Lecture

5/1/01
Tues.
Lecture

Section 5.4. The Fundamental Theorem of Calculus [I and II]

There are actually two parts to this theorem. We actually had exposure to one of them in Section 5.4

First, a couple of important properties we need to know.

Some More Properties of $\int_a^b f(x)\,dx$

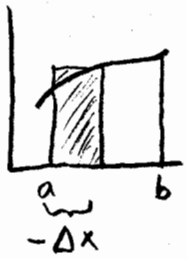
a. $$\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx}$$

Comes from LHS, RHS, or MS where

$$\Delta x = \frac{b-a}{n}$$

Comes from LHS, RHS, or MS where

$$-\Delta x = \frac{a-b}{n}$$



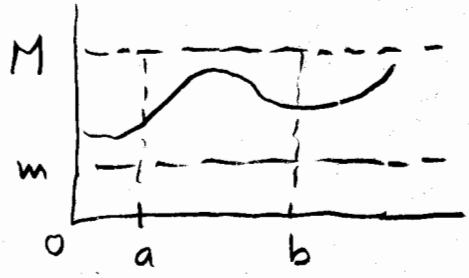
E.g., $$\int_0^5 x^2\,dx = -\int_5^0 x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^5 \qquad = -\left[\frac{x^3}{3}\right]_5^0$$

$$= \frac{(5)^3}{3} - \frac{(0)^3}{3} \qquad = -\left[\frac{(0)^3}{3} - \frac{(5)^3}{3}\right]$$

$$= \frac{125}{3} \qquad = \frac{125}{3}$$

b. If $m \le f(x) \le M$ on $[a, b]$



then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a$$

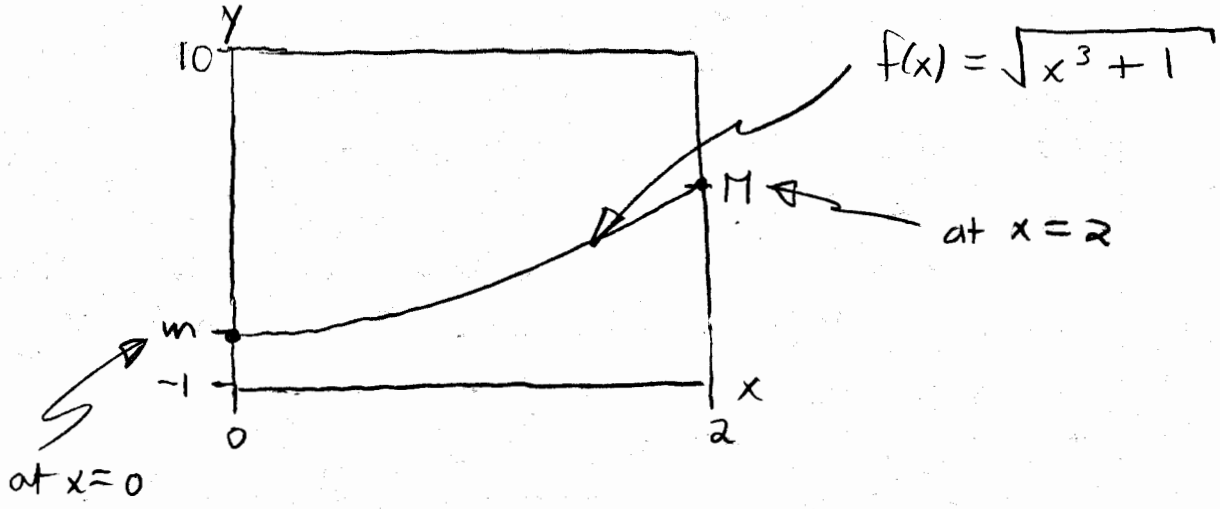
(comes from Property 2, p. 384, text, and from property that $\int_a^b c\,dx = c(b-a)$).

E.g., (HW Exercise 4, p. 390) Use Property 3 to estimate the value of the integral

$$\int_0^2 \sqrt{x^3+1}\,dx$$

CALCULATOR graph to find

$m =$ lowest point of $\sqrt{x^3+1}$ over $[0, 2]$
$M =$ highest point of $\sqrt{x^3+1}$ over $[0, 2]$



m and M are at the endpoints of $[0, 2]$. So

$$m = f(0) = \sqrt{(0)^3+1} = \sqrt{1} = 1$$
$$M = f(2) = \sqrt{(2)^3+1} = \sqrt{9} = 3$$

$$\therefore \quad m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \Rightarrow$$

$$1\cdot(2-0) \le \int_0^2 \sqrt{x^3+1}\,dx \le 3\cdot(2-0) \Rightarrow$$

$$\boxed{2 \le \int_0^2 \sqrt{x^3+1}\,dx \le 6}$$

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THE FUNDAMENTAL THEOREM OF CALCULUS, PART I (FTC I)

As long as $f(x)$ is continuous on $[a, b]$ then you can take the integral of $f(x)$ from a to b,

$$\int_a^b f(x)\,dx,$$

change "$f(x)\,dx$" to "$f(t)\,dt$" and then replace the upper limit b by the variable x between a and b,

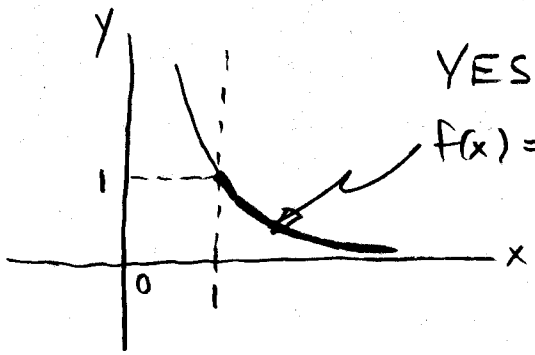
$$\int_a^x f(t)\,dt,$$

and you will end up with a NEW FUNCTION, say, $g(x)$:

$$\boxed{\underset{\text{A NEW FUNCTION}}{g(x)} = \int_a^x f(t)\,dt, \quad a \le x \le b}$$

E.g., Consider $f(x) = \frac{1}{x}$ on $[1, 10]$.

Is f cont. on $[0, 1]$?



YES,
$f(x) = \frac{1}{x}$ is cont. on $[1, 10]$

$$\int_1^{10} \frac{1}{x}\,dx \longrightarrow \int_1^{x} \frac{1}{t}\,dt = \Big[\ln|t|\Big]_1^{x}$$

$$= \ln|x| - \ln|1|$$

since $x > 0$, $|x| = x$

$$= \ln x - \ln 1$$

$$= \ln x - 0$$

$$= \ln x$$

Our new function $g(x)$! So

$$\boxed{g(x) = \ln x \quad , \quad 1 \le x \le 10}$$

NOTICE: $g'(x) = \frac{1}{x} = f(x)$. This leads us to the next thing.

Not only do we have a new function

$$g(x) = \int_a^x f(t)\,dt, \quad a \le x \le b,$$

but $g(x)$ is DIFFERENTIABLE and

$$\boxed{g'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x), \quad a \le x \le b}\ !$$

THIS SAYS TWO THINGS:

① $g'(x) = f(x) \Longrightarrow g$ is the ANTIDERIVATIVE of f on $[a, b]$.

② $\cancel{\frac{d}{dx}}\int_a^{\cancel{x}} f(t)\,dt = f(x) \Longrightarrow$ differentiation "UNDOES" integration and you get back f (but $f(x)$ now, not $f(t)$). Here the "a" can be anything and does not matter.

<u>Examples</u>. Use FTC I to find the derivative of the following functions.

1. (HW Exercise 11, p. 390.)

$$g(x) = \int_1^x (t^2-1)^{20}\, dt$$

Two ways to do:

① $$g(x) = \int_1^x \underbrace{(t^2-1)^{20}}_{f(t)}\, dt \Rightarrow$$ so g is an antiderivative of f

$$g'(x) = f(x) = \boxed{(x^2-1)^{20}}$$

② $$g'(x) = \cancel{\frac{d}{dx}} \cancel{\int_1^x} (t^2-1)^{20}\, dt \rightarrow (t^2-1)^{20}$$

cancel the $\frac{d}{dx}$ and the $\int_1^x$

$$\rightarrow \boxed{(x^2-1)^{20}}$$

change the t to an x

2. (HW Exercise 14, p. 390.)

$$F(x) = \int_x^2 \cos(t^2)\, dt$$

BEFORE ANYTHING, you must make sure the x is an UPPER, NOT LOWER, LIMIT:

$$F(x) = \int_x^2 \cos(t^2)\, dt = -\int_2^x \cos(t^2)\, dt$$

switch limits and add a negative

$$= \int_2^x -\cos(t^2)\, dt$$

pull the negative inside the integral to be part of the $\cos(t^2)$

Then

① $$F(x) = \int_2^x \underbrace{-\cos(t^2)}_{f(t)}\, dt \implies$$

$$F'(x) = f(x) = \boxed{-\cos(x^2)}$$

(2) $F'(x) = \cancel{\frac{d}{dx}} \int_2^{\cancel{x}} -\cos(t^2)\,dt \longrightarrow -\cos(t^2)$

$\longrightarrow \boxed{-\cos(x^2)}$

FTC II

As long as $f(x)$ is continuous on $[a, b]$ and $F'(x) = f(x)$, then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

or

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

or

$$\boxed{\int_a^b \frac{d}{dx} F(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)}$$

This time integration "UNDOES" differentiation and you get back $F(x)$, evaluated at a and b.

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$\therefore$ FTC I : $\cancel{\frac{d}{dx}}\cancel{\int_a^x} f(t)\,dt = f(x)$

FTC II : $\cancel{\int_a^b}\ \cancel{\frac{d}{dx}}\ F(x)\,dx = \Big[F(x)\Big]_a^b$