

## Lecture

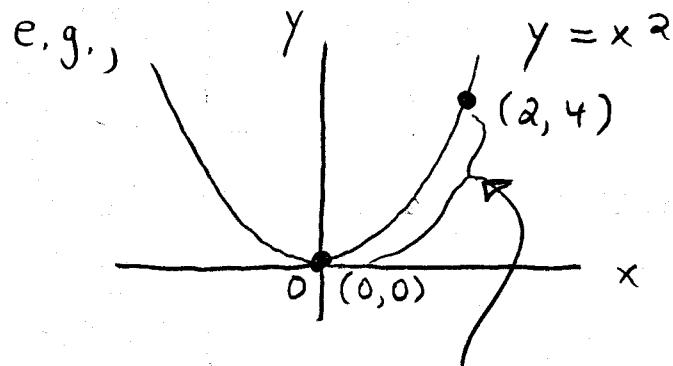
### Section 6.3. Arc Length

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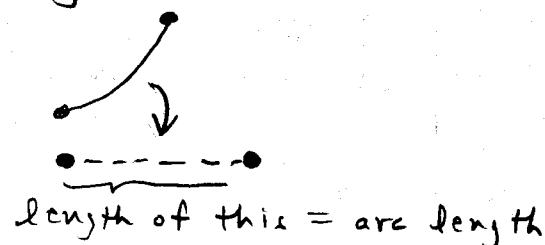
Thurs,

The definite integral  $\int_a^b f(x) dx$  can be used to measure the lengths of curves that are not necessarily straight lines.

Arc length = length of a segment of a curve (when the curve is straightened out)

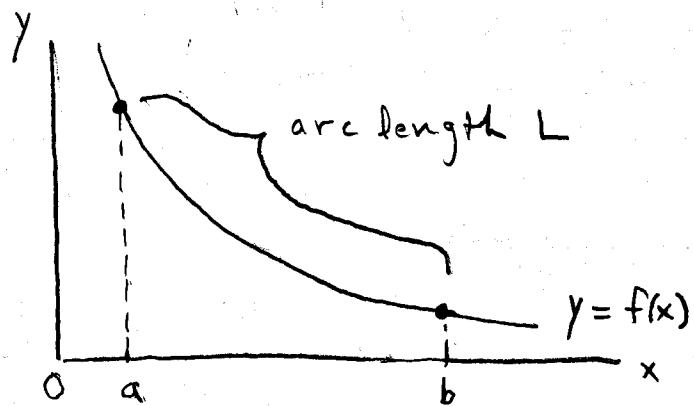


Q: How long would this curve segment be if it were stretched out to form a straight line?



### Rationale

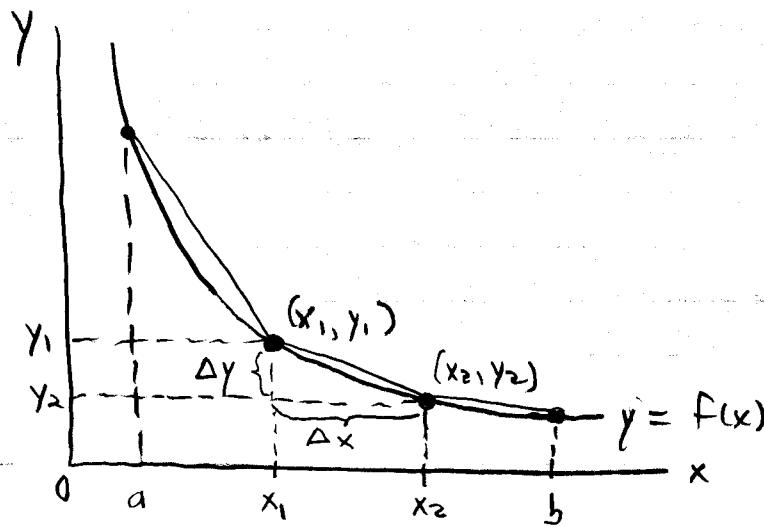
Let  $y = f(x)$  be a curve with arc length  $L$  from  $x=a$  to  $x=b$ :



Since we cannot actually take the curve segment from  $x=a$  to  $x=b$ , straighten it out, and then measure it, we need to do something else.

We approximate the curve by line segments, whose length we can easily measure.

E.g., we can divide the interval  $[a, b]$  into 3 subintervals ( $n=3$ ), where we have



$$\Delta y = y_2 - y_1 \quad \left\{ \begin{array}{l} \text{length} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ \Delta x = x_2 - x_1 \end{array} \right.$$

Will do some  
algebraic  
manipulations

$$= \sqrt{(\Delta x)^2 \left[ 1 + \frac{(\Delta y)^2}{(\Delta x)^2} \right]}$$

$$= \sqrt{1 + \left( \frac{\Delta y}{\Delta x} \right)^2} \cdot \Delta x$$

Then

$L = \text{SUM OF LENGTHS OF INFINITELY MANY LINE SEGMENTS } (n \rightarrow \infty)$

$$= \int_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$\frac{dy}{dx}$  is just the derivative of  $f(x)$ , or  $f'(x)$  !

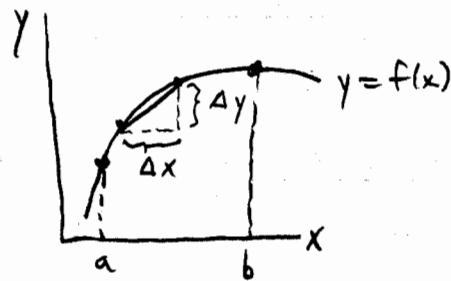
$$\boxed{L = \int_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx}$$

MEMORIZE

Unlike the integrals for AREA and VOLUME, you do not have to keep deriving this integral for each specific problem.

We actually have 2 formulas available to us to determine arc length :

① Curve given as  $y = f(x)$  (e.g.,  $y = x^2$ ) :



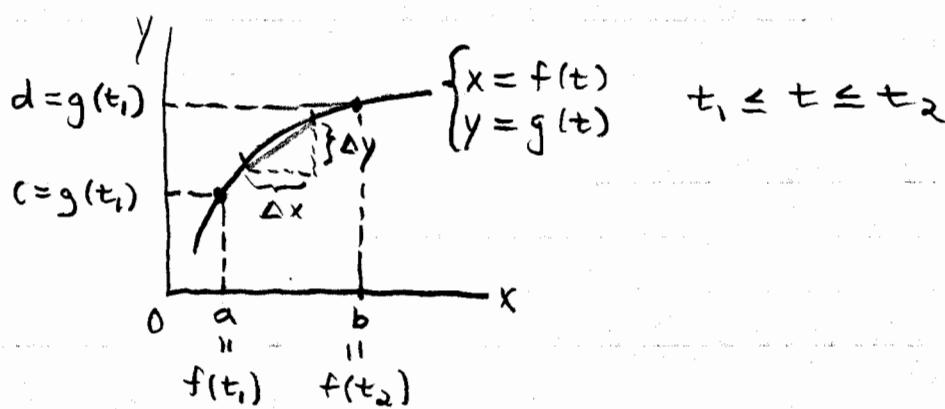
$$\text{length} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \cdot \Delta x \Rightarrow$$

**MEMORIZE**

$$\boxed{\text{ARC LENGTH} = \int_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx}$$

OMIT

② Curve given by something like  $x = y^2$  or  $x^3 + y^2 e^y = 1$ , which, in turn, should be represented by a parametric pair of equations  $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$  :



$$\begin{aligned} \text{length} &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \\ &= \sqrt{(\Delta t)^2 \left[ \frac{(\Delta x)^2}{(\Delta t)^2} + \frac{(\Delta y)^2}{(\Delta t)^2} \right]} \\ &= \sqrt{\left( \frac{\Delta x}{\Delta t} \right)^2 + \left( \frac{\Delta y}{\Delta t} \right)^2} \cdot \Delta t \Rightarrow \end{aligned}$$

MEMORIZE

$$\text{ARC LENGTH} = \int_{t=t_1}^{t=t_2} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} dt$$

Example.

1. Find the ARC LENGTH of the function

$$f(x) = \sqrt{4-x^2}$$

from  $x=0$  to  $x=2$ . (SEE Problem 3, HW#3.)

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First find  $f'(x) = \frac{dy}{dx} :$

$$\frac{dy}{dx} = \frac{d}{dx} \sqrt{4-x^2} = \frac{d}{dx} (4-x^2)^{1/2}$$

↓  
CHAIN RULE:

$$\frac{d}{dx} [g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

$$= \frac{1}{2}(4-x^2)^{-\frac{1}{2}} \cdot \frac{d}{dx}(4-x^2)$$

$$= \frac{1}{2}(4-x^2)^{-\frac{1}{2}} \cdot (-2x)$$

$$= -x(4-x^2)^{-\frac{1}{2}}$$

$$= -\frac{x}{\sqrt{4-x^2}}$$

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Then

$$\begin{aligned}\text{ARC LENGTH} &= \int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= \int_0^2 \sqrt{1 + \left(-\frac{x}{\sqrt{4-x^2}}\right)^2} dx \\ &= \int_0^2 \sqrt{1 + \frac{x^2}{4-x^2}} dx \\ &= \int_0^2 \sqrt{\frac{4-x^2}{4-x^2} + \frac{x^2}{4-x^2}} dx \\ &= \int_0^2 \sqrt{\frac{4-x^2+x^2}{4-x^2}} dx \\ &= \int_0^2 \sqrt{\frac{4}{4-x^2}} dx \\ &= \int_0^2 \frac{2}{\sqrt{4-x^2}} dx \\ &= 2 \int_0^2 \frac{1}{\sqrt{4-x^2}} dx\end{aligned}$$

↓ Use ENTRY #16 in TABLE OF INTEGRALS,

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1} \frac{u}{a} + C$$

↓ with  $a=2$  and  $u=x$

$$= 2 \sin^{-1}\left(\frac{x}{2}\right]_0^2$$

$$= 2 \sin^{-1}\left(\frac{2}{2}\right) - 2 \sin^{-1}\left(\frac{0}{2}\right)$$

$$= 2 \sin^{-1}(1) - 2 \sin^{-1}(0)$$

$$\downarrow \quad \sin \theta = 1 \Rightarrow \theta = \frac{\pi}{2} \Rightarrow \sin^{-1}(1) = \frac{\pi}{2}$$

$$\sin \theta = 0 \Rightarrow \theta = 0 \Rightarrow \sin^{-1}(0) = 0$$

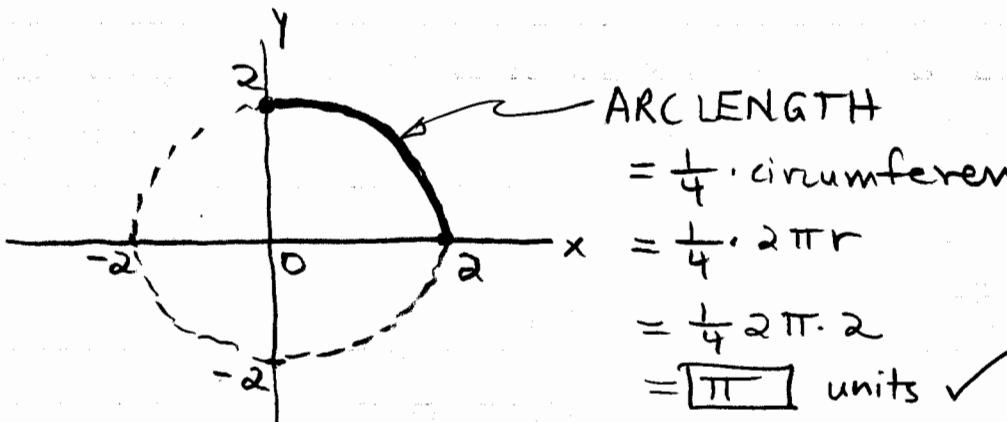
$$\downarrow = 2\left(\frac{\pi}{2}\right) - 2(0)$$

$$= \boxed{\pi} \text{ units}$$

NOTE : The graph of the curve  $y = \sqrt{4 - x^2}$  from  $x = 0$  to  $x = 2$  is the quarter circle with center  $(0, 0)$  and radius 2.

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Thus,



2. Find the ARC LENGTH of the function

$$f(x) = 1 - x$$

from  $x=0$  to  $x=1$ . (SEE Problem 4, HW #3.)

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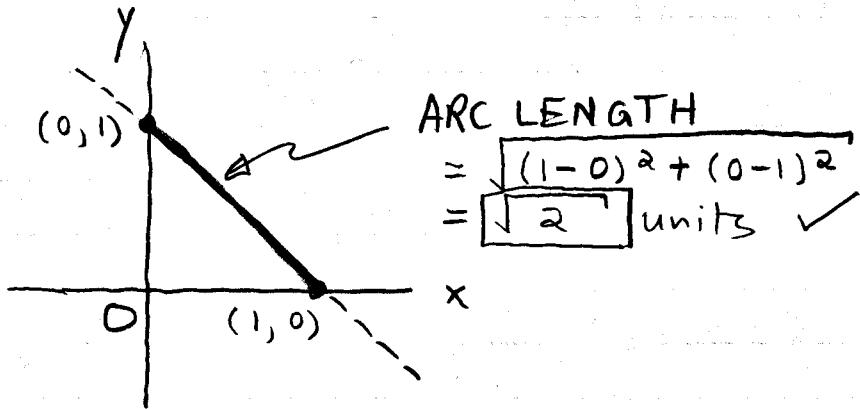
First find  $f'(x) = \frac{dy}{dx}$ :

$$\frac{dy}{dx} = \frac{d}{dx}(1-x) = -1$$

Then

$$\begin{aligned}\text{ARC LENGTH} &= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= \int_0^1 \sqrt{1 + (-1)^2} dx \\ &= \int_0^1 \sqrt{2} dx \\ &= \sqrt{2} \times \Big[ x \Big]_0^1 \\ &= \sqrt{2} \cdot 1 - \sqrt{2} \cdot 0 \\ &= \boxed{\sqrt{2}} \text{ units } \checkmark\end{aligned}$$

NOTE: The graph of the curve  
 $y = 1 - x$  from  $x = 0$  to  $x = 1$   
is the line segment from  
(0, 1) to (1, 0),



3.

(PP #3, Exercise 6, p. 471.)

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Graph the curve and find its exact length:

Thurs.

$$x = e^t + e^{-t}, \quad y = 5 - 2t, \quad 0 \leq t \leq 3$$

We will use the formula

$$\text{ARC LENGTH} = \int_{t=t_1}^{t=t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

First find  $\frac{dx}{dt}$  and  $\frac{dy}{dt}$ :

$$\textcircled{1} \quad \frac{dx}{dt} = \frac{d}{dt}(e^t + e^{-t}) = e^t - e^{-t}$$

CHAIN RULE says

$$\frac{d}{dt}(e^{at}) = ae^{at} \Rightarrow$$

$$\begin{aligned} \frac{d}{dt}(e^{-t}) &= \frac{d}{dt}(e^{(-1)t}) \\ &= (-1)e^{(-1)t} \\ &= -e^{-t} \end{aligned}$$

$$\textcircled{2} \quad \frac{dy}{dt} = \frac{d}{dt}(5 - 2t) = -2$$

Then

$$\begin{aligned} \text{ARC LENGTH} &= \int_{t=0}^{t=3} \sqrt{(e^t - e^{-t})^2 + (-2)^2} dt \\ &= \int_0^3 \sqrt{(e^t)^2 - 2e^t \cdot e^{-t} + (e^{-t})^2 + 4} dt \\ &\quad \left| \begin{array}{l} -(a^m)^n = a^{mn} = a^{nm} \Rightarrow \\ (e^t)^2 = e^{2t} \text{ and } (e^{-t})^2 = e^{-2t} \end{array} \right. \\ &\quad \left| \begin{array}{l} a^m \cdot a^n = a^{m+n} \Rightarrow \\ e^t \cdot e^{-t} = e^{t-t} = e^0 = 1 \end{array} \right. \\ &\downarrow \\ &= \int_0^3 \sqrt{e^{2t} - 2 + e^{-2t} + 4} dt \\ &= \int_0^3 \sqrt{e^{2t} + 2 + e^{-2t}} dt \\ &= (e^t)^2 + 2e^t \cdot e^{-t} + (e^{-t})^2 \\ &= (e^t + e^{-t})^2 \\ &= \int_0^3 \sqrt{(e^t + e^{-t})^2} dt \\ &= \int_0^3 (e^t + e^{-t}) dt \\ &\quad \left| \begin{array}{l} \int e^{at} dt = \frac{e^{at}}{a} + C \Rightarrow \\ \int e^{-t} dt = \frac{e^{-t}}{-1} + C \end{array} \right. \\ &\downarrow \end{aligned}$$

$$\begin{aligned} & \downarrow \\ & = e^t + \left[ \frac{e^{-t}}{-1} \right]_0^3 \\ & = e^t - e^{-t} \Big|_0^3 \\ & = (e^3 - e^{-3}) - (e^0 - e^{-0}) \\ & = e^3 - e^{-3} \quad \cancel{-1+X} \\ & = \boxed{e^3 - e^{-3}} \text{ units} \end{aligned}$$

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Thurs.

$$\approx \boxed{20.0357} \text{ units}$$

