Section 5.4: The Fundamental Theorem of Calculus [I and II]

These are actually two parts to this theorem. We actually had exposure to one of them in Section 5.4.

First, a couple of important properties we need to know.
Some More Properties of $\int_a^b f(x) \, dx$

\[ \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx \]

\( \Delta x = \frac{b-a}{n} \quad -\Delta x = \frac{a-b}{n} \)

Example: $\int_0^5 x^2 \, dx = -\int_5^0 x^2 \, dx$

\[ \left[ \frac{x^3}{3} \right]_0^5 - \left[ \frac{x^3}{3} \right]_5^0 \]

\[ \frac{5^3}{3} - \frac{0}{3} = \frac{125}{3} \]

\[ -\left( \frac{5^3}{3} - \frac{0}{3} \right) = -\frac{125}{3} \]
b. If \( m \leq f(x) \leq M \) on \([a, b]\)

\[
\begin{array}{c}
M \\
m \\
0 \quad a \quad b
\end{array}
\]

Then

\[ m(b-a) \leq \int_a^b f(x) \, dx \leq M(b-a) \]

(come from Property 2, p. 384, text, and from property that \( \int_a^b c \, dx = c(b-a) \)).

E.g., (HW Exercise 4, p. 390) Use Property 3 to estimate the value of the integral

\[
\int_0^2 \sqrt{x^2+1} \, dx
\]

**CALCULATOR** graph to find

\[ m = \text{lowest point of } \sqrt{x^2+1} \text{ over } [0, 2] \]

\[ M = \text{highest point of } \sqrt{x^2+1} \text{ over } [0, 2] \]
m and M are at the endpoints of \([0, 2]\). So

\[
m = f(0) = \sqrt{(0)^2 + 1} = \sqrt{1} = 1
\]

\[
M = f(2) = \sqrt{(2)^2 + 1} = \sqrt{5} = 3
\]

\[
m(b-a) \leq \int_{a}^{b} f(x) \, dx \leq M \cdot (b-a) \implies
\]

\[
1 \cdot (2-0) \leq \int_{0}^{2} \sqrt{x^2 + 1} \, dx \leq 3 \cdot (2-0) \implies
\]

\[
2 \leq \int_{0}^{2} \sqrt{x^2 + 1} \, dx \leq 6
\]
THE FUNDAMENTAL THEOREM OF CALCULUS, PART I (FTCI)

As long as \( f(x) \) is continuous on \([a, b]\) then you can take the integral of \( f(x) \) from \( a \) to \( b \),

\[
\int_a^b f(x) \, dx
\]

Change \( f(x) \, dx \) to \( f(t) \, dt \) and then replace the upper limit \( b \) by the variable \( x \) between \( a \) and \( b \),

\[
\int_a^x f(t) \, dt
\]

and you will end up with a NEW FUNCTION, say, \( g(x) \):

\[
g(x) = \int_a^x f(t) \, dt, \quad a \leq x \leq b
\]

A NEW FUNCTION

E.g., Consider \( f(x) = \frac{1}{x} \) on \([1, 10]\).

Is \( f \) cont. on \([0, 17]\)?
YES, \( f(x) = \frac{1}{x} \) is cont. on \([1, 10]\)

\[
\int_1^{10} \frac{1}{x} \, dx \quad \text{to} \quad \int_1^{10} \frac{1}{t} \, dt = [\ln t + 1]^t_1
\]

\[
= \ln 11 - \ln 1
\]

\[
= \ln x - \ln 1
\]

\[
= \ln x - 0
\]

\[
= \ln x
\]

Our new function \( g(x) \) is

\[
g(x) = \ln x, \quad 1 \leq x \leq 10
\]

NOTICE: \( g'(x) = \frac{1}{x} = f(x) \). This leads us to the next thing.
Not only do we have a new function
\[ g(x) = \int_a^x f(t) \, dt, \quad a \leq x \leq b, \]
but \( g(x) \) is DIFFERENTIABLE and
\[ g'(x) = \frac{d}{dx} \int_a^x f(t) \, dt = f(x), \quad a \leq x \leq b. \]

This says two things:

1. \[ g'(x) = f(x) \implies g \text{ is the antiderivative of } f \text{ on } [a,b]. \]
2. \[ \frac{d}{dx} \int_a^x f(t) \, dt = f(x) \implies \]
    differentiation "UNDOES" integration and you get back \( f \) (but not \( f(x) \) now), not \( f(t) \). Here the "a" can be anything and does not matter.

Examples. Use FTC I to find the derivative of the following functions.
1. (HW Exercise 11, p. 290.)

\[ g(x) = \int_1^x (t^2 - 1)^{20} \, dt \]

Two ways to do:

1. \[ g(x) = \int_1^x \frac{(t^2 - 1)^{20}}{f(t)} \, dt \]

   \[ g'(x) = f(x) = (x^2 - 1)^{20} \]

   So \( g \) is an antiderivative of \( f \).

2. \[ g'(x) = \frac{d}{dx} \int_1^x (t^2 - 1)^{20} \, dt \]

   \[ (t^2 - 1)^{20} \]

   Cancel the \( \frac{dx}{dt} \) and the \( \frac{dx}{dt} \).

   Change the \( t \) to \( x \).
2, (HW Exercise 14, p. 390.)

\[ F(x) = \int_x^2 \cos(t^2) \, dt \]

**BEFORE ANYTHING**, you must make sure the \( x \) is an \textit{UPPER}, NOT LOWER, LIMIT:

\[ F(x) = \int_x^2 \cos(t^2) \, dt = - \int_2^x \cos(t^2) \, dt \]

\[ \text{switch limits and add a negative} \]

\[ = \int_2^x - \cos(t^2) \, dt \]

\[ \text{pull the negative inside the integral to be part of the } \cos(t^2) \]

Then,

\[ F(x) = \int_2^x - \cos(t^2) \, dt \quad \Rightarrow \]

\[ f(t) \]

\[ F'(x) = f(x) = - \cos(x^2) \]
\[ F'(x) = \frac{d}{dx} \left[ \int_{2}^{x} -\cos(t^2) \, dt \right] \rightarrow -\cos(x^2) \]
As long as $f(x)$ is continuous on $[a, b]$ and $F'(x) = f(x)$, then

$$
\int_{a}^{b} f(x) \, dx = F(b) - F(a)
$$
or

$$
\int_{a}^{b} F'(x) \, dx = F(b) - F(a)
$$
or

$$
\left[ \int_{a}^{b} F(x) \, dx \right] = \left[ F(x) \right]_{a}^{b} = F(b) - F(a)
$$

This time, integration "UNDOES" differentiation and you get back $F(x)$, evaluated at $a$ and $b$. 

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\[ \frac{d}{dx} \int_a^x f(t) dt = f(x) \]

\[ \int_a^x \frac{d}{dx} F(x) dx = [F(x)]_a^b \]