p$

$(y_c + y_p)'$

$(y_c + y_p)''$

$$(y_c+y_p)'' - 7(y_c+y_p)' + 10\,(y_c+y_p) \stackrel{?}{=} 24e^x$$

$$y_c'' + y_p'' - 7y_c' - 7y_p' + 10\,y_c + 10y_p \stackrel{?}{=} 24e^x$$

$$\underbrace{(y_c'' - 7y_c'' + 10\,y_c)}_{\substack{\text{showed } = 0 \\ \text{in } \#1}} + \underbrace{(y_p'' - 7y_p' + 10\,y_p)}_{\substack{\text{showed } = 24e^x \\ \text{in } \#2}} \stackrel{?}{=} 24e^x$$

$$0 + 24e^x \stackrel{\checkmark}{=} 24e^x$$

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Section 4.2. Reduction of Order.

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We need to mention the following before we discuss this technique.

1) Defn. A soln. $y(x)$ of a DE on I is called a trivial soln. if $y(x) \equiv 0$ on I (i.e., $y(x) = 0$ for all x in I).
A soln. which is not a trivial soln. is called a nontrivial soln.

E.g., $y'' + y' + y = 0$

One soln. of this DE is the trivial soln.:

$$(0)'' + (0)' + 0 \stackrel{?}{=} 0$$
$$0 + 0 + 0 \stackrel{\checkmark}{=} 0$$

E.g., $y'' + y' + y = 1$

This DE has only nontrivial solns.

<u>Remark</u>. Usually one is interested in finding <u>nontrivial</u> solns. when solving DEs!

2) <u>Recall</u>: Given $\{f_1, f_2\}$ on I, and we will assume now that $f_1(x) \neq 0$ for all x in I, we have that

$$\frac{f_2(x)}{f_1(x)} = \begin{cases} \text{constant} = c \text{ on } I, & \text{if } \{f_1, f_2\} \text{ lin. dep.,} \\ \text{nonconstant} = u(x) \text{ on } I, & \text{if } \{f_1, f_2\} \text{ lin. indep.} \end{cases}$$

E.g., $\{x, 5x\}$ on $(0, \infty)$

$$\frac{5x}{x} = 5 \Rightarrow \{x, 5x\} \text{ lin. dep on } (0, \infty)$$

E.g., $\{x, x^2\}$ on $(0, \infty)$

$$\frac{x^2}{x} = x \Rightarrow \{x, x^2\} \text{ lin. indep. on } (0, \infty)$$

Method of Reduction of Order

Consider the following homogeneous linear second-order DE:

(*) $$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \quad \text{on } I.$$

Suppose we already <u>know a nontrivial solution</u>,

$$y_1(x),$$

Actually, we need to assume

$$y_1(x) \neq 0 \quad \text{FOR ALL } x \text{ IN } I.$$

We want to determine the general solution of (*). In order to do that, we need to <u>find a second solution</u>,

$$y_2(x),$$

where

$$\{y_1, y_2\} \text{ IS LINEARLY INDEPENDENT.}$$

BUT if $\{y_1, y_2\}$ is linearly independent, then we must have

$$\frac{y_1(x)}{y_2(x)} = u(x) = \text{nonconstant function on } I \Rightarrow$$

$$\boxed{y_2(x) = u(x)\,y_1(x)}.$$

So, to find $y_2(x)$, we need to substitute

$$u(x)\,y_1(x) \quad \text{into} \quad (*)$$

and find

$$u(x).$$

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In doing so, we <u>reduce</u> $(*)$ to a linear DE of order 1 for u'.

<u>REMARKS</u>: 1) Suppose $y_1(x) = 0$ for some x_0 in I. Then divide I up into 2 subintervals, not including x_0, find $y_2(x)$ on each subinterval, and it will usually turn out that $y_2(x)$ is defined and satifies the DE at $x = x_0$.

2) To find $y_1(x)$, we need other methods of solution of DEs like power series solutions of linear DEs (SEE Chapter 6 in the text).

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<u>Example</u>. (HW Exercise 21, p. 112.)

$$x^2y'' - xy' + y = 0 \; ; \; y_1 = x$$

Find y_2. Assume appropriate interval of validity.

<u>Assume $I = (0, \infty)$</u>, since will be dividing by x^2 (so do not want $x = 0$) and will be integrating $1/x$ to give $\ln|x|$ (so want $x > 0$).

Write DE in <u>standard form</u>:

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0 \quad \text{on } I = (0, \infty).$$

Let

$$y_2 = uy_1 \Rightarrow y_2 = ux.$$

Then, by the product rule,

$$\begin{aligned} y_2 &= ux \\ y_2' &= u'x + u \cdot 1 = u'x + u \\ y_2'' &= (u''x + u' \cdot 1) + u' = u''x + 2u' \end{aligned}$$

Substitute y_2' and y_2'' into DE:

$$y_2'' - \frac{1}{x} y_2' + \frac{1}{x^2} y_2 = 0 \quad \text{on } I = (0, \infty) \Rightarrow$$

$$(u''x + 2u') - \frac{1}{x}(u'x + u) + \frac{1}{x^2}(ux) = 0 \Rightarrow$$

$$u''x + 2u' - u' - \cancel{\frac{1}{x}u} + \cancel{\frac{1}{x}u} = 0 \Rightarrow$$

$$u''x + u' = 0.$$

Let

$$w = u'.$$

Then we have a DE which is reduced in order from 1 to 2:

$$w'x + w = 0 \Longrightarrow$$

PLACE IN STANDARD FORM
(can divide by x since $I = (0, \infty)$)

$$w' + \frac{1}{x} w = 0$$

INTEGRATING FACTOR

$$\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln|x|} \underset{\text{SINCE } I = (0,\infty)}{=} e^{\ln x} = x \Longrightarrow$$

$$x\left(w' + \frac{1}{x}w\right) = x(0) \Rightarrow$$

$$xw' + 1 \cdot w = 0 \Rightarrow$$

$$\frac{d}{dx}(xw) = 0 \Rightarrow$$

$$\int \frac{d}{dx}(xw)\,dx = \int 0\,dx \Rightarrow$$

$$xw = C_1 \Rightarrow$$

$$w = \frac{C_1}{x}.$$

Then, since $w = u'$,

$$u' = \frac{C_1}{x} \Rightarrow$$

$$\int u'\,dx = \int \frac{C_1}{x}\,dx \Rightarrow$$

$$u = C_1 \int \frac{1}{x}\,dx \Rightarrow$$

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$$u = C_1(\ln|x| + C_2) \underset{\text{SINCE } I = (0,\infty)}{\Rightarrow}$$

$$u = C_1 \ln x + C_2. \Rightarrow$$

At this point, you can choose C_1 and C_2 to be specific values since

(1) $y_2 = u y_1$ is to be a specific solution

(2) the general solution $y = c_1 y_1 + c_2 y_2$ will turn out the same regardless of what C_1 and C_2 are chosen to be.

So, choose

$$C_1 = 1, \quad C_2 = 0 \implies$$

$$u = \ln x .$$

Then, since $y_2 = u y_1 = u x$,

$$\boxed{y_2 = x \ln x}$$

General solution:

$$\boxed{y = c_1 x + c_2 x \ln x}$$

———— o ————

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Alternative to the Method of Reduction of Order: Using its Formula

The text gives a formula for $y_2(x)$ on p. 111 that you can use in place of making all the above computations:

$$y'' + P(x)\,y' + Q(x)\,y = 0 \;;\; y_1(x).$$

$$\boxed{y_2(x) = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx}$$

ABEL'S FORMULA

MEMORIZE THIS IF YOU USE THIS IN PLACE OF THE ABOVE COMPUTATIONS TO FIND $y_2(x)$.

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Section 4.3. Homogeneous Linear Equations with Constant Coefficients.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

CONSTANT SOLUTIONS

E.g., $7y'' + 2y' - 5y = 0$ ✓

$x^2 y'' + \frac{1}{x} y' + y = \cos x$ X

Example. (HW Exercise 17, p. 119.)

Find the general soln. of

$$3y'' + 2y' + y = 0.$$

Assume: The DE has a soln. of the form

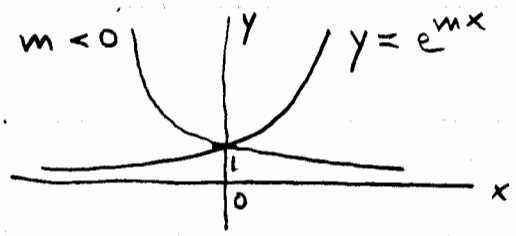
$$y = e^{mx}.$$

Substitute $y = e^{mx}$ into the DE and see what happens:

$$3m^2\cancel{e^{mx}} + 2me^{\cancel{mx}} + \cancel{e^{mx}} = 0 \Longrightarrow$$

Can divide both sides of this equation by e^{mx} since $e^{mx} \neq 0$ for all x and any m

$y = e^{mx}$, $m < 0$



$y = e^{mx}$, $m > 0$

$$\boxed{3m^2 + 2m + 1 = 0}$$ CHARACTERISTIC EQUATION

We just converted a DE problem into an ALGEBRA problem. We next solve for m.

DIGRESSION:

Recall: • Solns. or roots of the quadratic eq.

$$Ax^2 + Bx + C = 0$$

are always given by the quadratic formula

$$x_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Have one of the following:

(1) 2 distinct distinct real roots
$x_1 \neq x_2$ $(B^2 - 4AC > 0)$

(2) 1 repeated real root
$x_1 = x_2$ $(B^2 - 4AC = 0)$

(3) Complex conjugate pair of roots,
$x_1 = \alpha + i\beta$, $x_2 = \alpha - i\beta$
$(B^2 - 4AC < 0)$

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• $i \stackrel{\text{def}}{=} \sqrt{-1} \Rightarrow i^2 = -1$

• $z = a + ib$, a, b real
a = real part of z
b = imaginary part of z

Solving for m cont'd:

$$3m^2 + 2m + 1 = 0$$

$$m_{1,2} = \frac{-2 \pm \sqrt{4-12}}{6} = \frac{-2 \pm \sqrt{-8}}{6} = \frac{-2 \pm 2\sqrt{2}\,i}{6}$$

$$= -\frac{1}{3} \pm \frac{\sqrt{2}}{3} i$$

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$\therefore$ We get 2 (distinct) solns.:

$$y_1(x) = e^{m_1 x} = e^{\left(-\frac{1}{3} - \frac{\sqrt{2}}{3} i\right) x}$$

$$y_2(x) = e^{m_2 x} = e^{\left(-\frac{1}{3} + \frac{\sqrt{2}}{3} i\right) x}$$

($\{y_1, y_2\}$ is a lin. indep. set since

$$\frac{y_1}{y_2} = \frac{e^{\left(-\frac{1}{3} - \frac{\sqrt{2}}{3} i\right) x}}{e^{\left(-\frac{1}{3} + \frac{\sqrt{2}}{3} i\right) x}} = \frac{e^{-x/3}\, e^{i\sqrt{2}\,x/3}}{e^{-x/3}\, e^{i\sqrt{2}\,x/3}} = e^{-i\frac{\sqrt{2}}{3}x - i\frac{\sqrt{2}}{3}x}$$

$$= e^{-i\frac{2\sqrt{2}}{3}x} \neq \text{constant.} \;)$$

General Soln.:

$$y = c_1 e^{\left(-\frac{1}{3} - \frac{\sqrt{2}}{3} i\right) x} + c_2 e^{\left(-\frac{1}{3} + \frac{\sqrt{2}}{3} i\right) x}$$

However, BY CONVENTION, we want $y(x)$ to <u>appear</u> as the linear combination of of REAL, NOT UNREAL, FUNCTIONS (with the arbitrary constants now being real or complex).

To do this we consider

EULER'S FORMULA
(in complex analysis - accept on faith)

$$e^{i\theta} \stackrel{\text{def}}{=} \cos\theta + i\sin\theta \,, \quad \theta \text{ a real number}$$

$$\left(e^{-i\theta} = e^{i(-\theta)} \begin{aligned} &= \cos(-\theta) + i\sin(-\theta) \\ &= \cos\theta + i(-\sin\theta) \\ &= \cos\theta - i\sin\theta \end{aligned} \right)$$

$$\therefore \quad e^{\left(-\frac{1}{3} - \frac{\sqrt{2}}{3}i\right)x} = e^{-\frac{1}{3}x} \cdot e^{-\frac{\sqrt{2}}{3}ix}$$

$$= e^{-\frac{1}{3}x} e^{-i\frac{\sqrt{2}}{3}x}$$

$$= e^{-\frac{1}{3}x}\left(\cos\frac{\sqrt{2}}{3}x - i\sin\frac{\sqrt{2}}{3}x\right)$$

$$e^{\left(-\frac{1}{3} + \frac{\sqrt{2}}{3}i\right)x} = e^{-\frac{1}{3}x}\left(\cos\frac{\sqrt{2}}{3}x + i\sin\frac{\sqrt{2}}{3}x\right)$$

$$\therefore \quad y(x) = c_1 e^{-\frac{1}{3}x}\left(\cos\frac{\sqrt{2}}{3}x - i\sin\frac{\sqrt{2}}{3}x\right)$$
$$+ c_2 e^{-\frac{1}{3}x}\left(\cos\frac{\sqrt{2}}{3}x + i\sin\frac{\sqrt{2}}{3}x\right)$$

$$= \underbrace{(c_1 + c_2)}_{\text{REPLACE BY NEW } c_1} e^{-\frac{1}{3}x}\cos\frac{\sqrt{2}}{3}x$$
$$+ \underbrace{i(c_2 - c_1)}_{\text{REPLACE BY NEW } c_2} e^{-\frac{1}{3}x}\sin\frac{\sqrt{2}}{3}x \quad \Rightarrow$$

$$\boxed{y = c_1 e^{-\frac{1}{3}x}\cos\frac{\sqrt{2}}{3}x + c_2 e^{-\frac{1}{3}x}\sin\frac{\sqrt{2}}{3}x}$$

In general:

If $y_1 = e^{(\alpha - i\beta)x}$ and $y_2 = e^{(\alpha + i\beta)x}$

then the gen. soln. is written as

$$y = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x$$

Two Distinct Real Roots

Example. (HW Exercise 8, p. 119.)

Find the general solution of

$$y'' - 3y' + 2y = 0$$

$y = e^{mx}$: $y'' - 3y' + 2y = 0 \Rightarrow$

$$m^2 \cancel{e^{mx}} - 3m\cancel{e^{mx}} + 2\cancel{e^{mx}} = 0 \Rightarrow$$

$$m^2 - 3m + 2 = 0 \Rightarrow$$

$$(m-1)(m-2) = 0 \Rightarrow$$

$$m_1 = 1, \quad m_2 = 2 \Rightarrow$$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \Rightarrow$$

$$\boxed{y = c_1 e^{x} + c_2 e^{2x}}$$ DONE!

Repeated Roots

Example. (HW Exercise 9, p. 119.)

Find the general solution of

$$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0.$$

$y = e^{mx}$: $y'' + 8y' + 16y = 0 \Rightarrow$

$$m^2\cancel{e^{mx}} + 8m\cancel{e^{mx}} + 16\cancel{e^{mx}} = 0 \Rightarrow$$

$$m^2 + 8m + 16 = 0 \Rightarrow$$

$$(m+4)(m+4) = 0 \Rightarrow$$

$$m_1 = -4, \quad m_2 = -4$$

-4 IS A REPEATED ROOT

$\therefore$ $y_1 = e^{-4x}$ and $y_2 = e^{-4x}$ so that the general solution is

$$y = c_1 e^{-4x} + c_2 e^{-4x} \ ?$$

NO!

With $y_1 = e^{-4x}$ and $y_2 = e^{-4x}$, $\{y_1, y_2\}$ is linearly dependent, so no general solution can be formed from y_1 and y_2.

We have one solution,

$$y_1 = e^{-4x}.$$

We need a second solution, y_2.

It turns out that a second solution can still be found from $y_1 = e^{-4x}$ using the

REDUCTION OF ORDER FORMULA (4.2)

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

$$(y'' + P(x)y' + Q(x)y = 0)$$

So, we have

$$y_2 = e^{-4x} \int \frac{e^{-\int 8\,dx}}{\underbrace{(e^{-4x})^2}_{= e^{-4x\cdot 2} = e^{-8x}}}\,dx$$

$$= e^{-4x} \int \frac{e^{-8x}}{e^{-8x}}\, dx$$

$$= e^{-4x} \int 1\, dx$$

$$= e^{-4x} \left(x \cancel{4}/\cancel{4} \right) \Rightarrow$$

$$y_2 = xe^{-4x}$$

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$\therefore$ General soln.:

$$\boxed{y = c_1 e^{-4x} + c_2 x e^{-4x}}$$

<u>NOTE</u>: For second-order DEs, always "tag on" an x to make the second soln. distinct.

If you have something like

$$y^{(4)} + 16y''' + 96y'' + 256y' + 256y = 0 \Rightarrow$$

$$m^4 + 16m^3 + 96m^2 + 256m + 256 = 0 \Rightarrow$$

$$(m+4)^4 = 0 \Rightarrow$$

$$m_1 = m_2 = m_3 = m_4 = -4,$$

then "tag on" increasing powers of x:

$$y_1 = e^{-4x}, \quad y_2 = xe^{-4x}, \quad y_3 = x^2e^{-4x},$$

$$y_4 = x^3e^{-4x}.$$

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Section 4.4. Undetermined Coefficients – Superposition Approach.

Recall from Subsection 4.1.3:

If you have, say, a second-order nonhomogeneous linear DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \text{ on } I$$

then the following is true:

(1) $g(x) \neq 0$ (for at least one x in I)

(2) Its general soln. is of the form

$$y = \underbrace{y_c}_{\text{COMPLEMENTARY SOLN.}} + \underbrace{y_p}_{\text{PARICULAR SOLN.}}$$

SOLUTION TO CORRESPONDING HOMOG. DE

$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$

AND OF THE FORM

$y_c = c_1y_1 + c_2y_2$

ANY SPECIFIC SOLN. TO NONHOMOG. DE

(3) If, say,

$$g(x) = g_1(x) + g_2(x),$$

where $\{g_1, g_2\}$ lin. indep., and if

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g_1(x) \longrightarrow y_{p_1} \text{ is a part. soln.}$$

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g_2(x) \longrightarrow y_{p_2} \text{ is a part. soln.}$$

then

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) = g_1(x) + g_2(x)$$

$$\longrightarrow y_{p_1} + y_{p_2} \text{ is a part. soln.}$$

by the

SUPERPOSITION PRINCIPLE.

So, to find the general solution

$$y = y_c + y_p = c_1 y_1 + c_2 y_2 + y_{p_1} + y_{p_2}$$

of

$$a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = g(x) = g_1(x) + g_2(x) \quad \text{on } I,$$

we will do the following:

(1) y_c will be found using the "CHARACTERISTIC EQ. METHOD," where $a_2(x)$, $a_1(x)$, and $a_0(x)$ will all be constant coefficients.

(2) y_{p_1} and y_{p_2} will be found using the METHOD OF UNDETERMINED COEFFICIENTS, to be discussed.

– – – – – – o – – – – – –

First, some things you need to know before learning about the METHOD OF UNDETERMINED COEFFICIENTS.

NONHOMOGENEOUS LINEAR DE

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x)$$

must satisfy the following two conditions before the METHOD OF UNDETERMINED COEFFICIENTS is used:

1. $a_0(x), a_1(x), \ldots, a_{n-1}(x), a_n(x)$ are all CONSTANT COEFFICIENTS

2. Repeated differentiation of each $g_i(x)$ function produces a FINITE set of distinct (or linearly independent) functions

✓ E.g., $e^{-x} \xrightarrow[\text{REPEATEDLY DIFFERENTIATE}]{} \{\overset{f(x)}{e^{-x}}, \overset{f'(x)}{\cancel{-}e^{-x}}, \overset{f''(x)}{e^{-x}}, \overset{f'''(x)}{\cancel{-}e^{-x}}, \ldots\}$

$\longrightarrow \{e^{-x}\}$

UP TO A CONSTANT MULTIPLE, THE DISTINCT FUNCTIONS ARE

✓ E.g., $xe^{x} \xrightarrow[\text{REPEATEDLY DIFFERENTIATE}]{} \{xe^{x}, e^{x}+xe^{x}, 2e^{x}+xe^{x}, 3e^{x}+xe^{x}, \ldots\}$

$\longrightarrow \{e^{x}, xe^{x}\}$

BREAKING APART SUMS AND ELIMINATING CONSTANT MULTIPLES, THE DISTINCT FUNCTIONS ARE

✓ E.g., $x^2 \xrightarrow[\text{REPEATEDLY DIFFERENTIATE}]{} \{x^2, \cancel{2}x, \cancel{2}\cdot 1, \cancel{6}, \cancel{6}, \ldots\}$

$\xrightarrow[\text{UP TO A CONSTANT MULTIPLE, THE DISTINCT FUNCTIONS ARE}]{} \{x^2, x, 1\}$

✓ E.g., $\sin x \xrightarrow[\text{REPEATEDLY DIFFERENTIATE}]{} \{\sin x, \cos x, \cancel{4}\sin x, \cancel{4}\cos x, \sin x, \cos x, \ldots\}$

$\xrightarrow[\text{UP TO A CONSTANT MULTIPLE, THE DISTINCT FUNCTIONS ARE}]{} \{\sin x, \cos x\}$

✗ E.g., $\frac{1}{x} = x^{-1} \xrightarrow[\text{REPEATEDLY DIFFERENTIATE}]{} \{x^{-1}, -x^{-2}, 2x^{-3}, -6x^{-4}, \ldots\}$

$\xrightarrow[\text{UP TO A CONSTANT MULTIPLE, THE DISTINCT FUNCTIONS ARE}]{} \{x^{-1}, x^{-2}, x^{-3}, x^{-4}, \ldots\}$

<u>SUMMARY</u>: To use METHOD OF UNDETERMINED COEFFICIENTS, you need

(1) a DE with constant coefficients

(2) $g_i(x) = e^{ax}$ or
$a_n x^n + \cdots + a_1 x + a_0$ or
$\sin ax$ or
$\cos bx$ or
any combination of the above like

$x^2 e^x$ or
$\sin 3x\, e^{2x}$ or
$(x^3 + x^2 + 1)\cos x$

(3) $g_i \longrightarrow \underbrace{\{g_i, g_i', g_i'', g_i''', \ldots\}}_{\text{Called the "FAMILY GENERATED BY } g_i\text{"}}$

METHOD OF UNDETERMINED COEFFICIENTS

We demonstrate the method through typical examples.

Examples.

1. (HW Exercise 10, p. 130.)

Solve $y'' + 2y' = 2x + 5 - e^{-2x}$.

$$1 \cdot y'' + 2y' = \underbrace{(2x+5)}_{g_1(x)} + \underbrace{(-e^{-2x})}_{g_2(x)}$$

constant coefficients ✓

$g_1(x)$ polynomial ✓ $\quad$ $g_2(x)$ exponential ✓

We will find the general solution

$$y = y_c + y_p .$$

STEP 1. Find y_c (complementary soln.).

$y'' + 2y = 0$ Homog. ✓ Linear ✓ Const. Coeffs. ✓

$\Rightarrow$ use "CHAR. EQ. METHOD"

$y = e^{mx}$: $m^2 \cancel{e^{mx}} + 2m \cancel{e^{mx}} = 0 \Rightarrow$

$m^2 + 2m = 0 \Rightarrow$

$m(m+2) = 0 \Rightarrow$

$m_1 = 0, \; m_2 = -2 \Rightarrow$

$y_1 = e^{m_1 x} = e^{0 \cdot x} = 1, \; y_2 = e^{m_2 x} = e^{-2x} \Rightarrow$

$y_c = c_1 y_1 + c_2 y_2 = c_1 \cdot 1 + c_2 e^{-2x} = c_1 + c_2 e^{-2x}$.

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So,

$$y_1 = 1, \; y_2 = e^{-2x}, \quad y = c_1 + c_2 e^{-2x}$$

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STEP 2. Find y_p (a particular soln.)

- Again note that

$$g(x) = \underbrace{(2x+5)}_{g_1(x)} + \underbrace{(-e^{-2x})}_{g_2(x)}$$

- Determine the FAMILIES GENERATED BY $g_1(x) = 2x+5$ and $g_2(x) = -e^{-2x}$ (i.e., repeatedly differentiate $g_1(x)$ and $g_2(x)$):

$$g_1(x) = 2x+5 \longrightarrow \{2(x)+5\cdot(1),\ 2\cdot(1),\ 0,\ 0,\ \ldots\}$$

PICK OUT THE DISTINCT ELEMENTARY FUNCTIONS INVOLVED

$$\longrightarrow \{x,\ 1\}$$

$$g_2(x) = -e^{-2x} \longrightarrow \{-(e^{-2x}),\ 2(e^{-2x}),\ -4(e^{-2x}),\ \ldots\}$$

$$\longrightarrow \{e^{-2x}\}$$

- Split original nonhomog. DE into 2 nonhomog. DEs:

(1) $y'' + 2y' = 2x+5$

Let y_{p_1} denote a part. soln. of DE.

Set

$$y_{p_1} = A\underline{\underline{x}} + B\cdot\underline{\underline{1}} = Ax + B$$

From $\{x, 1\}$ for $g_1(x) = 2x+5$

(2) $y'' + 2y' = -e^{-2x}$.
Let y_{p_2} denote a part. soln. of DE.
Set

$$y_{p_2} = C\underline{\underline{e^{-2x}}}$$

From $\{e^{-2x}\}$ for $g_2(x) = -e^{-2x}$

- Separately compare y_{p_1}, y_{p_2} to y_c:

(1) $y_{p_1} = Ax + B\cdot 1$ <u>vs.</u> $y_c = c_1 \cdot 1 + c_2 e^{-2x}$

1 shows up in both y_{p_1} and y_c.
Need to make y_{p_1} distinct from y_c.
The following works:

Keep multiplying $y_{p_1} = Ax + B\cdot 1$ by x until no term in y_{p_1} duplicates any term in y_c.

So

$$x\, y_{p_1} = x(Ax + B) = Ax^2 + Bx \Rightarrow$$

NEW $\boxed{y_{p_1} = Ax^2 + Bx}$

(2) $y_{p_2} = Ce^{-2x}$ <u>vs:</u> $y_c = c_1 \cdot 1 + c_2 e^{-2x}$

e^{-2x} shows up in both y_{p_2} and y_c.
Need to make y_{p_2} distinct from y_c.
The following works:

$$x y_{p_2} = x(Ce^{-2x}) = Cxe^{-2x} \Rightarrow$$

$$\text{NEW } y_{p_2} = Cxe^{-2x}$$

- Now must determine A, B, C, the "UNDETERMINED COEFFICIENTS":

(1)
$$\underbrace{\begin{aligned} y_{p_1} &= Ax^2 + Bx \\ y_{p_1}' &= 2Ax + B \\ y_{p_1}'' &= 2A \end{aligned}}$$

← DO <u>NOT</u> HAVE A ABSORB THE 2

$$\Downarrow$$

$$y_{p_1}'' + 2y_{p_1}' = 2x + 5 \Rightarrow$$

$$2A + 2(2Ax + B) = 2x + 5 \Rightarrow$$

$$4Ax + (2A + 2B) = 2x + 5 \Rightarrow$$

$$\left.\begin{aligned} 4A &= 2 \\ 2A + 2B &= 5 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} A &= \tfrac{1}{2} \\ 2A + 2B &= 5 \end{aligned}\right\} \Rightarrow$$

$$\boxed{\begin{aligned} A &= \tfrac{1}{2} \\ B &= 2 \end{aligned}} \Rightarrow$$

$$y_{p_1} = \frac{1}{2}x^2 + 2x$$

KEEP THESE FIXED CONSTANTS
(y_{p_1} is being thrown into y_p, not y_c with arbitrary constants c_1 and c_2)

(2)
$$\underbrace{\begin{aligned} y_{p_2} &= Cxe^{-2x} \\ y_{p_2}' &= Ce^{-2x} - 2Cxe^{-2x} \\ y_{p_2}'' &= -2Ce^{-2x} - 2Ce^{-2x} + 4Cxe^{-2x} \\ &= -4Ce^{-2x} + 4Cxe^{-2x} \end{aligned}}$$

DO NOT HAVE C ABORB ANY CONSTANTS

$$y_{p_2}'' + 2y_{p_2}' = -e^{-2x} \implies$$

$$(-4Ce^{-2x} + 4C\cancel{xe^{-2x}}) + 2(Ce^{-2x} - 2C\cancel{xe^{-2x}}) = -e^{-2x} \implies$$

$$-2Ce^{-2x} = -e^{-2x} \implies$$

$$-2C = -1 \implies$$

$$\boxed{C = \frac{1}{2}} \implies$$

$$y_{p_2} = \frac{1}{2}xe^{-2x}$$

KEEP THIS FIXED CONSTANT

$$\therefore \; y_p = y_{p_1} + y_{p_2} = \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}$$

STEP 3. Write the general solution.

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$$y = y_c + y_p \Longrightarrow$$

$$\boxed{y = c_1 + c_2 e^{-2x} + \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}}$$

2. (HW Exercise 19, p. 130.)

Solve $y'' + 2y' + y = \sin x + 3\cos 2x$.

<u>OUTLINE</u>:

$$y'' + 2y' + y = \underbrace{\sin x}_{= g_1(x)} + \underbrace{3\cos 2x}_{= g_2(x)}$$

$g_1(x) = \sin x \longrightarrow \{\sin x, \cos x\}$

$g_2(x) = \cos 2x \longrightarrow \{\cos 2x, \sin 2x\}$

$\frac{d}{dx}(\cos 2x) = -2\sin 2x$

$y'' + 2y' + y = 0 \Longrightarrow$
$m^2 + 2m + 1 = 0 \Longrightarrow$
$(m+1)(m+1) = 0 \Longrightarrow$
$m_1 = -1,\ m_2 = -1 \Longrightarrow$

$y_1 = e^{-x},\ y_2 = xe^{-x} \Longrightarrow$

$$y_c = c_1 e^{-x} + c_2 x e^{-x}$$

$$y'' + 2y' + y = \sin x \Longrightarrow$$

$$y_{p_1} = A\sin x + B\cos x \Longrightarrow$$

DOES NOT DUPLICATE TERMS IN y_c

$$\left.\begin{array}{l} A = 0 \\ B = -\frac{1}{2} \end{array}\right\} \Longrightarrow$$

$$y_{p_1} = -\frac{1}{2}\cos x$$

$$y'' + 2y' + y = 3\cos 2x$$

$$y_{p_2} = C\sin 2x + D\cos 2x \Longrightarrow$$

DOES NOT DUPLICATE TERMS IN y_c

$$\left.\begin{array}{l} C = \frac{12}{25} \\ D = -\frac{9}{25} \end{array}\right\} \Longrightarrow$$

$$y_{p_2} = \frac{12}{25}\sin 2x - \frac{9}{25}\cos 2x$$

$\therefore$ Gen. soln. is

$$\boxed{y = c_1 e^{-x} + c_2 x e^{-x} - \frac{1}{2}\cos x + \frac{12}{25}\sin 2x - \frac{9}{25}\cos 2x}$$

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Section 4.6. Variation of Parameters.

Suppose you have a

NONHOMOGENEOUS LINEAR DE

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1 y' + a_0 y = \underbrace{g(x)}$$

$$= g_1(x) + \cdots + g_k(x)$$

BUT either

(1) $a_0(x), a_1(x), \ldots, a_n(x)$ are NOT ALL CONSTANT COEFFICIENTS

or

(2) $g_1(x), g_2(x), \ldots, g_k(x)$ DO NOT ALL GENERATE A FINITE FAMILY OF DERIVATIVES, i.e., they are not e^{bx}, $b_n x^n + \cdots + b_1 x + b_0$, $\sin bx$, $\cos bx$, or products of these

E.g., $\tan x \longrightarrow \underbrace{\{\overset{f(x)}{\sec^2 x}, \overset{f'(x)}{2\sec^2 x \tan x}, \overset{f''(x)}{4\sec^2 x \tan x + 2\sec^4 x}, \ldots\}}_{\text{all distinct (linearly independent)}}$

Then you CANNOT use the Method of Undetermined Coefficients to find y_p in $y = y_c + y_p$.

Instead you should use the

METHOD OF VARIATION OF PARAMETERS.

This method may remind you of the Method of Reduction of Order, but it is based on different ideas.

Motivation for the Method of Variation of Parameters for a Second-Order DE

[SEE 4.6. The text actually uses this motivation to compute y_p. I will use a formula that summarizes everything up.]

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \text{ on } I$$

Place in STANDARD FORM:

(*) $$y'' + P(x)y' + Q(x)y = f(x)$$ (defined on an I where $a_2(x)$ is never 0)

For simplicity, suppose we know y_c:

$$y'' + P(x)y' + Q(x)y = 0$$

General Soln.: $y_c = c_1 y_1 + c_2 y_2$

Generalize these ARBITRARY CONSTANTS

$$c_1, c_2$$

to ARBITRARY FUNCTIONS

$$u_1(x), u_2(x)$$

and write

$$y_p = u_1(x) y_1 + u_2(x) y_2$$

[Analogous to the motivation for the Method of Solving First-Order Linear Equations in 2.3 — We skipped this.]

<u>Question</u>: Will we be able to find $u_1(x)$, $u_2(x)$ such that y_p is a particular soln. of (*)?

<u>Answer</u>: The answer turns out to be YES AS LONG AS WE ASSUME

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0, \\ y_1' u_1' + y_2' u_2' = f(x). \end{cases}$$

(SEE p. 142 of the text for why we need to assume this.)

Cramer's rule then says we can solve this "system of 2 algebraic equations in 2 unknowns" u_1' and u_2' where

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$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\underbrace{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}_{\substack{\text{Wronskian of } y_1 \text{ and } y_2 \\ = W(y_1, y_2) \\ \neq 0 \text{ for all } x \text{ in } I}}} = \frac{-f(x)\, y_2}{y_1 y_2' - y_1' y_2},$$

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$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\underbrace{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}_{\substack{\text{Wronskian of } y_1 \text{ and } y_2 \\ = W(y_1, y_2) \\ \neq 0 \text{ for all } x \text{ in } I}}} = \frac{y_1\, f(x)}{y_1 y_2' - y_1' y_2}.$$

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To find u_1 and u_2, all we need to do is integrate u_1' and u_2' with respect to x:

$$u_1 = \int u_1' \, dx = \int \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \, dx ,$$

$$u_2 = \int u_2' \, dx = \int \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \, dx .$$

(u_1' and u_2' are integrable because we will assume they are both continuous.)

SUMMARY IN A FORMULA:

$$\boxed{\begin{aligned} y_p &= y_1 u_1 + y_2 u_2 \\ &= y_1 \int \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \, dx + y_2 \int \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \, dx \\ &= y_1 \int \frac{-f(x)\, y_2}{y_1 y_2' - y_1' y_2} \, dx - y_2 \int \frac{y_1 f(x)}{y_1 y_2' - y_1' y_2} \, dx \end{aligned}}$$

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<u>Example</u>. (HW Exercise 1, p. 146.)

Solve $y'' + y = \sec x$ using Variation of Parameters.
State interval over which general solution is defined.

<u>Note</u>: $\sec x = \dfrac{1}{\cos x}$

<u>General solution</u>: $y = y_c + y_p$

<u>STEP 1</u>. Find y_c using "CHARACTERISTIC EQUATION METHOD":

$$y'' + y = 0$$

$y = e^{mx}$: $y'' + y = 0 \Rightarrow$

$$m^2 \cancel{e^{mx}} + \cancel{e^{mx}} = 0 \Rightarrow$$

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$$m^2 + 1 = 0 \Rightarrow$$

$$m^2 = -1 \Rightarrow$$

$$m = \pm\sqrt{-1} = \pm i \Rightarrow$$

$$m_1 = -i \ , \ m_2 = i \Rightarrow$$

$$y_1 = e^{-ix}, \quad y_2 = e^{ix}$$

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Since

$$e^{-ix} = \boxed{\cos x} - i\boxed{\sin x}$$
$$e^{ix} = \boxed{\cos x} + i\boxed{\sin x}$$

let

$$y_1 = \cos x, \quad y_2 = \sin x$$

We absolutely must re-represent y_1 and y_2 as REAL functions, since we are going to find y_p in terms of y_1 and y_2 using REAL integration

Then

$$y_c = c_1 y_1 + c_2 y_2 \Rightarrow$$

$$y_c = c_1 \cos x + c_2 \sin x.$$

<u>STEP 2</u>. Find y_p using the METHOD OF VARIATION OF PARAMETERS:

Either go through the steps on p. 142 of the text OR use the FORMULA. We will use

the FORMULA here,

$$y_p = y_1 \int \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} dx + y_2 \int \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} dx$$

$$= \cos x \int \frac{\begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} dx + \sin x \int \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} dx$$

$$= \cos x \int \frac{-(\sec x)(\sin x)}{(\cos x)(\cos x) - (-\sin x)(\sin x)} dx$$

$$+ \sin x \int \frac{(\cos x)(\sec x)}{(\cos x)(\cos x) - (-\sin x)(\sin x)} dx$$

$$= \cos x \int \frac{-\frac{1}{\cos x} \cdot \sin x}{\cos^2 x + \sin^2 x} dx + \sin x \int \frac{\cos x \cdot \frac{1}{\cos x}}{\cos^2 x + \sin^2 x} dx$$

$$= \cos x \underbrace{\int \frac{-\tan x}{1} dx}_{} + \sin x \int \frac{1}{1} dx$$

SEE back of text, TABLE OF INTEGRALS, #12 :

$$\int \tan u \, du = -\ln|\cos u| + C$$

$$= -\cos x\left(-\ln|\cos x|\,\cancel{A}/\cancel{C_1}\right) + \sin x\left(x\,\cancel{A}/\cancel{C_2}\right)$$

$-C_1\cos x$ and $C_2\sin x$ will be "absorbed" by the terms $c_1\cos x$ and $c_2\sin x$ in the general solution

$$\begin{aligned} y &= y_c + y_p \\ &= c_1y_1 + c_2y_2 + y_p \\ &= c_1\cos x + c_2\sin x + y_p \end{aligned}$$

$$= \boxed{\cos x\,\ln|\cos x| + x\sin x}$$

STEP 3. General solution:

$$\begin{aligned} y &= y_c + y_p \\ &= c_1y_1 + c_2y_2 + y_1\int u_1'\,dx + y_2\int u_2'\,dx \Rightarrow \end{aligned}$$

$$\boxed{y = c_1\cos x + c_2\sin x + \cos x\,\ln|\cos x| + x\sin x}$$

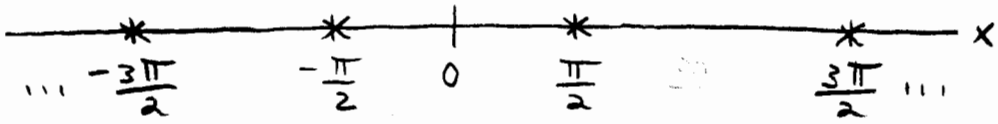
STEP 4. Interval over which general solution is defined:

The general soln,

$$y = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x$$

is defined

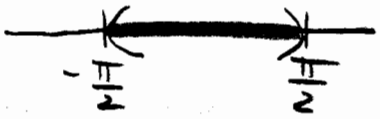
wherever $\cos x \neq 0 \Longrightarrow$
wherever $x \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots \Longrightarrow$



So, for I,

CHOOSE ANY OPEN INTERVAL BETWEEN THESE ZEROS OF $\cos x \Longrightarrow$

CHOOSE $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



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So, let

$$\boxed{I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$$

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$\boxed{\text{CHAPTER 7}}$ The Laplace Transform.

The <u>Laplace transform</u> is an improper integral used to transform difficult-to-solve problems into more readily solvable problems:

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt.$$

Lecture

Section 7.1. Definition of the Laplace Transform.

Function: An OPERATION that transforms one function into another number.

$$\text{E.g.,} \quad f(x) = x^2 \Rightarrow \underbrace{f(3) = 3^2 = 9}_{f:\ 3 \longmapsto 9}$$

Operator: An OPERATION that transforms one function into another function or into a number.

$$\text{E.g.,} \quad \underbrace{\frac{d}{dx}(x^2) = 2x}_{\frac{d}{dx}:\ x^2 \longrightarrow 2x}$$

$$\text{E.g.,} \quad \underbrace{\int x^2\,dx = \frac{x^2}{2} + C}_{\int dx:\ x^2 \longrightarrow \frac{x^2}{2} + C}$$

E.g., $\underbrace{\int_0^1 x^2\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}}_{\int_0^1 dx\,:\; x^2 \longrightarrow \frac{1}{2}}$

$\therefore$ Derivatives and integrals can be viewed as OPERATORS that transform functions into other functions or numbers.

<u>Linear Operator</u>: An operator which transforms a linear combination of functions into a linear combination of the operator acting on the functions

E.g., $\frac{d}{dx}\left[a f(x) + b\, g(x)\right] = a\left[\frac{d}{dx} f(x)\right] + b\left[\frac{d}{dx} g(x)\right]$

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E.g., $\int \left[a f(x) + b\, g(x)\right] dx = a \int f(x)\,dx + b \int g(x)\,dx$

$\therefore$ Derivatives and integrals can be viewed as LINEAR OPERATORS.

Integral Operators

One common type of INTEGRAL OPERATOR is an integral which transforms a function of one variable into a function of another variable:

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$$\underbrace{\int_{t=a}^{t=b} K(s,t)\, f(t)\, dt = F(s)}$$

$$\int_{t=a}^{t=b} K(s,t) \quad dt \; ; \; f(t) \rightarrow F(s)$$

How to "handle" the Laplace transform

E.g.,

$$\int_{t=0}^{t=1} \underbrace{e^{st}}_{K(s,t)} \underbrace{e^{t}}_{f(t)}\, dt = \int_{t=0}^{t=1} e^{st+t}\, dt$$

$$= \int_{t=0}^{t=1} e^{(s+1)t}\, dt$$

- t = variable of integration
- $s+1$ = a constant

$$= \left. \frac{e^{(s+1)t}}{s+1} \right|_{t=0}^{t=1}$$

$$= \frac{e^{(s+1)\cdot 1}}{s+1} - \frac{e^{(s+1)\cdot 0}}{s+1}$$

$$= \underbrace{\boxed{\frac{e^{s+1}}{s+1} - \frac{1}{s+1}}}_{F(s)}$$

The Laplace Transform

The LAPLACE TRANSFORM is an improper integral operator of the form

$$\int_{t=a}^{t=b} K(s,t)\, f(t)\, dt$$

where

(1) $K(s,t) = e^{-st}$

(2) $a = 0, \quad b = \infty$

Specifically, the Laplace transform looks like this:

$$\boxed{F(s) = \int_0^\infty f(t)\, e^{-st}\, dt}$$ LAPLACE TRANSFORM

The variable s is called a PARAMETER, which is held constant <u>during</u> integration but which can be allowed to vary <u>after</u> integration.

NOTATION:

$$\underbrace{\int_0^\infty f(t)\, e^{-st}\, dt}$$

$L\{f(t)\}$ or $\mathcal{L}\{f(t)\}$ or $F(s)$

We will use this notation and let $L\{f(t)\} = F(s)$

Piecewise Continuity and Exponential Order of Functions

If $f(t)$ is (1) piecewise continuous on $[0, \infty)$
(2) of exponential order
then the improper integral $\int_0^\infty f(t)\, e^{-st}\, dt$

exists and we can evaluate the Laplace transform of $f(t)$, $L\{f(t)\}$.

This is as far as I will go on talking about "piecewise continuity" and "exponential order of functions" — We will SKIP DEFINITION 7.2 and THEOREM 7.1 on pp. 260-262.

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Examples

What you need to know to do these examples:

(1) INTEGRATION BY PARTS.

$$\int_a^b uv'\,dt = uv\Big|_a^b - \int_a^b u'v\,dt$$

Given: $u(t)$ Find: $u'(t)$

Given: $v'(t)$ Find: $v(t) = \int v'(t)\,dt$

THEOREM 7.2, p. 264, text, has already worked out the Laplace transform of some commonly seen functions, so that you may avoid integration, and thus integration by parts, in these cases.

(2) TRIG IDENTITIES.

$$\sin 2t = 2\sin t\cos t$$
$$\cos 2t = \cos^2 t - \sin^2 t$$
$$\cos^2 t + \sin^2 t = 1$$
$$\cos^2 t = \frac{1}{2} + \frac{\cos 2t}{2}$$
$$\sin^2 t = \frac{1}{2} - \frac{\cos 2t}{2}$$

ALSO: $\sinh t = \dfrac{e^{t} - e^{-t}}{2}$

$\cosh t = \dfrac{e^{t} + e^{-t}}{2}$

<u>Examples</u>.

1. (HW Exercise 3, p. 265.)

Let

$$f(t) = \begin{cases} t, & 0 \le t < 1, \\ 1, & t \ge 1. \end{cases}$$

Do the actual integration to find $L\{f(t)\}$.

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$$L\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \underbrace{\int_0^1 t e^{-st}\, dt}_{①} + \underbrace{\int_1^\infty 1 \cdot e^{-st}\, dt}_{②}$$

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① Use INTEGRATION BY PARTS:

$$\int_a^b uv'\,dt = uv\Big|_a^b - \int_a^b u'v\,dt$$

Let $u = t$, $v' = e^{-st}$ (where we treat s like a constant here). Then

$u = t$ $\qquad v' = e^{-st}$

$u' = 1$ $\qquad v = \int v'\,dt = \int e^{-st}\,dt = \dfrac{e^{-st}}{-s}$

Use indefinite integral for finding v.

Need to assume $s \neq 0$ at this point.

$$\therefore \int_0^1 te^{-st}\,dt = t\cdot\frac{e^{-st}}{-s}\Big|_0^1 - \int_0^1 1\cdot\frac{e^{-st}}{-s}\,dt$$

$$= -\frac{te^{-st}}{s}\Big|_0^1 + \frac{1}{s}\int_0^1 e^{-st}\,dt$$

$$= -\frac{te^{-st}}{s}\Big|_0^1 + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]\Big|_0^1$$

$$= \left[\left(-\frac{1\cdot e^{-s\cdot 1}}{s}\right) - \left(-\frac{0\cdot e^{-s\cdot 0}}{s}\right)\right] + \frac{1}{s}\left[\left(\frac{e^{-s\cdot 1}}{-s}\right) - \left(\frac{e^{-s\cdot 0}}{-s}\right)\right]$$

$$= -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}$$

(2) Use INTEGRATION OF AN IMPROPER INTEGRAL:

$$\int_1^\infty 1 \cdot e^{-st}\,dt = \int_1^\infty e^{-st}\,dt.$$

— Treat s like a constant here

$$= \left.\frac{e^{-st}}{-s}\right|_1^\infty$$

$$= \left(\frac{e^{-s\cdot\infty}}{-s}\right) - \left(\frac{e^{-s\cdot 1}}{-s}\right)$$

— For $e^{-s\cdot\infty}$ to be finite, we must have $s>0$. So let $s>0$. Then

$$e^{-s\cdot\infty} = e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$$

$$= \frac{0}{-s} + \frac{e^{-s}}{s}$$

$$= \frac{e^{-s}}{s}$$

$$\therefore \; L\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \cancel{-\frac{e^{-s}}{s}} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} + \cancel{\frac{e^{-s}}{s}}$$

$$= \boxed{\frac{1}{s^2} - \frac{e^{-s}}{s^2}}, \quad s > 0$$

OMIT

2. (HW Exercise 5, p. 265.)

Let

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi, \\ 0, & t \ge \pi. \end{cases}$$

Do the actual integration to find $L\{f(t)\}$.

$$L\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \underbrace{\int_0^\pi \sin t\, e^{-st}\, dt}_{①} + \underbrace{\int_\pi^\infty 0 \cdot e^{-st}\, dt}_{②}$$

(1) Use INTEGRATION BY PARTS (twice):

$$\int_a^b uv'\,dt = uv\Big|_a^b - \int_a^b u'v\,dt$$

Let $u = \sin t$, $v' = e^{-st}$ (where we treat s like a constant here). Then

$$u = \boxed{\sin t} \qquad v' = e^{-st}$$
$$u' = \boxed{\cos t} \qquad v = \int v'\,dt = \int e^{-st}\,dt = \frac{e^{-st}}{-s} \quad \cancel{+C}$$

Use indefinite integral for finding v.

Need to assume $s \neq 0$ at this point.

$$\therefore \int_0^\pi \sin t\, e^{-st}\,dt = \sin t \cdot \frac{e^{-st}}{-s}\Big|_0^\pi - \int_0^\pi \cos t \cdot \frac{e^{-st}}{-s}\,dt$$
$$= \frac{\sin t\, e^{-st}}{-s}\Big|_0^\pi + \frac{1}{s}\underbrace{\int_0^\pi \cos t\, e^{-st}\,dt}_{(1a)}$$

(1a) $$\int_0^\pi \cos t\, e^{-st}\,dt = uv\Big|_0^\pi - \int_0^\pi u'v\,dt$$

$$u = \boxed{\cos t} \qquad v' = e^{-st}$$
$$u' = \boxed{-\sin t} \qquad v = \int v'\,dt = \int e^{-st}\,dt = \frac{e^{-st}}{-s} \quad \cancel{+C}$$

$$= \cos t \cdot \frac{e^{-st}}{-s}\Big|_0^{\pi} - \int_0^{\pi} -\sin t \cdot \frac{e^{-st}}{-s}\,dt$$

$$= \frac{\cos t\, e^{-st}}{-s}\Big|_0^{\pi} - \frac{1}{s}\int_0^{\pi} \sin t\, e^{-st}\,dt$$

$$\therefore \int_0^{\pi} \sin t\, e^{-st}\,dt = \frac{\sin t\, e^{-st}}{-s}\Big|_0^{\pi} + \frac{1}{s}\left[\frac{\cos t\, e^{-st}}{-s}\Big|_0^{\pi} - \frac{1}{s}\int_0^{\pi} \sin t\, e^{-st}\,dt\right] \Longrightarrow$$

$$\left(1 + \frac{1}{s^2}\right)\int_0^{\pi} \sin t\, e^{-st}\,dt = \frac{\sin t\, e^{-st}}{-s}\Big|_0^{\pi} + \frac{\cos t\, e^{-st}}{-s^2}\Big|_0^{\pi} \Rightarrow$$

$$\frac{s^2+1}{s^2}\int_0^{\pi} \sin t\, e^{-st}\,dt = \left[\left(\frac{\overbrace{\sin \pi}^{=0}\, e^{-s\cdot\pi}}{-s}\right) - \left(\frac{\overbrace{\sin 0}^{=0}\, e^{-s\cdot 0}}{-s}\right)\right] + \left[\left(\frac{\overbrace{\cos \pi}^{=-1}\, e^{-s\cdot\pi}}{-s^2}\right) - \left(\frac{\overbrace{\cos 0}^{=1}\, e^{-s\cdot 0}}{-s^2}\right)\right]$$

$$= \frac{e^{-s\pi}}{s^2} + \frac{1}{s^2} \Longrightarrow$$

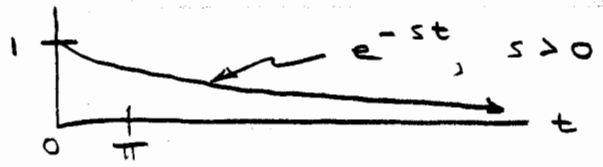
$$\int_0^{\pi} \sin t\, e^{-st}\,dt = \frac{s^{\cancel{2}}}{s^2+1}\left(\frac{e^{-s\pi}}{\cancel{s^2}} + \frac{1}{\cancel{s^2}}\right)$$

$$= \frac{e^{-s\pi}+1}{s^2+1}$$

(2) $\int_{\pi}^{\infty} 0 \cdot e^{-st}\,dt = \int_{0}^{\infty} 0\,dt = C\Big|_{0}^{\infty} = C - C = 0$

For $0 \cdot e^{-st}$ to be equal to 0, we must have $s > 0$. So let $s > 0$. Then

$0 \cdot e^{-st} = 0$ from $t = \pi$ to $t = \infty$



$$\therefore L\{f(t)\} = \int_0^{\pi} f(t)\, e^{-st}\, dt$$

$$= \frac{e^{-s\pi}+1}{s^2+1} + 0$$

$$= \boxed{\frac{e^{-s\pi}+1}{s^2+1}}, \quad s > 0$$

OMIT

3. (HW Exercise 28, p. 266.)

Let

$$f(t) = t^2 - e^{-9t} + 5,$$

Use the table in THEOREM 7.2, p. 264, text, to find $L\{f(t)\}$.

$$L\{f(t)\} = L\{t^2 - e^{-9t} + 5\}$$

$$= L\{t^2 + (-1)e^{-9t} + 5(1)\}$$

— Use the LINEARITY of L:
$L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}$

$$= L\{t^2\} + (-1)L\{e^{-9t}\} + 5L\{1\}$$

— By the TABLE:
$L\{1\} = \frac{1}{s}$, $L\{t^n\} = \frac{n!}{s^{n+1}}$,

$L\{e^{at}\} = \frac{1}{s-a}$

$$= \frac{2!}{s^{2+1}} - \frac{1}{s-(-9)} + 5 \cdot \frac{1}{s}$$

$$= \boxed{\frac{2}{s^3} - \frac{1}{s+9} + \frac{5}{s}}, \quad s > 0$$

Lecture

Section 7.2. Inverse Transform.

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Laplace transform operator:

$$L\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt = F(s)$$

Inverse (Laplace) transform operator:

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$$L^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{a-\infty i}^{a+\infty i} F(s)e^{st}\,ds = f(t)$$

where (1) a is a sufficiently large positive number
(2) depending on conditions, one of the following situations holds:
(a) $L^{-1}\{F(s)\}$ does not exist, i.e., the integral diverges
(b) $L^{-1}\{F(s)\}$ exists but is NEVER unique, i.e., $L^{-1}\{F(s)\}$ is equal to more than one piecewise continuous $f(t)$

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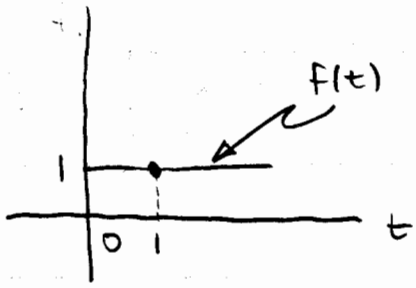
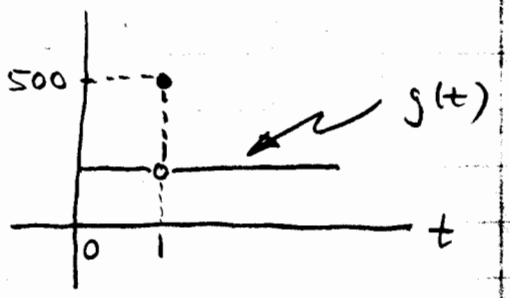
<u>Example</u>. (SEE HW Exercise 35, p. 273.)

Let

$$f(t) = 1, \quad 0 \le t < \infty,$$

$$g(t) = \begin{cases} 1, & 0 \le t < 3, \quad 3 < t < \infty, \\ 500, & t = 3 \end{cases}$$





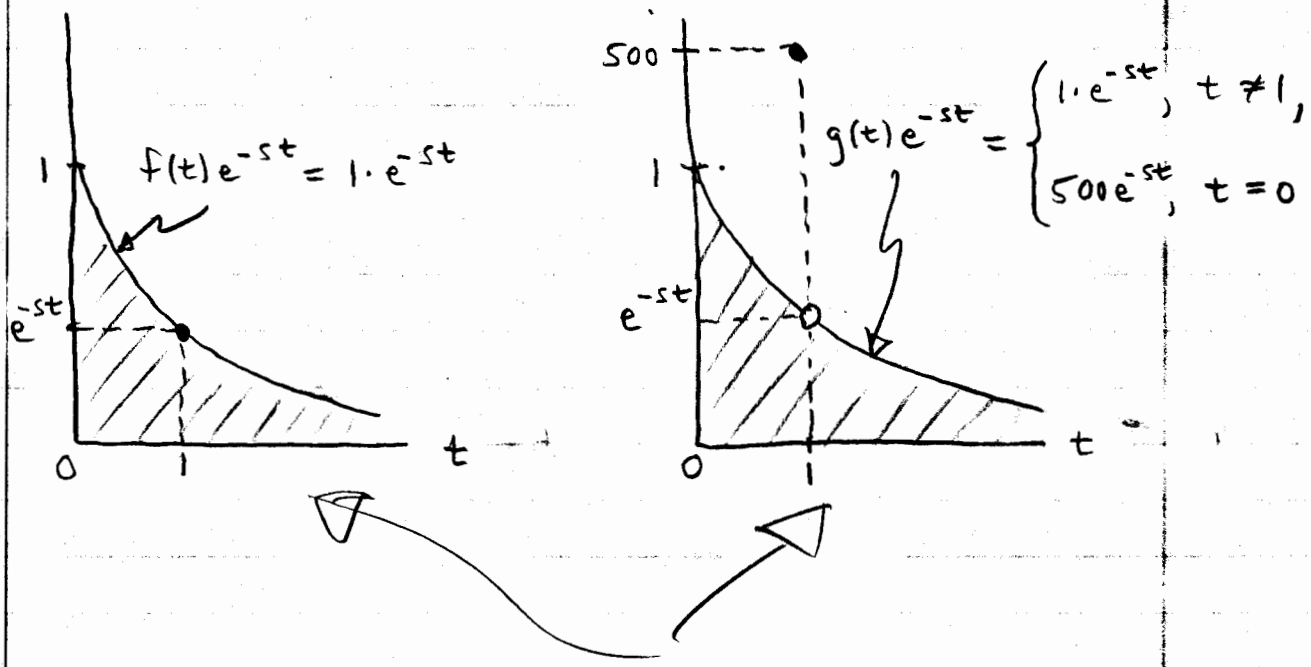
$f(t)$ is continuous on $[0, \infty)$

$g(t)$ is piecewise continuous, but not continuous, on $[0, \infty)$ where

(1) $g(t)$ is cont. on $[0, 1)$ and cont. on $(1, \infty)$

(2) $\lim_{t \to 1^-} g(t) = 1 < \infty$ and

$\lim_{t \to 1^+} g(t) = 1 < \infty$.



Still same area under these curves from $t = 0$ to $t = \infty$

$$\therefore \quad L\{g(t)\} = L\{f(t)\} = L\{1\} = \frac{1}{s}$$

by THEOREM 7.2, p. 264, text.

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Lecture

Evaluation of $L^{-1}\{F(s)\}$

To evaluate $L^{-1}\{F(s)\}$, instead of using ACTUAL INTEGRATION, which would involve COMPLEX ANALYSIS THEORY, we will use either

(1) the table in THEOREM 7.3, p. 267, text, or
(2) the table of Laplace transforms at the FRONT OF THE TEXT or in APPENDIX III.

E.g., $L^{-1}\{\frac{1}{s}\} = 1$

E.g., $L^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

———— o ————

Very often, $F(s)$ does not quite match any function given on the right-hand side of a TABLE OF LAPLACE TRANSFORMS

	$f(t)$	$L\{f(t)\} = F(s)$
1.	1	$\frac{1}{s}$
2	t	$\frac{1}{s^2}$
	$\vdots$	$\vdots$

In this case, we look for the possibility of MODIFYING $F(s)$ so that it will match an item in the table.

We need to know about two things in order to attempt this modification.

(1) The LINEARITY of the INVERSE TRANSFORM:

Since L^{-1} is an integral (just like L is), it is a linear operator. So

$$L^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha L^{-1}\{F(s)\} + \beta L^{-1}\{G(s)\}$$

E.g., From the table of Laplace transforms, we have

$$L^{-1}\left\{\frac{1}{s}\right\} = 1 \text{ and } L^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}.$$

Then

$$L^{-1}\left\{\frac{5}{s} - \frac{7}{s-2}\right\} = 5L^{-1}\left\{\frac{1}{s}\right\} - 7L^{-1}\left\{\frac{1}{s-2}\right\}$$
$$= 5(1) - 7e^{2t}$$
$$= \boxed{5 - 7e^{2t}}$$

(2) PARTIAL FRACTION DECOMPOSITION:

E.g., $$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)}$$
$$= \frac{1/2}{x-1} + \frac{-1/2}{x+1}$$

I will review as I go through examples.

———o———

Examples.

1. (HW Exercise 25, p. 273).

$$L^{-1}\left\{\frac{s}{(s-2)(s-3)(s-6)}\right\} = ?$$

First perform PARTIAL FRACTION DECOMPOSITION on $\frac{s}{(s-2)(s-3)(s-6)}$, before evaluating $L^{-1}\left\{\frac{s}{(s-2)(s-3)(s-6)}\right\}$:

$$\frac{s}{(s-2)(s-3)(s-6)} = \frac{A}{s-2} + \frac{B}{s-3} + \frac{C}{s-6}$$

3 "linear factors" (s^1 - constant).

- NEED TO SOLVE FOR A, B, C
- MULTIPLY BOTH SIDES BY $(s-2)(s-3)(s-6)$

$$s = A(s-3)(s-6) + B(s-2)(s-6) + C(s-2)(s-3)$$

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Now, this equation should HOLD FOR ALL VALUES OF s.
Therefore, we can plug into this equation any values of s that might make it

convenient to solve for A, B, C.
In particular, try $s = 2, 3, 6$

$$s = A(s-3)(s-6) + B(s-2)(s-6) + C(s-2)(s-3)$$

$s = 2$: $2 = A(-1)(-4) + B(0)(-4) + C(0)(-1) \Rightarrow$
$2 = 4A \Rightarrow$
$\boxed{A = \frac{1}{2}}$

$s = 3$: $3 = A(0)(-3) + B(1)(-3) + C(1)(0) \Rightarrow$
$3 = -3B \Rightarrow$
$\boxed{B = -1}$

$s = 6$: $6 = A(3)(0) + B(4)(0) + C(4)(3) \Rightarrow$
$6 = 12C \Rightarrow$
$\boxed{C = \frac{1}{2}}$

Then

$$L^{-1}\left\{\frac{s}{(s-2)(s-3)(s-6)}\right\}$$

$$= L^{-1}\left\{\frac{A}{s-2}+\frac{B}{s-3}+\frac{C}{s-6}\right\}$$

$$= L^{-1}\left\{\frac{\frac{1}{2}}{s-2}+\frac{-1}{s-3}+\frac{\frac{1}{2}}{s-6}\right\}$$

By LINEARITY of L^{-1}

$$= \frac{1}{2}L^{-1}\left\{\frac{1}{s-2}\right\}+(-1)L^{-1}\left\{\frac{1}{s-3}\right\}+\frac{1}{2}L^{-1}\left\{\frac{1}{s-6}\right\}$$

From TABLE OF LAPLACE TRANSFORMS, item # 11 : $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

$$= \boxed{\frac{1}{2}e^{2t} - e^{3t} + \frac{1}{2}e^{6t}}$$

2. (HW Exercise 9, p. 272.)

$$L^{-1}\left\{\frac{1}{4s+1}\right\} = ?$$

"$\frac{1}{4s+1}$" looks similar to item #11 in the TABLE OF LAPLACE TRANSFORMS at the front of the text:

$$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

A little algebra will make "$\frac{1}{4s+1}$" match item #11:

$$\frac{1}{4s+1} = \frac{\frac{1}{4}}{\frac{1}{4}} \cdot \frac{1}{4s+1} = \frac{\frac{1}{4}(1)}{\frac{1}{4}(4s+1)}$$

$$= \frac{\frac{1}{4}}{s+\frac{1}{4}} = \frac{\frac{1}{4}}{s-\left(-\frac{1}{4}\right)}$$

$$\therefore \; L^{-1}\left\{\frac{1}{4s+1}\right\} = L^{-1}\left\{\frac{\frac{1}{4}}{s-(-\frac{1}{4})}\right\}$$

— By LINEARITY of L^{-1}

$$= \frac{1}{4} L^{-1}\left\{\frac{1}{s-(-\frac{1}{4})}\right\}$$

From TABLE, item # 11:

$$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}, \quad a = -\frac{1}{4}$$

$$= \boxed{e^{-\frac{1}{4}t}}$$

3. (HW Exercise 29, p. 273.)

$$L^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = ?$$

"$\frac{1}{s^2(s^2+4)}$" looks similar to item #31 in the TABLE OF LAPLACE TRANSFORMS at the front of the text:

$$L^{-1}\left\{\frac{k^3}{s^2(s^2+k^2)}\right\} = kt - \sin kt$$

We need to identify what k should be in "$\frac{1}{s^2(s^2+4)}$":

$$k^2 = 4 \Rightarrow k = 2$$

We are "missing" $k^3 = 2^3 = 8$ in "$\frac{1}{s^2(s^2+4)}$"

so we multiply and divide "$\frac{1}{s^2(s^2+4)}$" by 8:

$$\frac{1}{s^2(s^2+4)} = \frac{8}{8}\cdot\frac{1}{s^2(s^2+4)}$$
$$= \frac{1}{8}\cdot 8\cdot\frac{1}{s^2(s^2+4)}$$
$$= \frac{1}{8}\cdot\frac{8}{s^2(s^2+4)}$$

$$\therefore\ L^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = L^{-1}\left\{\frac{1}{8}\cdot\frac{8}{s^2(s^2+4)}\right\}$$

By LINEARITY of L^{-1}

$$= \frac{1}{8}L^{-1}\left\{\frac{8}{s^2(s^2+4)}\right\}$$

$$= \frac{1}{8}L^{-1}\left\{\frac{2^3}{s^2(s^2+2^2)}\right\}$$

From TABLE, item #31;

$$L^{-1}\left\{\frac{k^3}{s^2(s^2+k^2)}\right\} = kt - \sin kt,$$

$k = 2$

$$= \frac{1}{8}(2t - \sin 2t)$$

$$= \boxed{\frac{1}{4}t - \frac{1}{8}\sin 2t}$$

Lecture

Section 7.4. Transforms of Derivatives...

Suppose $L\{f(t)\} = F(s)$. Then it can be shown, using integration by parts on

$$\int_0^\infty f'(t)\, e^{-st}\, dt,$$

that

$$\boxed{L\{f'(t)\} = sF(s) - f(0).}$$

Lecture

Section 7.5. Applications.

We can pull the ideas from Sections 7.1, 7.2, and 7.4 together and solve an initial value problem.

Strategy: Transform an IVP into an ALGEBRAIC PROBLEM.

$$\begin{tikzcd}
\text{DE in } t \arrow[r, "L"] & \text{ALGEBRAIC EQ. in } s \arrow[d, "\text{solve}"] \\
\text{solution in } t & \text{solution in } s \arrow[l, "L^{-1}"']
\end{tikzcd}$$

———— o ————

<u>Example</u>. (HW Exercise 1, p. 301.)

Solve the IVP

$$\begin{cases} y' - y = 1, \\ y(0) = 0 \end{cases}$$

Let $L\{y(t)\} = Y(s)$

Then $L\{y'(t)\} = sY(s) - \underbrace{y(0)}_{=0}$ and $L^{-1}\{Y(s)\} = y(t)$.

<u>STEP 1</u>. "Laplace transform" $y' - y = 1$:

$$L\{y' - y\} = L\{1\} \Longrightarrow$$

$$L\{y'(t)\} - L\{y(t)\} = L\{1\} \Longrightarrow$$

By LINEARITY

$L\{1\} = \frac{1}{s}$ from TABLE, item #1

$$[sY(s) - \underbrace{y(0)}_{=0}] - Y(s) = \frac{1}{s} \Longrightarrow$$

$$sY(s) - Y(s) = \frac{1}{s} \Longrightarrow$$

$$Y(s)\,(s-1) = \frac{1}{s} \Longrightarrow$$

$$Y(s) = \underbrace{\frac{1}{s(s-1)}}$$

$$\frac{1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1} \Longrightarrow$$

$$1 = A(s-1) + Bs$$

$s=0$: $\quad 1 = A(0-1) + B(0) \Longrightarrow$

$1 = -A \Longrightarrow$

$A = -1$

$s=1$: $\quad 1 = A(1-1) + B(1) \Longrightarrow$

$B = 1$

$$\therefore\ Y(s) = \frac{-1}{s} + \frac{1}{s-1}$$

<u>STEP 2</u>. "Inverse transform" $Y(s) = \frac{-1}{s} + \frac{1}{s-1}$:

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{-1}{s} + \frac{1}{s-1}\right\} \Longrightarrow$$

— By LINEARITY

$$y(t) = -L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{1}{s-1}\right\}$$

$L^{-1}\left\{\frac{1}{s}\right\} = 1$ and $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$, $a = 1$, from TABLE, items # 1, 11

$$= -(1) + e^{1 \cdot t}$$

$$= \boxed{-1 + e^{t}}$$

Lecture

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Section 4.5. Indeterminate Forms and L'Hôpital's Rule

- Pronounced LOHPEETAHL'S
- Alternative spelling: L'Hospital's

Method of finding limits of functions using derivatives.

Recall:

1. $f(x) = \dfrac{x^2 - 4}{x + 2}$

Directly evaluate $f(x)$ at $x = -2$: $f(-2) = \dfrac{0}{0}$

indeterminate form (and $\neq 0$ or 1)

So indirectly look at $f(x)$ near $x = -2$:

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{x^2 - 4}{x + 2} = \lim_{x \to -2} \frac{\cancel{(x+2)}(x-2)}{\cancel{x+2}}$$

FACTOR/CANCEL } ALGEBRA

$$= \lim_{x \to -2} (x - 2) = -2 - 2 = \boxed{-4}$$

$f(x)$ approaches -4 as x approaches -2

3. $f(x) = \dfrac{6x^3+1}{12x^3+5}$

What happens to $f(x)$ as $x \to +\infty$?

$$f(\infty) = \frac{6(\infty)^3+1}{12(\infty)^3+5} = \frac{6(\infty)+1}{12(\infty)+5} = \frac{\infty+1}{\infty+1} = \frac{\infty}{\infty}$$

indeterminate form
(and $\neq \infty$ or 1)

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{6x^3+1}{12x^3+5} = \lim_{x\to\infty} \frac{6x^3+1}{12x^3+5} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

MULTIPLY TOP AND BOTTOM BY $\frac{1}{x^3}$ } ALGEBRA

$$= \lim_{x\to\infty} \frac{6 + \frac{1}{x^3}\to 0}{12 + \frac{5}{x^3}\to 0} = \frac{6+0}{12+0} = \frac{6}{12} = \boxed{\frac{1}{2}}$$

There are other functions like these but one cannot do algebra on them to obtain a limit.
Instead, one uses L'Hôpital's rule.

<u>L'Hopital's rule (without proof).</u>

Suppose you are given the following limit to evaluate

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$ (a can be a finite number or $\pm\infty$)

and either

$$\frac{f(a)}{g(a)} = \frac{0}{0} \quad \text{or} \quad \frac{f(a)}{g(a)} = \frac{\pm\infty}{\pm\infty}$$

$\frac{+\infty}{+\infty}$, $\frac{+\infty}{-\infty}$, $\frac{-\infty}{+\infty}$, $\frac{-\infty}{-\infty}$

Then you can do the following:

$$\lim_{x \to a} \frac{f(x)}{g(x)} \underset{\uparrow}{=} \lim_{x \to a} \frac{f'(x)}{g'(x)} = \text{ANSWER}$$

TAKE THE DERIVATIVE OF f(x) AND OF g(x) SEPARATELY

[<u>Proof</u> uses what is called the "generalized mean value theorem":

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x)}{g'(x)}; \quad x \in (a,b)$$

which relates f and g to f' and g'. We skipped the "mean value theorem,"

$$\frac{f(b)-f(a)}{b-a} = f'(x), \quad x \in (a,b)$$

in Section 4.3.]

Examples.

1. $\lim_{x \to 0} \frac{\sin x}{x} = ?$ (know from Section 3.4, p. 219, that $\lim_{x \to 0} \frac{\sin x}{x} = 1$ using geometry – we skipped this)

$f(x) = \sin x$, $g(x) = x$

$\frac{f(0)}{g(0)} = \frac{\sin 0}{0} = \frac{0}{0}$ indeterminate form

$\therefore$ Can use L'Hôpital's rule to evaluate limit:

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(x)} = \lim_{x \to 0} \frac{\cos x}{1}$$

NOW PLUG $x = 0$ INTO THIS

$$= \frac{\cos 0}{1} = \frac{1}{1} = \boxed{1} \checkmark$$

2. $\lim_{x \to 0} \frac{\cos x - 1}{x^2} = ?$

$f(x) = \cos x - 1$, $g(x) = x^2$

$\frac{f(0)}{g(0)} = \frac{\cos 0 - 1}{0^2} = \frac{1-1}{0} = \frac{0}{0}$ indeterminate form

$\therefore$ Can use L'Hôpital's rule to evaluate limit:

$$\lim_{x \to 0} \frac{\cos x - 1}{x^2} = \lim_{x \to 0} \frac{\frac{d}{dx}(\cos x - 1)}{\frac{d}{dx}(x^2)} = \lim_{x \to 0} \frac{-\sin x}{2x}$$

IF PLUG $x=0$ INTO THIS, AGAIN GET $\frac{0}{0}$:

$$\frac{-\sin 0}{0^2} = \frac{-0}{0} = \frac{0}{0}$$

SO APPLY L'HÔPITAL'S RULE AGAIN

$$= \lim_{x \to 0} \frac{\frac{d}{dx}(-\sin x)}{\frac{d}{dx}(2x)} = \lim_{x \to 0} \frac{-\cos x}{2}$$

NOW PLUG $x=0$ INTO THIS

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$$= \frac{-\cos 0}{2} = \frac{-1}{2} = \boxed{-\frac{1}{2}}$$

3. $\lim_{x \to \infty} e^{-x} \cdot \ln x = ?$

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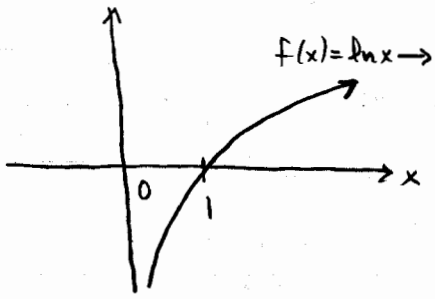
Rewrite $e^{-x} \cdot \ln x$ as

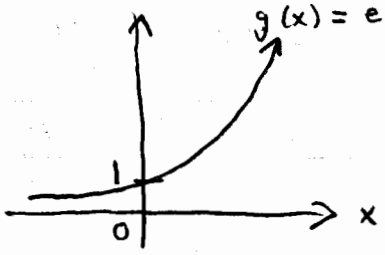
$$e^{-x} \cdot \ln x = \frac{1}{e^x} \cdot \ln x = \frac{\ln x}{e^x}$$

$f(x) = \ln x, \quad g(x) = e^x$

Observe:



$f(x) = \ln x \to \infty$ as $x \to \infty \Rightarrow f(\infty) = \ln(\infty) = \infty$



$g(x) = e^x \to \infty$ as $x \to \infty \Rightarrow g(\infty) = e^{\infty} = \infty$

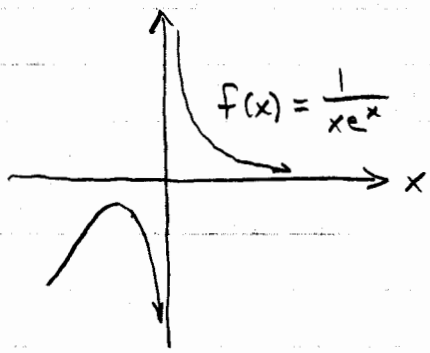
$$\frac{f(\infty)}{g(\infty)} = \frac{\ln(\infty)}{e^{\infty}} = \frac{\infty}{\infty} \quad \text{indeterminate form}$$

$\therefore$ Can use L'Hôpital's rule to evaluate limit:

$$\lim_{x\to\infty} e^{-x}\cdot \ln x = \lim_{x\to\infty} \frac{\ln x}{e^x} = \lim_{x\to\infty} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(e^x)}$$

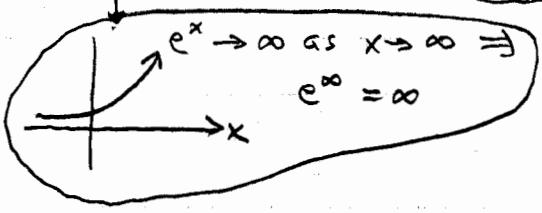
$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{e^x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{e^x}{1}} = \lim_{x\to\infty} \frac{1}{x}\cdot\frac{1}{e^x}$$

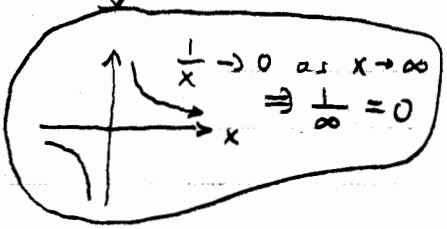
$$= \lim_{x\to\infty} \frac{1}{\underbrace{xe^x}} = 0$$



denominator gets larger as x gets larger, so fraction goes to 0:

$$\frac{1}{\infty\, \underbrace{e^\infty}} = \frac{1}{\infty\cdot\infty} = \frac{1}{\underbrace{\infty}} = 0$$





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