Lecture

Section 6.2. Volumes.

3/21/02 Thurs. Instead of measuring 2-dimensional areas, we are going to measure 3-dimensional volumes using one of two techniques:

I. Disk/Washer Method
- a "slicing" method

II. Cylindrical Shell Method
- alternative to slicing method when inconvenient to use slicing method

Introduction

Your text starts out by measuring

VOLUME BY THE METHOD OF CROSS SECTIONS

We will start out by measuring

VOLUME OF A SOLID OF REVOLUTION AROUND THE x-AXIS OR y-AXIS

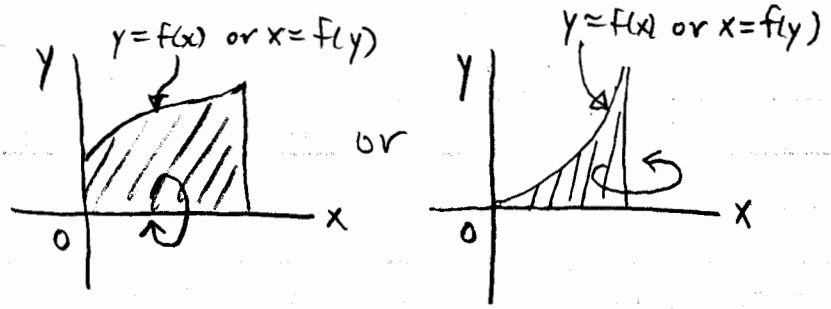
We will take a look at two basic techniques of measuring a solid of revolution:

I. <u>Disk or Washer (= Annular Ring) Method</u>

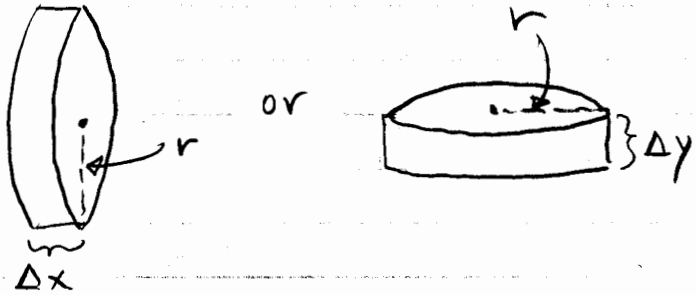
This technique is used to measure the <u>volume of solids</u> that can be created in many ways, including:

A. A region (in the xy-plane) between 2 curves is

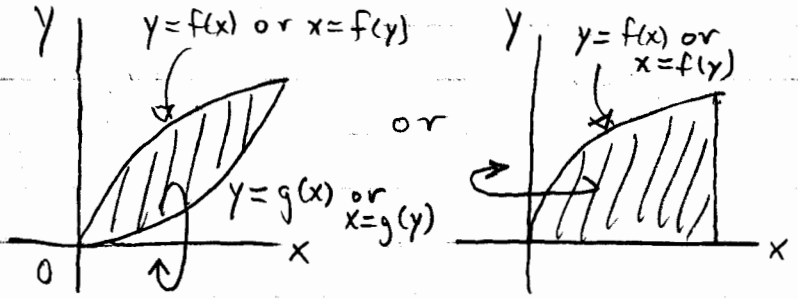
revolved around a horizontal or vertical line (like the x-axis or y-axis and no "hole" in the solid is produced:



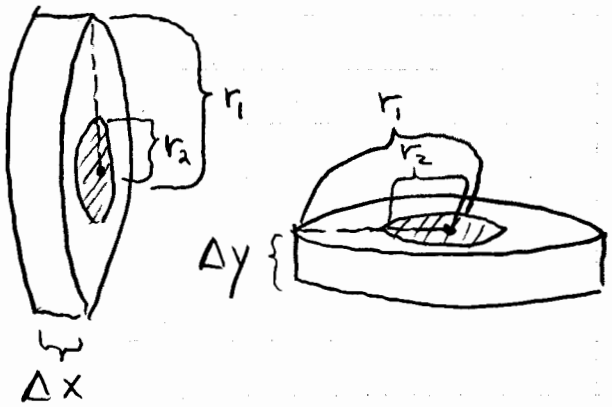
The <u>solid of revolution</u> is then sliced up into <u>disks</u>, whose volumes are actually measured:



B. A region (in the xy-plane) between 2 curves is revolved around a horizontal or vertical line (like the x- or y-axis) and a "hole" in the solid is produced:



The <u>solid of revolution</u> is then sliced up into washers (= annular rings = disks with holes in them):



II. Cylindrical Shell Method (When the Disk or Washer Method Fails)

This technique is used to measure the volume of solids when the Disk/Washer Method cannot be used, i.e., when one cannot solve for x given a curve

$$y = f(x),$$

and one needs to because the integral one must use is of the form

$$\int_{y=c}^{y=d} F(y)\, \underline{\underline{dy}}\ ;$$

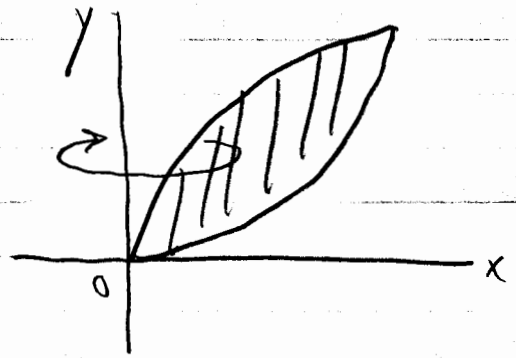
or one cannot solve for y given a curve

$$x = f(y),$$

and one needs to because the integral one must use is of the form

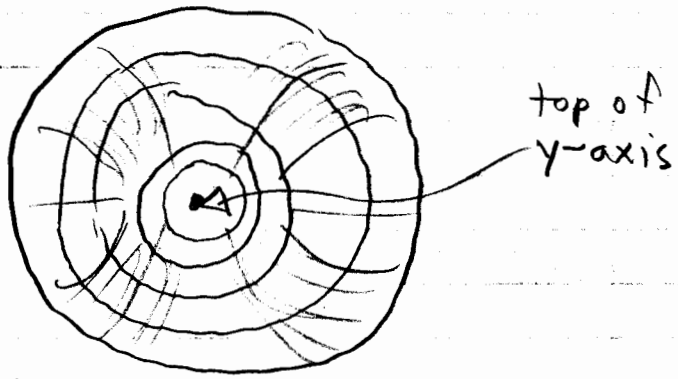
$$\int_{x=a}^{x=b} F(x)\, \underline{\underline{dx}}.$$

Ultimately, one does not "slice up" the solid of revolution, but "cuts it up" into concentric cylinders (like tin cans with a little thickness to them):



$\Downarrow$

<u>Looking down into solid cut up into concentric cylinders</u>



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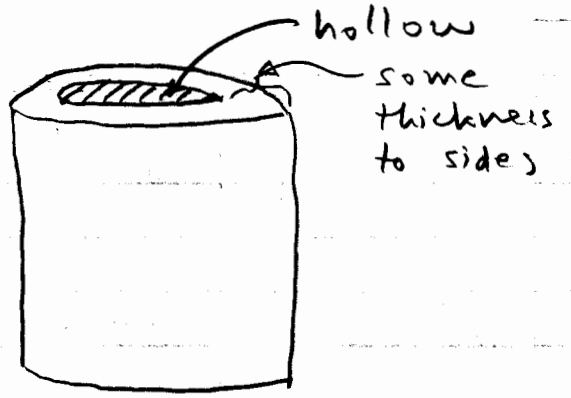
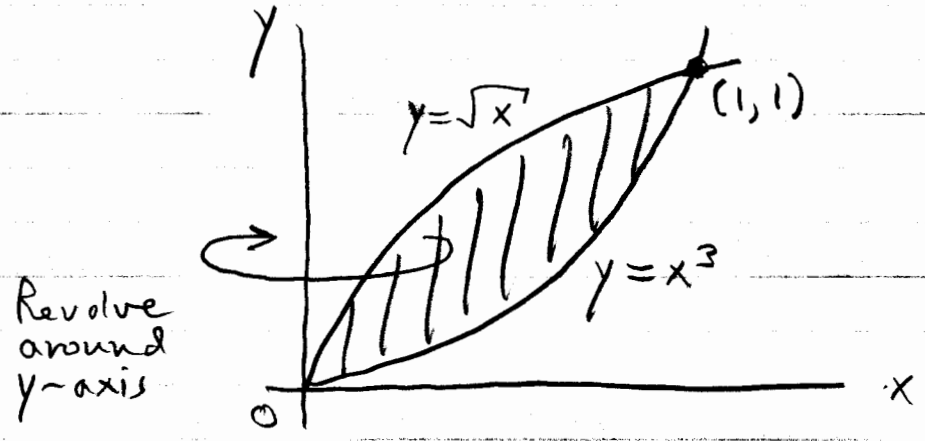
Example of a cylinder

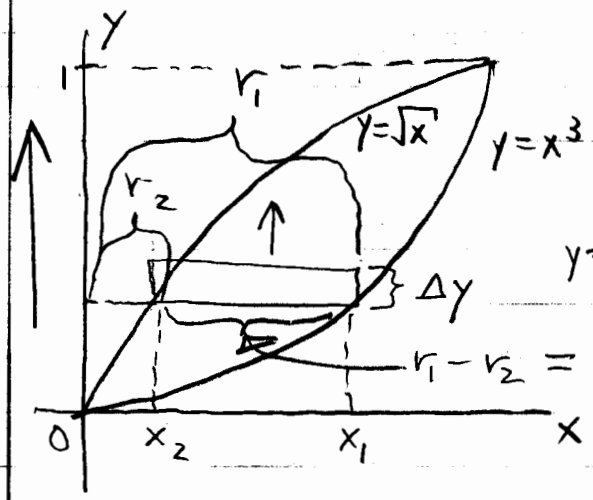


SEE pp. 261 - 265

Example of Washer vs. Cylindrical Shell Method.



Washer



$\Rightarrow x = \sqrt[3]{y}$ $\quad$ $\Rightarrow x = y^2$

$y = x^3$ $\quad$ $y = \sqrt{x}$

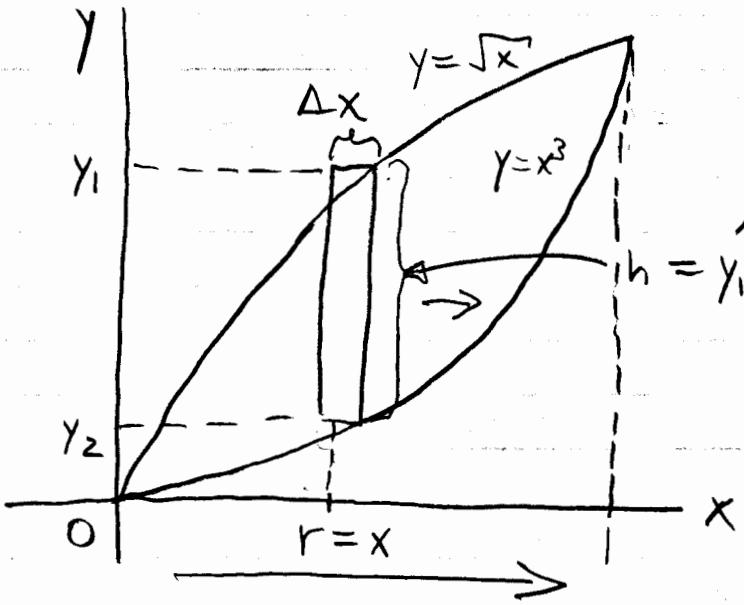
$$r_1 - r_2 = x_1 - x_2 = \overset{=y^{1/3}}{\sqrt[3]{y}} - y^2$$

$$\Downarrow \quad r_1 - r_2 = \overset{=y^{1/3}}{\sqrt[3]{y}} - y^2$$

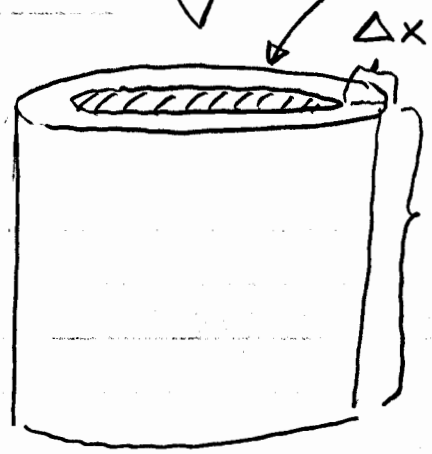
$$\text{VOLUME of disk} = \pi r_1^2 h - \pi r_2^2 h$$
$$= \pi y^{2/3} \Delta y - \pi y^4 \Delta y$$

$$\therefore \text{VOLUME of solid} = \int_0^1 \pi (y^{3/2} - y^4)\, dy$$

Cylindrical



$$h = \underbrace{y_1}_{=\sqrt{x}} - \underbrace{y_2}_{=x^3} = \sqrt{x} - x^3$$



Circumference $= 2\pi r = 2\pi x$

$$h = \underbrace{\sqrt{x}}_{=x^{1/2}} - x^3$$

$$= x(x^{1/2} - x^3) = x^{3/2} - x^4$$

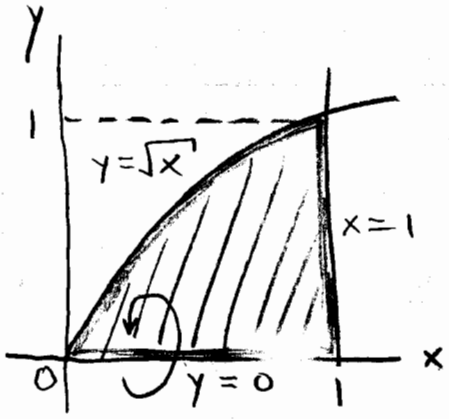
$$\text{VOLUME of disk} = 2\pi r h \Delta x = 2\pi \overbrace{x(\sqrt{x} - x^3)} \Delta x$$

$$\therefore \text{VOLUME of solid} = \int_0^1 2\pi\left(x^{3/2} - x^4\right) dx$$

I. Disk / Washer Method

<u>Examples</u> Find the volume of the SOLID obtained by rotating the REGION bounded by the given CURVES about the specified LINE.

1. $y = \sqrt{x}$, $x = 1$, $y = 0$; about the x-axis.
(SEE Problem 2(a), (b), HW #3.)



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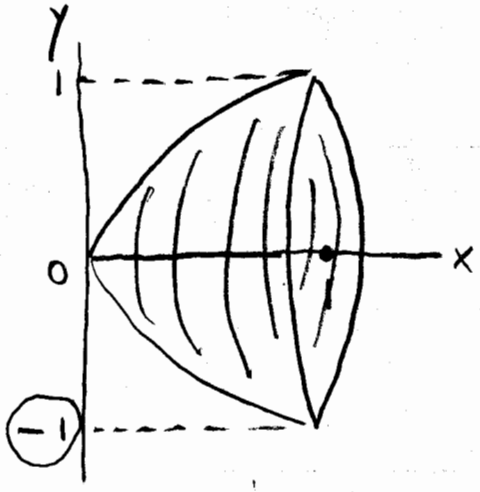
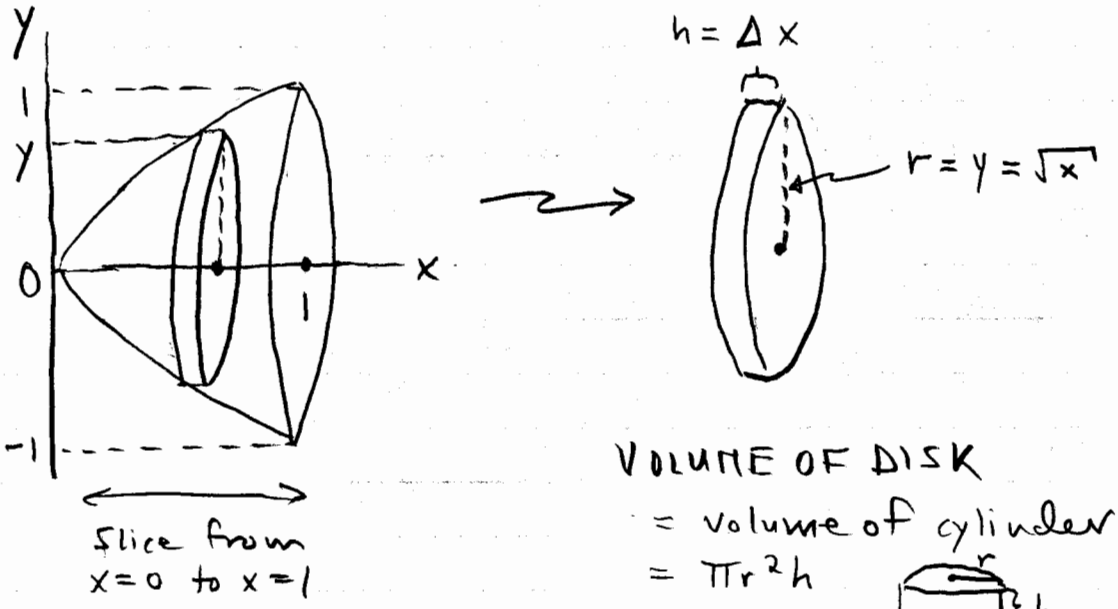


Since rotation is about x-axis, slice VOLUME up into DISKS <u>perpendicular to x-axis</u>:



VOLUME OF DISK

= volume of cylinder

$= \pi r^2 h$

$= \pi (\sqrt{x})^2 \Delta x$

$= \pi x \Delta x$

∴ VOLUME OF SOLID

≈ SUM OF VOLUMES OF n DISKS FROM x = 0 to x = 1

$$= \sum_{i=1}^{n} \pi x_i \Delta x \xrightarrow[\text{as } n \to \infty]{} \int_0^1 \pi x \, dx$$

$\therefore$ VOLUME OF SOLID

$$= \int_0^1 \pi x \, dx$$

$$= \pi \left(\frac{x^2}{2} \right) \Big]_0^1$$

$$= \frac{\pi x^2}{2} \Big]_0^1$$

$$= \frac{\pi (1)^2}{2} - \frac{\pi (0)^2}{2}$$

3/21/02
Thurs.

$$= \boxed{\frac{\pi}{2}} \text{ units}^3$$

[Be more explicit in instructions.]

2. $y = \sqrt{x}$, $x = 1$, $y = 0$; about the y-axis.
(SEE Problem 2(c), HW #3.)

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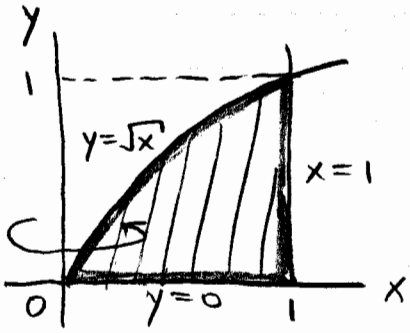

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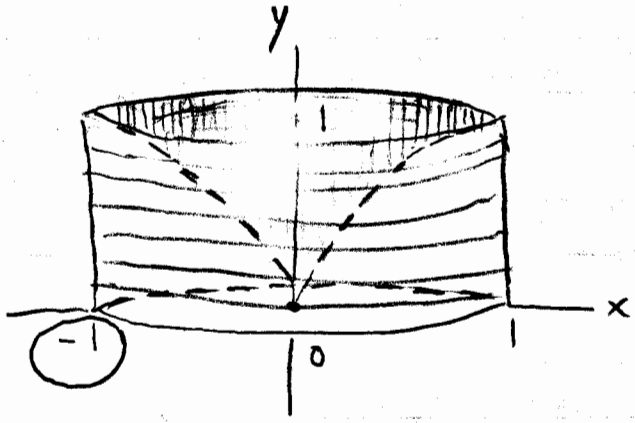

Since rotation is now about y-axis, slice VOLUME up into ANNULI or WASHERS <u>perpendicular to y-axis</u>:

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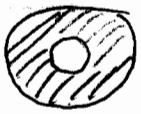
ANNULUS = 
(same as WASHER)

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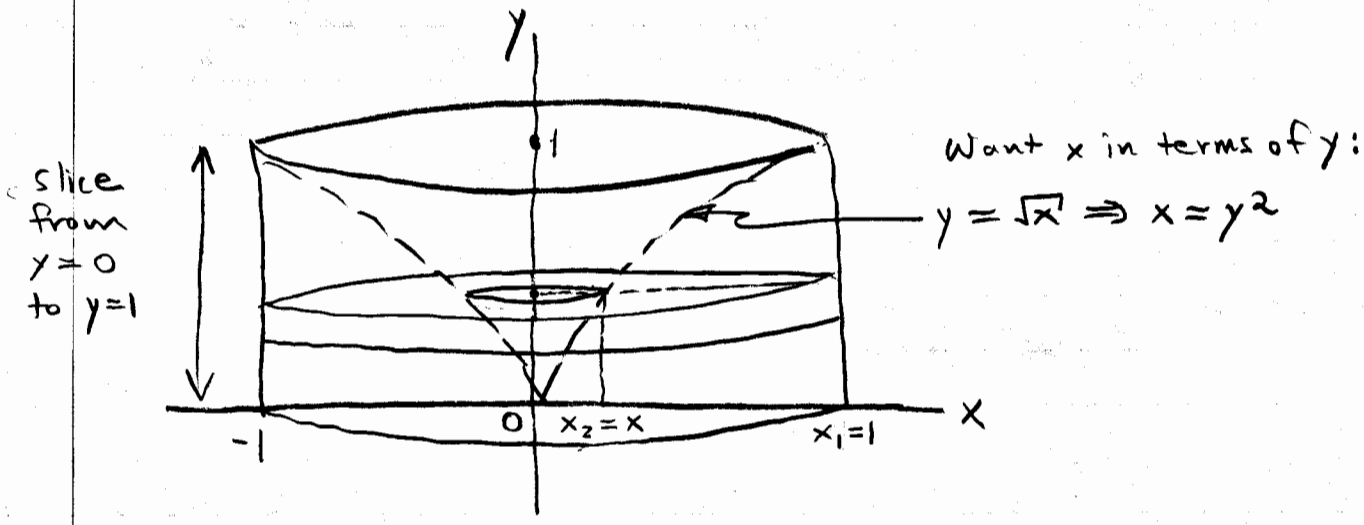
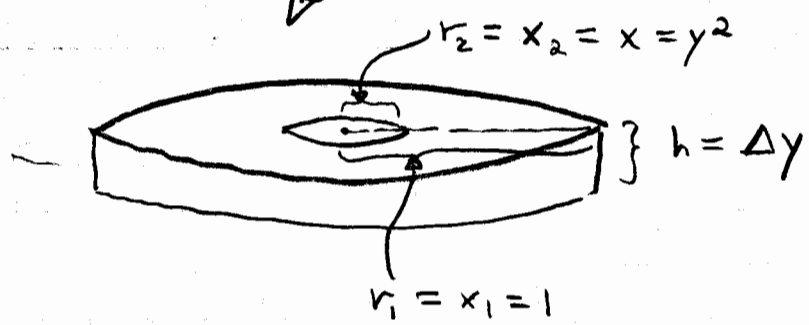
Will show how to go directly from region to schematic to integral





VOLUME OF ANNULUS
= volume of outer cylinder −
volume of inner cylinder

$$= \pi r_1^2 h - \pi r_2^2 h$$

$$= \pi (r_1^2 - r_2^2) h$$

$$= \pi\left[(1)^2 - (y^2)^2\right]\Delta y$$

$$= \pi(1 - y^4)\,\Delta y$$

$\therefore$ VOLUME OF SOLID

= SUM OF VOLUMES OF INFINITELY MANY SUCH ANNULI FROM $y = 0$ TO $y = 1$

$$= \int_0^1 \pi(1 - y^4)\,dy$$

$$= \pi\int_0^1 1\,dy - \pi\int_0^1 y^4\,dy$$

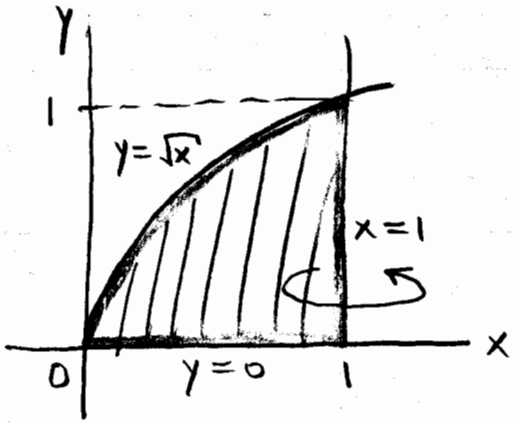
$$= \pi y - \frac{\pi y^5}{5}\Big]_0^1$$

$$= \left[\pi(1) - \frac{\pi(1)^5}{5}\right] - \left[\pi(0) - \frac{\pi(0)^5}{5}\right]$$

$$= \pi - \frac{\pi}{5} = \frac{5\pi}{5} - \frac{\pi}{5} = \boxed{\frac{4\pi}{5}}\ \text{units}^3$$

[Be more explicit in instructions.]

3. $y=\sqrt{x}$, $x=1$, $y=0$; about the line $x=1$.



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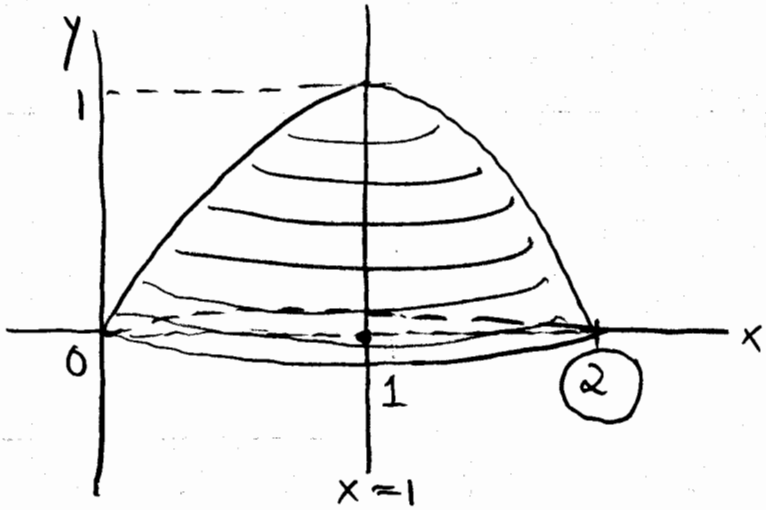
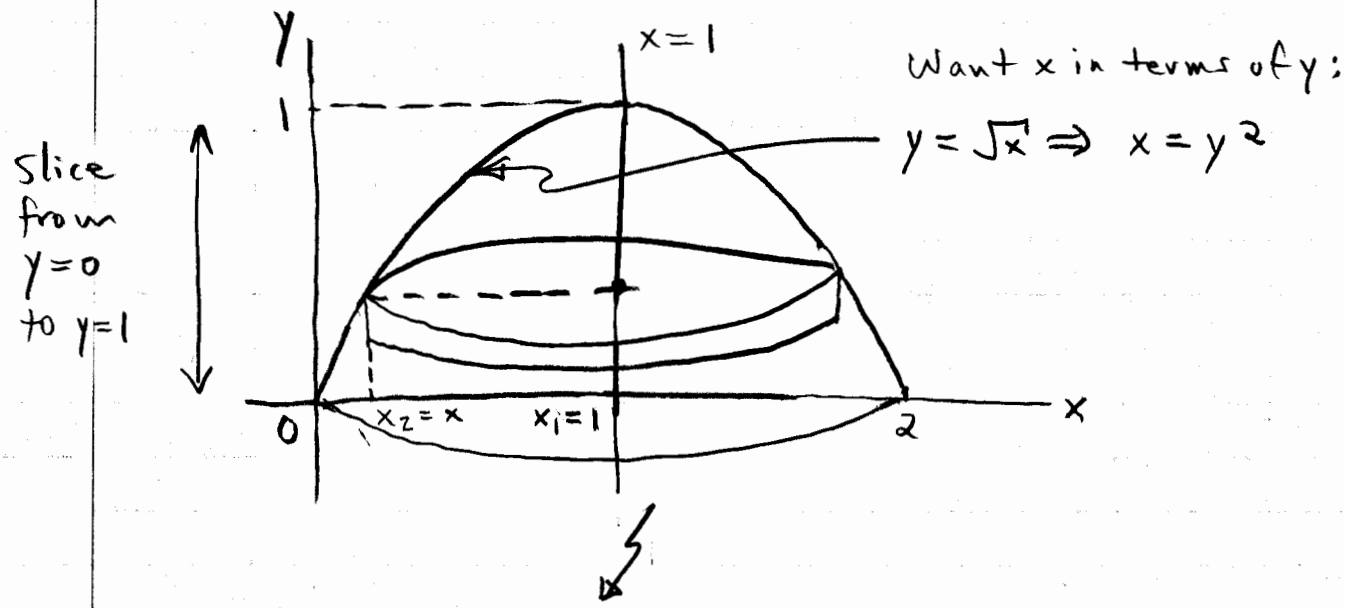


Since rotation is about the line $x=1$ (which is vertical and parallel to y-axis), slice VOLUME up into DISKS <u>perpendicular to y-axis</u>:



$r = x_1 - x_2 = 1 - x = 1 - y^2$

VOLUME OF DISK

$$= \pi r^2 h$$
$$= \pi (1-y^2)\,\Delta y$$
$$= \pi (1 - 2y^2 + y^4)\,\Delta y$$

$\therefore$ VOLUME OF SOLID

= SUM OF VOLUMES OF INFINITELY MANY SUCH DISKS FROM $y=0$ TO $y=1$

$$= \int_0^1 \pi(1-2y^2+y^4)\,dy$$

$$= \pi\int_0^1 1\,dy - 2\pi\int_0^1 y^2\,dy + \pi\int_0^1 y^4\,dy$$

$$= \pi y - \frac{2\pi y^3}{3} + \frac{\pi y^5}{5}\Big]_0^1$$

$$= \left[\pi(1) - \frac{2\pi(1)^3}{3} + \frac{\pi(1)^5}{5}\right] - \left[\pi(0) - \frac{2\pi(0)^3}{3} + \frac{\pi(0)^5}{5}\right]$$

$$= \pi - \frac{2\pi}{3} + \frac{\pi}{5}$$

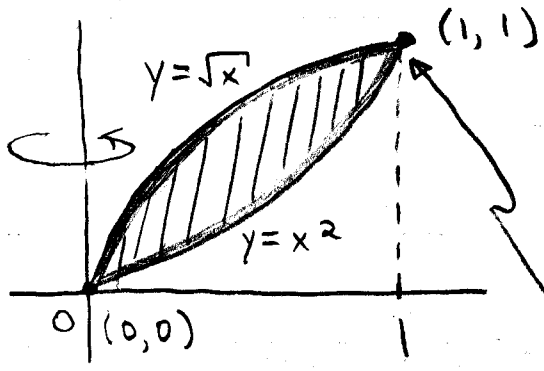
$$= \frac{15\pi}{15} - \frac{10\pi}{15} + \frac{3\pi}{15} = \boxed{\frac{8\pi}{15}} \text{ units}^3$$

II. Cylindrical Shell Method

Example. Find the volume of the SOLID obtained by rotating the REGION bounded by the given CURVES about the specified line:

[Be more explicit in instructions.]

$y = \sqrt{x}$, $y = x^2$; about the y-axis.
(SEE Problem 2(d), HW #3.)



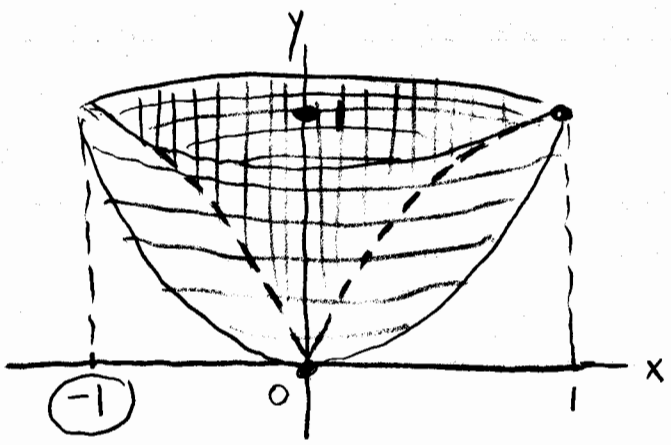
SET $y=\sqrt{x}$ EQUAL TO $y=x^2$ to obtain x- and y-coordinates of points of intersection:

$$y = y \Rightarrow$$
$$\sqrt{x} = x^2 \Rightarrow$$
$$(\sqrt{x})^2 = (x^2)^2 \Rightarrow$$
$$x = x^4 \Rightarrow$$
$$x^4 - x = 0 \Rightarrow$$
$$x(x^3 - 1) = 0$$

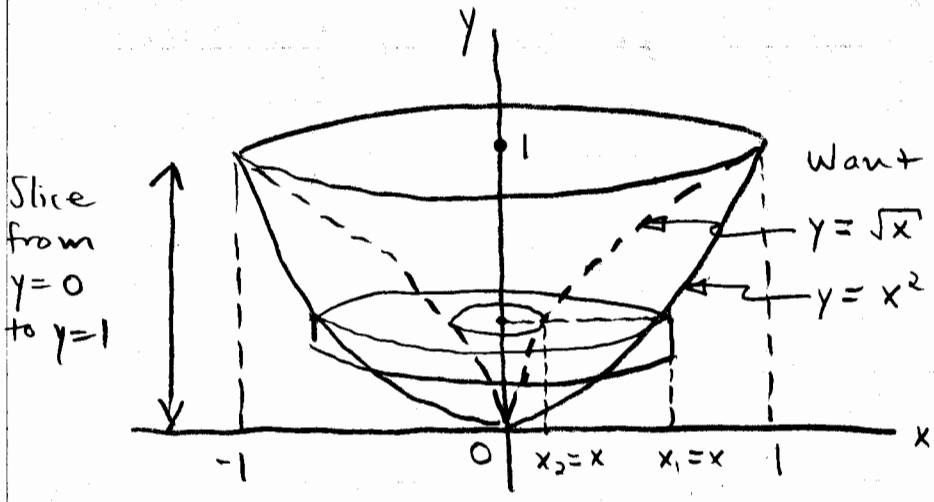
$x = 0$ $\quad x^3 - 1 = 0 \Rightarrow$
$x^3 = 1 \Rightarrow$
$x = 1$

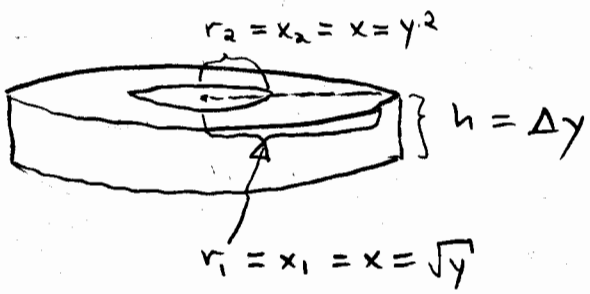
$y = \sqrt{0} = 0 \Rightarrow y = 0$

$y = \sqrt{1} = 1 \Rightarrow y = 1$

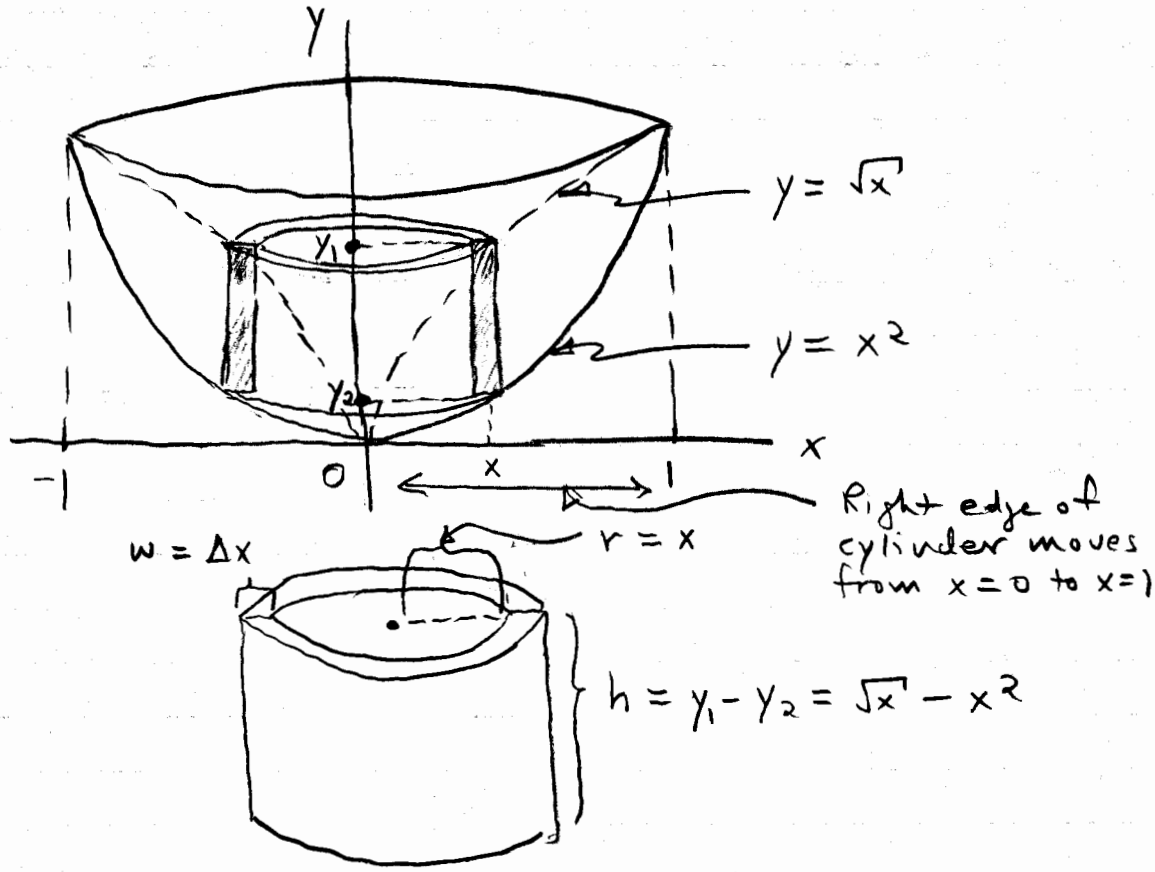


Since rotation is about y-axis, can slice VOLUME up into ANNULI or WASHERS <u>perpendicular to y-axis</u>:

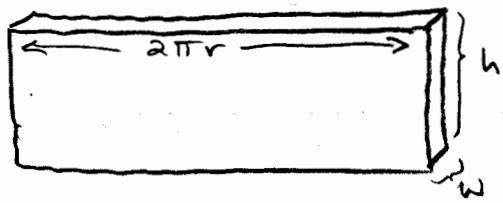




ALTERNATIVELY, and especially when cannot express x in terms of y for the radii (e.g., $y = x^2 + x^3 \Rightarrow x = ???$), can create CYLINDRICAL SHELLS rather than slice volume:



VOLUME OF CYLINDRICAL SHELL
= surface area × width



$= 2\pi r h w$

$$= 2\pi x\,(\sqrt{x} - x^2)\,\Delta x$$

$$= 2\pi\, x\,(x^{1/2} - x^2)\,\Delta x$$

$$= 2\pi\,(x^{3/2} - x^3)\,\Delta x$$

$\therefore$ VOLUME OF SOLID

= SUM OF VOLUMES OF INFINITELY MANY SUCH CYLINDRICAL SHELLS FROM $x=0$ TO $x=1$

$$= \int_0^1 2\pi\,(x^{3/2} - x^3)\,dx$$

$$= 2\pi \int_0^1 x^{3/2}\,dx - 2\pi \int_0^1 x^3\,dx$$

$$= 2\pi \frac{x^{5/2}}{5/2} - 2\pi \frac{x^4}{4}\Big]_0^1$$

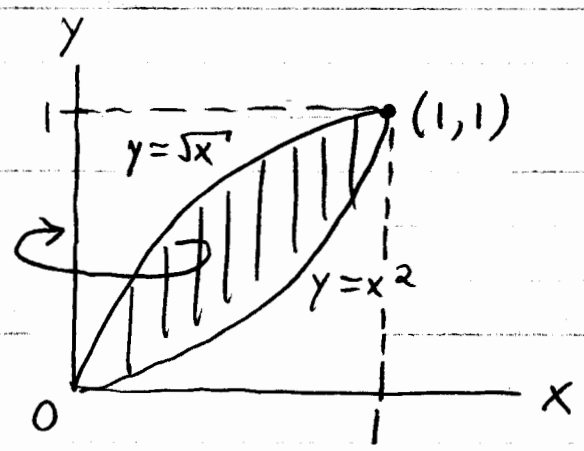
$$= \frac{4\pi x^{5/2}}{5} - \frac{\pi x^4}{2}\Big]_0^1$$

$$= \left(\frac{4\pi}{5} - \frac{\pi}{2}\right) - (0 - 0)$$

$$= \frac{8\pi}{10} - \frac{5\pi}{10} = \boxed{\frac{3\pi}{10}} \text{ units}^3$$

Discussion of Previous Example:

We summarize using the WASHER METHOD vs. the CYLINDRICAL SHELL METHOD on finding the SOLID OF REVOLUTION of



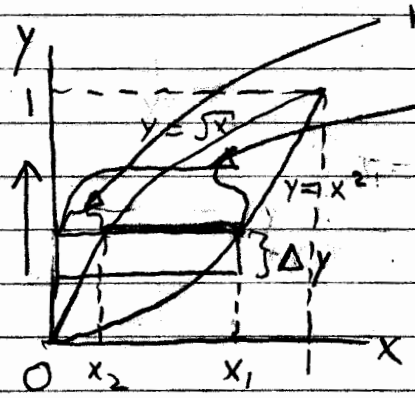
$\boxed{\text{Washer}}$

$y = \sqrt{x} \Rightarrow y^2 = x$

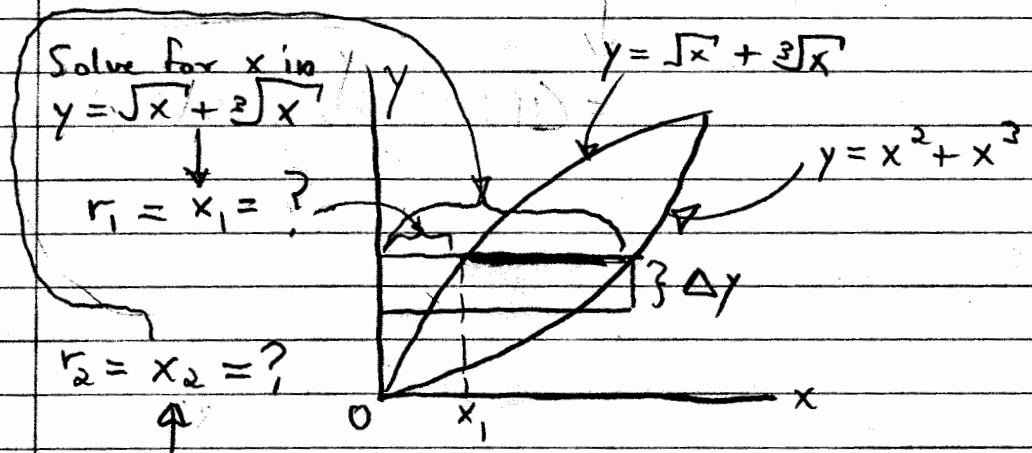
$r_2 = x_2 = y^2$

$r_1 = x_1 = \sqrt{y}$

$y = x^2 \Rightarrow \sqrt{y} = x$

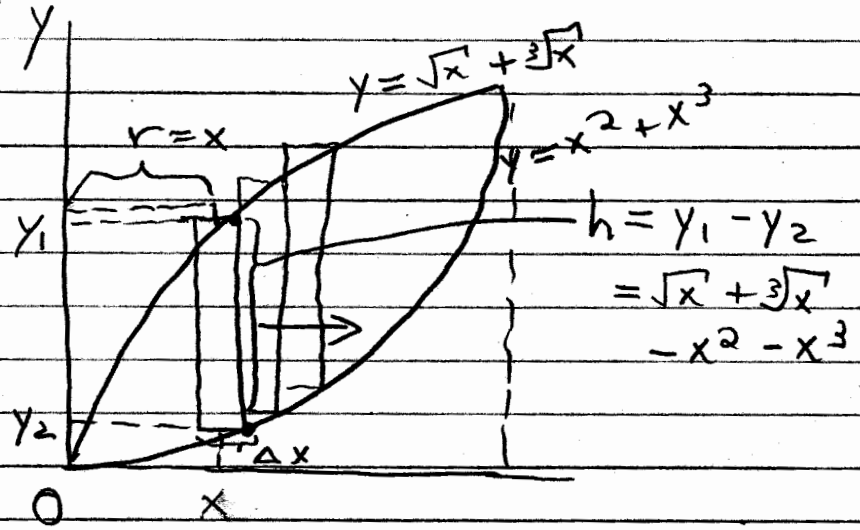


Now, if I had chosen to do the example



then we would have had no choice but to use the cylindrical method which allows one to keep everything in terms of x;

$\boxed{\text{Cylindrical}}$



In any event, the results we obtained were;

$$\int_0^1 \pi(y-y^4)\,dy = \frac{\pi y^2}{2} - \frac{\pi y^5}{5}\Big]_0^1$$

$$= \frac{\pi}{2} - \frac{\pi}{5}$$

$$= \frac{5\pi}{10} - \frac{2\pi}{10}$$

$$= \boxed{\frac{3\pi}{10}}$$

$2\pi r h$

$$\int_0^1 2\pi x(\sqrt{x} - x^2)\cdot dx$$

$$= \int_0^1 2\pi\left(x^1 \cdot x^{1/2} - x^1 \cdot x^2\right) dx$$

$$= \int_0^1 2\pi\left(x^{3/2} - x^3\right) dx$$

$$= 2\pi\left(\frac{2}{5}x^{5/2} - \frac{x^4}{4}\right)\Big]_0^1$$

$$= \frac{4\pi x^{5/2}}{5} - \frac{\pi x^4}{2}\Big]_0^1$$

$$= \frac{4\pi}{5} - \frac{\pi}{2}$$

$$= \frac{8\pi}{10} - \frac{5\pi}{2}$$

$$= \boxed{\frac{3\pi}{10}}$$