Section 5.4. The Fundamental Theorem of Calculus [I and II]

**THE FUNDAMENTAL THEOREM OF CALCULUS (FTC):**

Differentiation and integration are opposite processes.

**Recall:** The idea of "opposite processes" came up with inverse functions.

E.g., $e^x$ and $\ln x$ are inverse functions with

$$e^{\ln x} = x$$ and $$\ln e^x = x$$
To talk about the FTC, we will look at this definite integral:

\[
\int_{a}^{x} f(t) \, dt
\]

**Upper Limit:** make it the variable \( x \)

**Lower Limit:** keep as a number

Then

\[
F'(x) = f(x)
\]

\[
\int_{a}^{x} f(t) \, dt = F(x) - F(a) \quad \Rightarrow \quad g(x) = \int_{a}^{x} f(t) \, dt
\]

"Have created" a definite integral that can be viewed as a \text{FUNCTION OF} \( x \)
Example. \( g(x) = \int_1^x t^2 \, dt = \frac{t^3}{3} \bigg|_1^x = \frac{x^3}{3} - \frac{1}{3} \)

1st. As a function of \( x \), can plug in various values of \( x \):

\[
g(0) = \int_1^0 t^2 \, dt = \frac{t^3}{3} \bigg|_1^0 = \left(0 \right)^3 - \left(1 \right)^3 = 0 - \frac{1}{3} = -\frac{1}{3}
\]

\[
g(1) = \int_1^1 t^2 \, dt = \frac{t^3}{3} \bigg|_1^1 = \left(1 \right)^3 - \left(1 \right)^3 = 0 - \frac{1}{3} = -\frac{1}{3}
\]

\[
g(2) = \int_1^2 t^2 \, dt = \frac{t^3}{3} \bigg|_1^2 = \left(2 \right)^3 - \left(1 \right)^3 = 8 - \frac{1}{3} = \frac{23}{3}
\]

2nd. As of function of \( x \), possibly can find its derivative:

\[
g'(x) = \frac{d}{dx} \left( \int_1^x t^2 \, dt \right)
\]

\[
= \frac{d}{dx} \left( \frac{x^3}{3} - \frac{1}{3} \right)
\]

\[
= \frac{d}{dx} \left( \frac{x^3}{3} \right) - \frac{d}{dx} \left( \frac{1}{3} \right)
\]

\[
= \frac{1}{3} \frac{d}{dx} (x^3) - 0
\]

\[
= \frac{1}{3} \cdot 3x^2 = x^2
\]
I. "Prove that
\[ \frac{d}{dx} \left( \int_a^x f(t) \, dt \right) = f(x) - f(a) \]

II. \[ \frac{d}{dx} \left( \int_0^x \frac{f(t)}{t} \, dt \right) = f(x) \]

III. \[ \frac{d}{dx} \left( \int_0^x f(t) \, dt \right) = f(x) \]

IV. \[ \frac{d}{dx} \left( \int_a^b f(t) \, dt \right) = f(x) \]

V. f(x) continuous on [a, b] and a < x < b.
\[ \int_a^x \frac{d}{dt} \left( f(t) \right) dt = \left. f(t) \right|_a^x = f(x) - f(a) \]

\( f(t) \) is the antiderivative of \( \frac{d}{dt} (f(t)) \) because

\[ f'(t) = \frac{d}{dt} \left( f(t) \right) \]

WHAT'S INSIDE THE INTEGRAL
**NOTES:**

1. **FTC II** can also be written as
   \[ \int_a^b \frac{d}{dt}(f(t)) \, dt = f(b) - f(a) \]
   or
   \[ \int_a^b f'(t) \, dt = f(b) - f(a) \]
   or
   \[ \int_a^b F'(t) \, dt = F(b) - F(a) \]

2. **Q:** Why must \( f(x) \) be continuous on \([a, b]\)?

   **Short Simple Answer:** So \( g(x) = \int_a^x f(t) \, dt \) can be differentiable and can take its derivative
   \[ g'(x) = \frac{d}{dx} \left( \int_a^x f(t) \, dt \right) \]
Long Complicated Answer: This can be found on p. 389 of the text in a "read proof" of the FTC I. We will not cover it. It is covered in more advanced courses again.
Examples. Find the derivative of the function.

1. (Exercise 7, p. 386.) \( g(x) = \int_0^x \sqrt{1 + 2t} \, dt \)

Call the integral "\( g(x) \)" since integral is a function of \( x \):
\[
\int_0^x \sqrt{1 + 2t} \, dt = F(x) - F(0)
\]

\( g'(x) = \frac{d}{dx} g(x) = \frac{d}{dx} \left( \int_0^x (\sqrt{1 + 2t} \, dt) \right) \)

plug this out and replace \( t \) by \( x \)

(He 0 makes no difference)

\[
g'(x) = \sqrt{1 + 2x}
\]
2. (Exercise 9, p. 386.) \( g(y) = \int_0^y t^2 \sin t \, dt \)

\[
g'(y) = \frac{d}{dy} g(y) = \frac{d}{dy} \left( \int_0^y t^2 \sin t \, dt \right)
\]

Pluck out and replace \( t \) by \( y \)

(the 2 makes no difference)

\[
= y^2 \sin y \checkmark
\]
3. (Exercise 10, p. 386.) \[ F(x) = \int_x^{10} \tan \theta \, d\theta. \]

To apply FTC, need \( x \) as UPPER LIMIT OF INTEGRATION.

So first must apply following property (that we skipped over in Sect. 5.2):

\[ \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx \]

**Proof.**

\[ \int_a^b f(x) \, dx = F(b) - F(a) \]
\[ = -(F(a) - F(b)) \]
\[ = -\int_b^a f(x) \, dx \checkmark \]

\[ F'(x) = \frac{d}{dx} F(x) = \frac{d}{dx} \left( \int_x^{10} \tan \theta \, d\theta \right) \]
\[ = \frac{d}{dx} \left( -\int_0^x \tan \theta \, d\theta \right) \]
\[ = -\frac{d}{dx} \left( \int_0^x \tan \theta \, d\theta \right) \]
\[-19-\]

\[= - (\tan x)\]

\[= \sqrt{-\tan x}\]
4. (Exercise 13, p. 386.) \[ y = \int_3^{\sqrt{x}} \frac{\cos t}{t} \, dt \]

To apply FTC, need to replace \( \sqrt{x} \) by a \( u \) so no longer have the square root. However, in doing so, we "pay a price," and this "price" is called the \textit{Chain Rule}.

\[ y' = \frac{dy}{dx} = \frac{d}{dx} \left( \int_3^{\sqrt{x}} \frac{\cos t}{t} \, dt \right) \]

1. Replace \( \sqrt{x} \) by \( u : \frac{u = \sqrt{x}}{u} \)
2. Replace \( \frac{d}{dx} \left( \frac{u}{3} \right) \) by \( \frac{du}{dx} \)

3. "Tog on" a \( \frac{du}{dx} \)

\text{CHAIN RULE}

\[ \frac{d}{du} \left( \int_3^{u} \frac{\cos t}{t} \, dt \right) \cdot \frac{du}{dx} \]

\[ = \frac{\cos u}{u} \cdot \frac{du}{dx} \]

\[ = \frac{d}{dx} \left( \frac{\cos \sqrt{x}}{\sqrt{x}} \right) = \frac{d}{dx} \left( \frac{1}{2}x^{-\frac{1}{2}} \right) = \frac{1}{2}x^{-\frac{3}{2}} \]

\[ = \frac{\cos \sqrt{x}}{\sqrt{x}} \cdot \frac{1}{2}x^{-\frac{1}{2}} \]

Replace \( u \) by \( \sqrt{x} \), where \( u = \sqrt{x} \).
\[ -20a - \]

\[ \frac{\cos \sqrt{x}}{\sqrt{x}} \cdot \frac{1}{2} \cdot x^{-\frac{1}{2}} \]

\[ \frac{\cos \sqrt{x}}{\sqrt{x}} \cdot \frac{1}{2 \sqrt{x}} \]

Recall: \( \sqrt{x} \cdot \sqrt{x} = \sqrt{x^2} = x \)

\[ \boxed{\frac{\cos \sqrt{x}}{2x}} \]