Lecture

Section 5.4. The Fundamental Theorem of Calculus [I and II]

THE FUNDAMENTAL THEOREM OF CALCULUS (FTC):

Differentiation and integration are opposite processes.

Recall: The idea of "opposite processes" came up with inverse functions.

E.g., e^x and $\ln x$ are inverse functions with

$$e^{\ln x} = x \quad \text{and} \quad \ln e^x = x$$

To talk about the FTC, we will look at this definite integral:

UPPER LIMIT: make it the variable x

DUMMY VARIABLE: make it different from x

$$\int_a^x f(t)\,dt$$

LOWER LIMIT: keep as a number

Then

$F' = f$

$$\int_a^x f(t)\,dt = F(t)\Big]_a^x = \underbrace{F(x) - F(a)}_{\text{set } = g(x)} \Rightarrow$$

$$g(x) = \int_a^x f(t)\,dt$$

$\therefore$ Have "created" a definite integral that can be viewed as a FUNCTION OF x

<u>Example</u>. $g(x) = \int_1^x t^2\,dt = \left.\frac{t^3}{3}\right]_1^x = \frac{x^3}{3} - \frac{1}{3}$

1st As a function of x, can plug in various values of x:

$$g(0) = \int_1^0 t^2\,dt = \left.\frac{t^3}{3}\right]_1^0 = \frac{(0)^3}{3} - \frac{(1)^3}{3} = 0 - \frac{1}{3} = -\frac{1}{3}$$

$$g(1) = \int_1^1 t^2\,dt = \left.\frac{t^3}{3}\right]_1^1 = \frac{(1)^3}{3} - \frac{(1)^3}{3} = \frac{1}{3} - \frac{1}{3} = 0$$

$$g(2) = \int_1^2 t^2\,dt = \left.\frac{t^3}{3}\right]_1^2 = \frac{(2)^3}{3} - \frac{(1)^3}{3} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

2nd As of function of x, possibly can find its derivative;

$$\begin{aligned}
g'(x) &= \frac{d}{dx}\left(\int_1^x t^2\,dt\right) \\
&= \frac{d}{dx}\left(\frac{x^3}{3} - \frac{1}{3}\right) \\
&= \frac{d}{dx}\left(\frac{x^3}{3}\right) - \frac{d}{dx}\left(\frac{1}{3}\right) \\
&= \frac{1}{3}\frac{d}{dx}(x^3) - 0 \\
&= \frac{1}{\cancel{3}}(\cancel{3}x^2) \\
&= x^2
\end{aligned}$$

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<u>FTC</u>: $f(x)$ continuous on $[a, b]$ and $a \le x \le b$.

I. $$\boxed{\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)}$$

$$\left(\cancel{\frac{d}{dx}}\left(\cancel{\int_a^x} f(t)\,dt\right) = f(x)\right)$$

II. $$\boxed{\int_a^x \frac{d}{dt}\left(f(t)\right)\,dt = f(x) - f(a).}$$

$$\left(\cancel{\int_a^x \frac{d}{dt}}\left(f(t)\right)\,dt = f(x) - f(a)\right)$$

"<u>Proof</u>."

I: $$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = \frac{d}{dx}\left(F(x) - F(a)\right)$$

$F'(x) = f(x)$

$$= \frac{d}{dx}F(x) - \frac{d}{dx}F(a)$$

$$= F'(x) - 0$$

$$= f(x) \checkmark$$

II: $\int_a^x \frac{d}{dt}\left(f(t)\right)dt = f(t)\Big]_a^x = f(x) - f(a)$

$f(t)$ is the antiderivative of $\frac{d}{dt}\left(f(t)\right)$ because

$$f'(t) = \underbrace{\frac{d}{dt}\left(f(t)\right)}_{\text{WHAT'S INSIDE THE INTEGRAL}}$$

NOTES:

1. FTC II can also be written as

$$\int_a^b \frac{d}{dt}\left(f(t)\right)\, dt = f(b) - f(a)$$

or

$$\int_a^b f'(t)\, dt = f(b) - f(a)$$

or

$$\int_a^b F'(t)\, dt = F(b) - F(a)$$

2. Q: Why must $f(x)$ be continuous on $[a, b]$?

Short Simple Answer: So $g(x) = \int_a^x f(t)\, dt$ can be differentiable and can take its derivative

$$g'(x) = \frac{d}{dx}\left(\int_a^x f(t)\, dt\right)$$

Long Complicated Answer: This can be found on p. 389 of the text in a "real proof" of the FTC I. We will not cover it. It is covered in more advanced courses again.

<u>Examples</u>. Find the derivative of the function.

1. (Exercise 7, p. 386.) $g(x) = \int_0^x \sqrt{1+2t}\, dt$

Call the integral "$g(x)$" since integral is a function of x:

$$\int_0^x \sqrt{1+2t}\, dt = \underbrace{F(x)}_{\text{function of } x} - \underbrace{F(0)}_{\text{constant}}$$

$$g'(x) = \frac{d}{dx} g(x) = \cancel{\frac{d}{dx}}\left(\cancel{\int_0^x} \sqrt{1+2t}\, dt\right)$$

pluck this out and replace t by x (the 0 makes no difference)

$$= \boxed{\sqrt{1+2x}} \checkmark$$

2. (Exercise 9, p. 386.) $g(y) = \int_2^y t^2 \sin t \, dt$

$$g'(y) = \frac{d}{dy} g(y) = \cancel{\frac{d}{dy}}\left(\cancel{\int_2^y} t^2 \sin t \, dt\right)$$

pluck out and replace t by y
(the 2 makes no difference)

$$= \boxed{y^2 \sin y} \checkmark$$

3. (Exercise 10, p. 386.) $F(x) = \int_x^{10} \tan\theta \, d\theta$.

To apply FTC, need x as UPPER LIMIT OF INTEGRATION.

So first must apply following property (that we skipped over in Sect. 5.2):

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

<u>Proof</u>. $\int_a^b f(x)\,dx = F(b) - F(a)$

$$= -(F(a) - F(b))$$

$$= -\int_b^a f(x)\,dx \checkmark$$

$$F'(x) = \frac{d}{dx}F(x) = \frac{d}{dx}\left(\int_x^{10} \tan\theta \, d\theta\right)$$

$$= \frac{d}{dx}\left(-\int_{10}^{x} \tan\theta \, d\theta\right)$$

$$= -\cancel{\frac{d}{dx}}\left(\cancel{\int_{10}^{x}} \tan\theta \, d\theta\right)$$

$$= -(\tan x)$$

$$= \boxed{-\tan x} \checkmark$$

4. (Exercise 13, p. 386.) $y = \int_3^{\sqrt{x}} \frac{\cos t}{t}\, dt$

To apply FTC, need to replace $\sqrt{x}$ by a u so no longer have the square root. However, in doing so, we "pay a price," and this "price" is called the CHAIN RULE.

$$y' = \frac{dy}{dx} = \frac{d}{dx}\left(\int_3^{\sqrt{x}} \frac{\cos t}{t}\, dt\right)$$

$$\underbrace{\begin{array}{l} \text{① Replace } \sqrt{x} \text{ by } u: \boxed{u = \sqrt{x}} \\ \text{② Replace } \frac{d}{dx}(\ ) \text{ by } \frac{d}{du}(\ ) \\ \text{③ "Tag on" a } \frac{dx}{du} \end{array}}_{\text{CHAIN RULE}}$$

$$= \frac{d}{du}\left(\int_3^{u} \frac{\cos t}{t}\, dt\right) \cdot \frac{du}{dx}$$

$$= \frac{\cos u}{u} \cdot \underbrace{\frac{du}{dx}}_{= \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2}}$$

$$= \frac{\cos u}{u} \cdot \frac{1}{2}x^{-1/2}$$

Replace u by $\sqrt{x}$, where $u = \sqrt{x}$

1/15/02
Tues.

$$= \frac{\cos\sqrt{x}}{\sqrt{x}} \cdot \frac{1}{2} x^{-1/2}$$

$$= \frac{\cos\sqrt{x}}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$$

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Thurs.

Recall: $\sqrt{x}\sqrt{x} = \sqrt{x \cdot x} = \sqrt{x^2} = x$

1/17/02
Thurs.

$$= \boxed{\frac{\cos\sqrt{x}}{2x}} \checkmark$$