Lecture

Section 6.2. Volumes.

3/21/02 Instead of measuring 2-dimensional areas, thus we are going to measure 3-dimensional volumes using one of two techniques:

I. Disk/Washer Method
   - a "slicing" method

II. Cylindrical Shell Method
    - alternative to slicing method when inconvenient to use slicing method
Introduction

Your text starts out by measuring

VOLUME BY THE METHOD
OF CROSS SECTIONS

We will start out by measuring

VOLUME OF A SOLID OF
REVOLUTION AROUND THE
x-axis... OR y-axis

We will take a look at two basic techniques of measuring a solid of revolution:

I. Disk or Washer (= Annular Ring)

Method

This technique is used to measure the volume of solids that can be created in many ways, including:

A. A region (in the xy-plane) between 2 curves is
revolved around a horizontal or vertical line (like the x-axis or y-axis) and no "hole" in the solid is produced:

\[ y = f(x) \text{ or } x = f(y) \]

The solid of revolution is then sliced up into disks, whose volumes are actually measured:

\[ \Delta x \]

B. A region (in the xy-plane) between 2 curves is revolved around a horizontal or vertical line (like the x- or y-axis) and a "hole" in the solid is produced:
The solid of revolution is then sliced up into washers \((=\) annular rings \(=\) disks with holes in them\):
II. Cylindrical Shell Method (When the Disk or Washer Method Fails)

This technique is used to measure the volume of solids when the disk/washer method cannot be used, i.e., when one cannot solve for $x$ given a curve $y = f(x)$, and one needs to because the integral one must use is of the form

$$\int_{y=c}^{y=d} F(y) \, dy$$

or one cannot solve for $y$ given a curve $x = f(y)$, and one needs to because the integral one must use is of the form

$$\int_{x=a}^{x=b} F(x) \, dx$$
Ultimately, one does not "slice up" the solid of revolution, but "cuts it up" into concentric cylinders (like tin cans with a little thickness to them):
Example of a cylinder

hollow
some thickness to sides
Example of Washer vs. Cylindrical Shell Method.

Washer

\[ y = x^2 \quad y = \sqrt{x} \quad (1, 1) \]

Revolve around \( y \)-axis

\[ r_1 - r_2 = x_1 - x_2 = 3\sqrt{y} - y^2 \]

\[ VOLUME = \pi \int_0^1 (r_1^2 - r_2^2) \, dy = \pi \int_0^1 (3y^{3/2} - y^4) \, dy \]
Cylindrical

\[ y = \sqrt{x}, \quad y = x^2 \]

\[ h = y_1 - y_2 = \sqrt{x^3} - x^3 \]

\[ C = 2\pi r = 2\pi x \]

\[ h = \sqrt{x^3} - x^3 \]

\[ = x(x^{\frac{3}{2}} - x) = x^{\frac{5}{2}} - x^4 \]

\[ \text{Volume of disk} = \pi r^2 h \Delta x = 2\pi x (\sqrt{x^3} - x^3) \Delta x \]

\[ \text{Volume of solid} = \int_0^1 2\pi (x^{\frac{3}{2}} - x^4) \, dx \]
I. Disk/Washer Method

Examples Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

1. \( y = \sqrt{x}, \ x = 1, \ y = 0 \); about the x-axis.
   (See Problem 2(a), (b), HW #3.)

\[
\begin{align*}
&y = \sqrt{x} \\
&x = 1 \\
&y = 0
\end{align*}
\]
Since rotation is about \( x \)-axis, slice \( \text{VOLUME} \) up into \( \text{DISKS} \) perpendicular to \( x \)-axis:

\[
Y
\]

\[
\text{slice from } x = 0 \text{ to } x = 1
\]

\[
\begin{align*}
 h &= \Delta x \\
 r &= y = \sqrt{x} \\
 \text{Volume of Disk} &= \text{Volume of Cylinder} \\
 &= \pi r^2 h \\
 &= \pi (\sqrt{x})^2 \Delta x \\
 &= \pi x \Delta x
\end{align*}
\]

\[
\Rightarrow \text{Volume of Solid} = \sum_{i=1}^{n} \pi x_i \Delta x \quad \xrightarrow{\text{as } n \to \infty} \int_{0}^{1} \pi x \, dx
\]
\[ \text{VOLUME OF SOLID} \]
\[ = \int_0^1 \pi x \, dx \]
\[ = \pi \left[ \frac{x^3}{2} \right]_0^1 \]
\[ = \frac{\pi}{2} \left( 1^3 - 0^3 \right) \]
\[ = \frac{\pi}{2} \text{ units}^3 \]

3/21/02
Thus,

\[ \boxed{\frac{\pi}{2} \text{ units}^3} \]
Will show later on.

Same as $H$

Annulus:

Perpendicular to $y$-axis.

Slice volume up into annular or washers.

Since relation is now about $y$-axis.

$3/25/02$

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Volume of Annulus

= volume of outer cylinder − volume of inner cylinder

= \pi r_1^2 h - \pi r_2^2 h

= \pi (r_1^2 - r_2^2) h
\[
\begin{align*}
&= \pi \left[ (1)^2 - (y^2)^2 \right] \Delta y \\
&= \pi \ (1 - y^4) \Delta y \\
&\text{\textsuperscript{\textdegree} \ VOLUME \ OF \ SOLID} \\
&= \sum \text{sum of volumes of infinitely many such annuli from } y = 0 \ TO \ y = 1 \\
&= \int_0^1 \pi (1 - y^4) \, dy \\
&= \pi \int_0^1 1 \, dy - \pi \int_0^1 y^4 \, dy \\
&= \pi y \bigg|_0^1 - \frac{\pi y^5}{5} \bigg|_0^1 \\
&= \left[ \pi(1) - \frac{\pi(1)^5}{5} \right] - \left[ \pi(0) - \frac{\pi(0)^5}{5} \right] \\
&= \pi - \frac{\pi}{5} = \frac{\pi}{5} - \frac{\pi}{5} = \boxed{\frac{4\pi}{5}} \ \text{units}^3
\end{align*}
\]
3. \( y = \sqrt{x}, \ x = 1, \ y = 0 \) about the line \( x = 1 \).
Since rotation is about the line $x=1$ (which is vertical and parallel to $y$-axis), slice VOLUME up into DISKS perpendicular to $y$-axis:

Slice from $y=0$ to $y=1$

欲将 $x$ 用 $y$ 表示，因 $y = \sqrt{x} \Rightarrow x = y^2$

$r = x_1 - x_2 = 1 - x = 1 - y^2$

$V_{\text{volume of disk}} = \pi r^2 h$

$= \pi (1 - y^2) \Delta y$

$= \pi (1 - 2y^2 + y^4) \Delta y$
VOLUME OF SOLID

\[
\text{Sum of volumes of infinitely many such disks from } y = 0 \text{ to } y = 1
\]

\[
= \int_0^1 \pi (1 - y^2 + y^4) \, dy
\]

\[
= \pi \int_0^1 1 \, dy - 2\pi \int_0^1 y^2 \, dy + \pi \int_0^1 y^4 \, dy
\]

\[
= \pi y - \frac{2\pi y^3}{3} + \frac{\pi y^5}{5} \bigg|_0^1
\]

\[
= \left[ \pi (1) - \frac{2\pi (1)^3}{3} + \frac{\pi (1)^5}{5} \right] - \left[ \pi (0) - \frac{2\pi (0)^3}{3} + \frac{\pi (0)^5}{5} \right]
\]

\[
= \pi - \frac{2\pi}{3} + \frac{\pi}{5}
\]

\[
= \frac{15\pi}{15} - \frac{10\pi}{15} + \frac{3\pi}{15} = \frac{8\pi}{15} \text{ units}^3
\]
II. Cylindrical Shell Method

Example. Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line:

\[ y = \sqrt{x}, \quad y = x^2 \] about the y-axis.

(See Problem 2(a), HW #3.)
Since rotation is about $y$-axis, can slice \textit{volume} up into \textit{annuli} or \textit{washers} perpendicular to $y$-axis.

Slice from $y = 0$ to $y = 1$.

Want $x$ in terms of $y$:

- $y = \sqrt{x} \Rightarrow x = y^2$
- $y = x^2 \Rightarrow x = \pm \sqrt{y} = \sqrt{y}$

$r_2 = x_2 = x = y^2$

$r_1 = x_1 = x = \sqrt{y}$

$h = \Delta y$
ALTERNATIVELY, and especially when cannot express \( x \) in terms of \( y \) for the radii (e.g., \( y = x^2 + x^3 \Rightarrow x = \) ?? ), can create CYLINDRICAL SHELLS rather than slice volume:

\[ y = \sqrt{x} \]

\[ y = x^2 \]

\[ w = \Delta x \]

\[ r = x \]

Right edge of cylinder moves from \( x = 0 \) to \( x = \) 

\[ h = y_1 - y_2 = \sqrt{x} - x^2 \]

VOLUME OF CYLINDRICAL SHELL

\[ = \text{surface area} \times \text{width} \]

\[ = 2\pi rh \Delta x \]
\[ \begin{align*}
\text{Volume of Solid} &= \text{Sum of volumes of infinitely many such cylindrical shells from } x = 0 \text{ to } x = 1 \\
&= \int_0^1 2\pi x (x^{\frac{3}{2}} - x^2) \, dx \\
&= 2\pi \left[ \int_0^1 x^{\frac{3}{2}} \, dx - \int_0^1 x^3 \, dx \right] \\
&= 2\pi \left[ \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{x^4}{4} \right]_0^1 \\
&= \left( \frac{4\pi}{5} - \frac{\pi}{2} \right) - \left( 0 - 0 \right) 
\end{align*} \]
\[264a - \] 

\[= \frac{8\pi}{10} - \frac{5\pi}{10} = \boxed{\frac{3\pi}{10}} \text{ units}^2\]
Discussion of Previous Example:

We summarize using the washer method vs. the cylindrical shell method on finding the solid of revolution of the region bounded by the curves $y = 3x^2$ and $y = x^2$ for $0 \leq x \leq 1$. The region is rotated about the y-axis.
Washer

\[ y = r^2 \Rightarrow y^2 = x \]

\[ r_2 = x_2 = (y^2) \]

\[ r_1 = x_1 = \sqrt{y} \]

\[ y = x^2 \Rightarrow \sqrt{y} = x \]

Now, if I had chosen to do the example

\[ y = \sqrt{x^2 + 2x} \]

\[ r_1 = x_1 = ? \]

\[ r_2 = x_2 = ? \]

\[ y = x^2 + x^3 \]

Solve for \( x \) in

\[ y = x^2 + x \]

Then we would have had no choice but to use the cylindrical method, which allows one to keep everything in terms of \( x \).
In any event, the results we obtained were:

\[
\int_0^1 \pi (y - y^4) \, dy = \frac{\pi y^2}{2} - \frac{\pi y^5}{5} \bigg|_0^1
\]

\[
= \frac{\pi}{2} - \frac{\pi}{5}
\]

\[
= \frac{5 \pi}{10} - \frac{2 \pi}{10}
\]

\[
= \frac{3 \pi}{10}
\]
\[ 2\pi r h \]

\[ \int_0^1 2\pi x \left( \sqrt{x^2 - x^3} \right) \, dx \]

\[ = \int_0^1 2\pi \left( x^\frac{1}{2} - x^3 \right) \, dx \]

\[ = 2\pi \left( \frac{2}{5} x^{\frac{5}{2}} - \frac{x^4}{4} \right) \bigg|_0^1 \]

\[ = \frac{4\pi}{5} - \frac{\pi}{2} \]

\[ = \frac{8\pi}{10} - \frac{5\pi}{2} \]

\[ = \frac{3\pi}{10} \]