

## Lecture

### Section 8.7. Taylor and MacLaurin Series.

4/25/02

Thurs. Some familiar functions like

$$e^x, \ln x, \sin x, \cos x, \arctan x$$

can be expressed in the form of a power series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

which are all true

FOR ALL  $x$  ! (Radius of Convergence:  $R = \infty$   
Interval of convergence:  $-\infty < x < \infty$ )

Another function equal to a power series, but  
only for some and not all  $x$  is the following:

(still start with  $x^0$ )

$$1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} \underbrace{1 \cdot x^{n-1}}_{\substack{a \\ r}} = \frac{1}{1-x}$$

if  $|x| < 1$

GEO. SERIES

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

ONLY IF  
 $|r| < 1$

$$\therefore \boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots}$$

$\underbrace{= f(x)}$

which is true for  $|x| < 1$  (or  $-1 < x < 1$ )

$\underbrace{\text{Radius of conv. } R}_{\text{Interval of Conv. } |x| < R}$

### FACTS :

1. Every function  $f(x)$  has "associated with it" a special power series  $T(x)$ :

$$f(x) : T(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

2. Not every function  $f(x)$  actually equals the power series  $T(x)$  associated with it.

Definition. The power series  $T(x)$  associated with a function  $f(x)$  is given by the formula

$$\begin{aligned} T(x) &= c_0 + c_1 (x-a) + c_2 (x-a)^2 \\ &\quad + c_3 (x-a)^3 + c_4 (x-a)^4 + \dots \\ \frac{f'(a)}{1!} (x-a)^1 &\leftarrow \\ \frac{f^{(0)}(a)}{0!} (x-a)^0 &\leftarrow \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 \\ &\quad + \frac{f'''(a)}{3!} (x-a)^3 + \frac{f^{(4)}(a)}{4!} (x-a)^4 + \dots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \end{aligned}$$

and is called the

**TAYLOR SERIES OF THE FUNCTION  
 $f(x)$  ABOUT  $x=a$**

If  $a=0$ , then

$$T(x) = f(0) + F'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$
$$= \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

and this TAYLOR SERIES is specifically called the

### MACLAURIN SERIES

TAYLOR SERIES have associated with them a

RADIUS OF CONVERGENCE  $R$

INTERVAL OF CONVERGENCE  
 $|x-a| < R$

REMARKS:

1. SEE p. 611-612 of the text where it is shown that

IF  $f(x)$  actually equals a power series,  
i.e.,  $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$

THEN  $c_0, c_1, c_2, \dots$  must be

$$c_0 = f(a), \quad c_1 = f'(a), \quad c_2 = \frac{f''(a)}{2!}, \dots$$

2. Usually, one does the following with Taylor Series  $T(x)$

① Find  $T(x)$  associated with  $f(x)$

② Try to show

$$f(x) = T(x)$$

OMIT

(SEE pp. 613-615 of text if you are interested)

We will only look for Maclaurin Series

Examples. Find the MACLAURIN SERIES for  $f(x)$ .

1.  $f(x) = e^x$

$$f(x) : T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$
$$= f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \dots$$

Need to find  $f(0), f'(0), f''(0), f'''(0)$ , etc.  
until able to generalize and write  
down what  $f^{(n)}(0)$  is :

$$f(x) = e^x \Rightarrow f(0) = e^0 = 1 \quad \textcircled{1}$$

$$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1 \quad \textcircled{1}$$

$$f''(x) = e^x \Rightarrow f''(0) = e^0 = 1 \quad \textcircled{1}$$

$$\therefore f^{(n)}(x) = e^x \Rightarrow \boxed{f^{(n)}(0) = 1}$$

$$\therefore e^x : T(x) = 1 + 1 \cdot x + \frac{1}{2!} x^2 + \dots + \frac{1}{n!} x^n + \dots$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{1}{n!} x^n}$$

Q. (PP#5, Prob. 3, p. 621.)

$$f(x) = \cos x$$

$$\begin{aligned} f(x) : T(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 \\ &\quad + \frac{f'''(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4 + \dots \end{aligned}$$

Find  $f(0), f'(0), f''(0)$ , etc. and stop when you have found a pattern:

$$f(x) = \cos x \Rightarrow f(0) = \cos(0) = 1$$

4/25/02  
Thurs.  $f'(x) = -\sin x \Rightarrow f'(0) = -\sin(0) = -0 = 0$

↑  $f''(x) = -\cos x \Rightarrow f''(0) = -\cos(0) = -1$

$f'''(x) = -(-\sin x) = \sin x \Rightarrow f'''(0) = \sin(0) = 0$

4/30/02  
Tues.  $f^{(4)}(x) = \cos x \Rightarrow f^{(4)}(0) = 1$

$$f^{(5)}(x) = -\sin x \Rightarrow f^{(5)}(0) = 0$$

$$f^{(6)}(x) = -\cos x \Rightarrow f^{(6)}(0) = -1$$

$$f^{(7)}(x) = -(-\sin x) = \sin x \Rightarrow f^{(7)}(x) = 0$$

Things repeat!  
Repetition every 4 terms!

$$\begin{aligned}\therefore \cos x : T(x) &= 1 + 0 \cdot x + \frac{-1}{2!} x^2 + \frac{0}{3!} x^3 \\ &\quad + \frac{1}{4!} x^4 + \frac{0}{5!} x^5 + \frac{-1}{6!} x^6 + \frac{0}{7!} x^7 \\ &\quad + \dots \\ &= \boxed{1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \dots} \\ &\quad \text{↑} \quad \text{↑} \quad \text{↑} \\ &\quad \text{all even!} \\ &\quad \text{↓} \\ &\quad \text{alternating pluses and minuses}\end{aligned}$$

3. (PP# 5, Prob. 4, p. 621)

$$f(x) = \sin 2x$$

$$\begin{aligned} F(x) : T(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 \\ &\quad + \frac{f'''(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4 + \dots \end{aligned}$$

Just find  $f(0), f'(0), f''(0), f'''(0), f^{(4)}(0)$ , and  $f^{(5)}(0)$  and then stop;

$$f(x) = \sin 2x \Rightarrow f(0) = \sin(2 \cdot 0) = \sin(0) = 0 \quad (1)$$

$$f'(x) = 2 \cos 2x \Rightarrow f'(0) = 2 \cos(2 \cdot 0) = 2 \cos(0) = 2 \cdot 1 = 2 \quad (2)$$

$$\frac{d}{dx} \sin ax = a \cos ax \quad \frac{d}{dx} \cos ax = -a \sin ax$$

$$\begin{aligned} f''(x) &= 2(-2 \sin 2x) = -4 \sin 2x \Rightarrow \\ f''(0) &= -4 \sin(2 \cdot 0) = -4 \sin(0) = -4 \cdot 0 = 0 \quad (3) \end{aligned}$$

$$\begin{aligned} f'''(x) &= -4(2 \cos 2x) = -8 \cos 2x \Rightarrow \\ f'''(0) &= -8 \cos(2 \cdot 0) = -8 \cos(0) = -8 \cdot 1 = -8 \quad (4) \end{aligned}$$

$$f^{(4)}(x) = -8(-2 \sin 2x) = 16 \sin 2x \Rightarrow f^{(4)}(0) = 0 \quad (5)$$

$$f^{(5)}(x) = 16(2 \cos 2x) = 32 \cos 2x \Rightarrow f^{(5)}(0) = 32 \quad (6)$$

$$\therefore \sin 2x; T(x) = 0 + 2x + \frac{0}{2!}x^2 + \frac{-8}{3!}x^3$$

$$+ \frac{0}{4!}x^4 + \frac{32}{5!}x^5 + \dots$$

$$= \boxed{2x - \frac{8}{3!}x^3 + \frac{32}{5!}x^5 \dots}$$

probably  
a minus